2A Readiness Guide

Introduction

This assessment is intended to help students who have been in the Explorer's (level 1) class decide if they are ready to enter the Novice (level 2A) class. It can also be used to help a student coming to the Academy for the first time decide if they are ready to jump right into the 2A class or if they should try the level 1 class first. First, let us highlight some of the critical aspects of the 2A class that differentiate it from Explorer's.

- The emphasis in 2A shifts toward less concrete problems, students will be required to think abstractly in cases such as "an arbitrary triangle with angles α, β, γ" and "the sum of all $\frac{1}{n^2+n}$ where n is all positive integers".
- Topics in 2A are generally more conceptual and understanding them requires abstract thought.
- There is a greater emphasis in 2A on proofs, mathematical writing, and notation than is present in Explorer's.
- Students will likely find the problems in 2A much more challenging than they found them in Explorer's, not just in terms of level of topic. The nature of these problems often requires more thought and perseverance and possibly multiple attempts at different approaches before the right approach is found.

Of course, mathematical knowledge must also be considered. If you believe your child is ready for the conceptual and abstract thinking involved in the 2A class and has the persistence to pursue challenging problems for which there is no quick solution, give them the following test to gauge their mathematical aptitude. The test is divided into two sections:

1. **Fundamental Skills** - These are to test basic skills the student will need as prerequisites for the topics to be covered in 2A. A student considering 2A should ideally be able to complete all of these with relative ease.
2. **Problem Solving** - These are to test their problem solving abilities, a student considering 2A should be able to complete at least 3 of the problems.

Fundamental Skills

1. Factor:

   (a) $a^2 - 2ab + b^2$

   (b) $x^2 + 13x + 36$

2. Simplify: $\dfrac{(1+x)(2+y)(y-z)}{2(2-y)(z-y)(x+1)}$

3. Solve the system:

$$2a + 3b = 93$$

$$3a + 2b = 97$$

4. Find the prime factorization of 7920.

5. Find a function that describes the sequence: $3, 6, 12, 24, 48, ...$

6. A triangle ABC has $\angle B = 63^\circ$ and $\angle C = 51^\circ$. The angle bisectors of $\angle B$ and of $\angle C$ intersect at I. Find $\angle AIB$.

7. Simplify: $\dfrac{4^{13}}{8^7}$

Problem Solving

1. Find all primes that are one less than a perfect square. Prove that the ones you listed are the only possible primes that fit this criteria.

2. If $x + \frac{1}{x} = 1$, compute $x^3 + \frac{1}{x^3}$.

3. If $a^2 + ab = 5$ and $b^2 + ab = 4$, find $\frac{a}{b}$.

4. Find all integers n such that $n^2 + 4n$ is a perfect square.

5. Find the 5050th term of the sequence $1, 2, 2, 3, 3, 3, 4, 4, 4, 4, ...$

Solutions

Fundamental Skills

1. (a) $(a-b)^2$
   (b) $(x+4)(x+9)$

2. $-\dfrac{2+y}{2(2-y)}$

3. $a=21, b=17$

4. $2^4 \cdot 3^2 \cdot 5 \cdot 11$

5. $f(x)=3\cdot 2^{x-1}$ or $a_n = 3\cdot 2^{n-1}$

6. $123°$

7. 2^5

Problem Solving

1. Write this as $p=x^2-1$ where p is the desired prime number and x is some positive integer. This is factorable as $p=(x-1)(x+1)$, since p is prime one of the factors must be 1 and the other must be p. $x-1=1$ and $p=x+1$ yields $p=3$ as the only possible result.

2. Cubing both sides of the given equation yields $(x+\frac{1}{x})^3 = x^3+3x+\frac{3}{x}+\frac{1}{x^3} = x^3+\frac{1}{x^3}+3(x+\frac{1}{x}) = 1^3$. Substituting the given equation for $x+\frac{1}{x}$ yields $x^3+\frac{1}{x^3}+3(1)=1$, thus
   $x^3+\frac{1}{x^3}=-2$

3. Factoring both equations yields $a(a+b)=5$ from the first and $b(a+b)=4$ from the second. Dividing the first by the second: $\dfrac{a(a+b)}{b(a+b)} = \dfrac{5}{4}$, $\dfrac{a+b}{a+b}$ cancels so the result yielded is
   $\frac{a}{b}=\frac{5}{4}$.

4. There are a number of ways of algebraically getting to a solution, one with minimal casework. We can write this as $n^2+4n=x^2$ where x is some positive integer. Add 4 to both sides yields $n^2+4n+4=x^2+4$, the left side can be factored as $(n+2)^2=x^2+4$, rearranging yields $(n+2)^2-x^2=4$, factor the left hand side to get $(n+2-x)(n+2+x)=4$, the two factors must be either 1 and 4 or both 2. The second case yields $x=0, n=0$, the first case yields $n+2-x=1, n+2+x=4$. Adding the two together yields $2n+4=5$ thus $n=\frac{1}{2}$, not an integer solution.
   The only solution is $n=0$.

5. Note the pattern, for each value n in the sequence we can write it n times. This means the total number of terms of the sequence where n is the last term written

would be $1 + 2 + 3 + 4 + ... + n = \frac{n(n+1)}{2}$. At this point one can plug in values for n until one gets 5050 or more, which happens when $n = 100$
So 100 is the 5050th term.