AwesomeMath Academy Novice: Problem Solving 2 Placement Test

Introduction

This assessment is intended to help students decide if the Novice: Problem Solving 2 class is a good fit for them. The test is divided into two sections:

- **Prerequisite Skills** - These are to test basic skills the student will need as *prerequisites* for the topics to be covered in Novice: Problem Solving 2. A student considering this class should be able to complete all of these with relative ease. A student having difficulty with this section may want to look toward the Beginner: Problem Solving 1 class, as these skills will be taught there.
- **Problem Solving** - These are to test the student's problem solving abilities. A student who completes 0-2 of these problems should find Novice: Problem Solving 2 to be the right difficulty for them. A student who completes 3 or more might consider looking at the Intermediate: Problem Solving 3 class instead.

Prerequisite Skills

1. Factor:

    (a) $a^2 - 2ab + b^2$

    (b) $x^2 + 13x + 36$

2. Simplify: $\dfrac{(1+x)(2+y)(y-z)}{2(2-y)(z-y)(x+1)}$

3. Solve the system:

$$2a + 3b = 93$$
$$3a + 2b = 97$$

4. Find the prime factorization of 7920.

5. Find a function that describes the sequence: $3, 6, 12, 24, 48, ...$

6. A triangle ABC has $\angle B = 63°$ and $\angle C = 51°$. The angle bisectors of $\angle B$ and of $\angle C$ intersect at I. Find $\angle BIC$.

7. Simplify: $\dfrac{4^{13}}{8^7}$

Problem Solving

1. A point P is chosen in the interior of triangle ABC so that when lines are drawn through P parallel to the sides of $\triangle ABC$, they divide ABC into three triangles and three quadrilaterals. The three triangles have areas 4, 9, and 49. Find the area of ABC.

2. If $x + \frac{1}{x} = 1$, compute $x^3 + \frac{1}{x^3}$.

3. The expression $(x+y+z)^{2006} + (x-y-z)^{2006}$ is simplified by expanding it and combining like terms. How many terms are in the simplified expression?

4. What is the size of the largest subset of $\{1, 2, 3, \ldots, 1000\}$ such that no pair of distinct elements has a sum divisible by 7?

5. Find the 5050th term of the sequence $1, 2, 2, 3, 3, 3, 4, 4, 4, 4, ...$

Solutions

Prerequisite Skills

1. (a) $(a-b)^2$
   (b) $(x+4)(x+9)$

2. $-\dfrac{2+y}{2(2-y)}$

3. $a = 21, b = 17$

4. $2^4 \cdot 3^2 \cdot 5 \cdot 11$

5. $f(x) = 3 \cdot 2^{x-1}$ or $a_n = 3 \cdot 2^{n-1}$

6. $123°$

7. 2^5

Problem Solving

1. Since all the small triangles are made by parallel lines, a short angle chase shows all of the small triangles are similar to triangle ABC, with the same orientation. Further, the three quadrilaterals that are formed are parallelograms, with sides touching the triangles. Using that opposite sides of a parallelogram are equal, we can show that any side of triangle ABC is the sum of the corresponding sides of the three small triangles. Then, since the area ratio between the small triangles is $4 : 9 : 49$, the linear ratio between the sides would be $2 : 3 : 7$. Then the linear ratio between the smallest triangle and ABC is $2 : 12$, meaning the area ratio is $4 : 144$. Triangle ABC has area 144.

2. Cubing both sides of the given equation yields $(x+\frac{1}{x})^3 = x^3+3x+\frac{3}{x}+\frac{1}{x^3} = x^3+\frac{1}{x^3}+3(x+\frac{1}{x}) = 1^3$. Substituting the given equation for $x+\frac{1}{x}$ yields $x^3+\frac{1}{x^3}+3(1) = 1$, thus
   $x^3 + \frac{1}{x^3} = -2$

3. It's much easier to start by making the substitution $a = y + z$ so the expression becomes $(x+a)^{2006} + (x-a)^{2006}$, then consider what happens within our binomial theorem for such an expansion, any term with an odd power of a will cancel between the two expansions. Once you've expanded and canceled, you must realize that each a^k will produce $k + 1$ terms.
   $1 + 3 + 5 + ... + 2007 = 1004^2 = 1008016$

4. Consider each number's remainder upon division by 7, any two numbers whose remainders sum to 7 will have a sum divisible by 7. Dividing 1000 by 7, we get 142 with remainder 6. This means that there will be 142 each of numbers with remainder $1, 2, 3, 4, 5$ or 6. We can take a full set of remainder n to place in our subset so long as we do not take any numbers with a remainder $7 - n$, so taking

the first three remainders and not the last three nearly maximizes our subset. The only thing that remains is to take exactly one multiple of 7, as two multiples of 7 would have their sum divisible by 7.
So the largest subset is $3 \cdot 142 + 1 = 427$.

5. Note the pattern, for each value n in the sequence we can write it n times. This means the total number of terms of the sequence where n is the last term written would be $1 + 2 + 3 + 4 + ... + n = \frac{n(n+1)}{2}$. At this point one can plug in values for n until one gets 5050 or more, which happens when $n = 100$
So 100 is the 5050th term.