AwesomeMath Academy Problem Solving 2 Placement Test

Introduction

This assessment is intended to help students decide if the Novice: Problem Solving 2 class is a good fit for them. The test is divided into two sections:

- **Prerequisite Skills** - These are to test basic skills the student will need as *prerequisites* for the topics to be covered in Problem Solving 2. A student considering this class should be able to complete all of these with relative ease. A student having difficulty with this section may want to look toward the Problem Solving 1 class, as these skills will be taught there.

- **Problem Solving** - These are to test the student's problem solving abilities. A student who completes 0-2 of these problems should find Problem Solving 2 to be the right difficulty for them. A student who completes 3 or more might consider looking at the Problem Solving 3 class instead.

Prerequisite Skills

1. Factor:

    (a) $a^2 - 2ab + b^2$

    (b) $x^2 + 13x + 36$

2. Simplify: $\dfrac{(1+x)(2+y)(y-z)}{2(2-y)(z-y)(x+1)}$

3. Solve the system:

$$2a + 3b = 93$$
$$3a + 2b = 97$$

4. Find the prime factorization of 7920.

5. Find a function that describes the sequence: $3, 6, 12, 24, 48, ...$

6. A triangle ABC has $\angle B = 63°$ and $\angle C = 51°$. The angle bisectors of $\angle B$ and of $\angle C$ intersect at I. Find $\angle BIC$.

7. Simplify: $\dfrac{4^{13}}{8^7}$

Problem Solving

1. A point P is chosen in the interior of triangle ABC so that when lines are drawn through P parallel to the sides of $\triangle ABC$, they divide ABC into three triangles and three quadrilaterals. The three triangles have areas 4, 9, and 49. Find the area of ABC.

2. Let x and y be positive integers such that $7x^5 = 11y^{13}$. The minimum value of x can be written in the form $a^c b^d$, where a, b, c, d are positive integers. Compute $a + b + c + d$.

3. Consider the polynomial $P(x) = 9x^4 - 3x^3 - 101x^2 + 195x - 100$, and let r_1, r_2, r_3, r_4 be its roots. Compute

$$(r_1 + r_2 + r_3)(r_1 + r_2 + r_4)(r_1 + r_3 + r_4)(r_2 + r_3 + r_4)$$

4. The expression $(x + y + z)^{2006} + (x - y - z)^{2006}$ is simplified by expanding it and combining like terms. How many terms are in the simplified expression?

5. Find the 5050th term of the sequence $1, 2, 2, 3, 3, 3, 4, 4, 4, 4, ...$

6. Two circles, C_1 and C_2, intersect at points X and Y. The center of circle C_1, called point O, lies on circle C_2 as well. From point Z on circle C_2 chords are drawn to points Y, O, and X, of lengths 7, 11, and 13 respectively. Find the radius of the circle C_1.

Solutions

Prerequisite Skills

1. (a) $(a-b)^2$
   (b) $(x+4)(x+9)$
2. $-\dfrac{2+y}{2(2-y)}$
3. $a = 21, b = 17$
4. $2^4 \cdot 3^2 \cdot 5 \cdot 11$
5. $f(x) = 3 \cdot 2^{x-1}$ or $a_n = 3 \cdot 2^{n-1}$
6. $123°$
7. 2^5

Problem Solving

1. Since all the small triangles are made by parallel lines, a short angle chase shows all of the small triangles are similar to triangle ABC, with the same orientation. Further, the three quadrilaterals that are formed are parallelograms, with sides touching the triangles. Using that opposite sides of a parallelogram are equal, we can show that any side of triangle ABC is the sum of the corresponding sides of the three small triangles. Then, since the area ratio between the small triangles is $4:9:49$, the linear ratio between the sides would be $2:3:7$. Then the linear ratio between the smallest triangle and ABC is $2:12$, meaning the area ratio is $4:144$. Triangle ABC has area 144.

2. We may note by the given equation that $7|y$ and $11|x$, notably these are the only prime factors that need be present among the two sides of the equation. From $7|y$ we find $7^{13}|y^{13}$, so $7^{13}|7x^5$. It must be then that $7|x$ as well. Similarly we find $11|y$. Let $x = 7^c 11^d$ and $y = 7^e 11^f$. Plugging these into the given equation, we find $7^{5c+1}11^{5d} = 7^{13e}11^{13f+1}$. Then $5c+1 = 13e$ has minimal value 26 when $c = 5$ and $5d = 13f + 1$ has minimal value 40 when $d = 8$. We conclude that $x = 7^5 11^8$ so $a + b + c + d = 31$

3. By Viete's Relations, $r_1 + r_2 + r_3 + r_4 = \frac{1}{3}$. Then the desired product is $\frac{1}{3} - r_1)(\frac{1}{3} - r_2)(\frac{1}{3} - r_3)(\frac{1}{3} - r_4)$. Note that by Root Factor Theorem, $P(x) = 9(x - r_1)(x - r_2)(x - r_3)(x - r_4)$ so we need only compute $\frac{1}{9}P(\frac{1}{3}) = -\frac{176}{9}$

4. It's much easier to start by making the substitution $a = y + z$ so the expression becomes $(x+a)^{2006} + (x-a)^{2006}$, then consider what happens within our binomial theorem for such an expansion, any term with an odd power of a will cancel between the two expansions. Once you've expanded and canceled, you must realize that each

a^k will produce $k+1$ terms.
$1+3+5+...+2007 = 1004^2 = 1008016$

5. Note the pattern, for each value n in the sequence we can write it n times. This means the total number of terms of the sequence where n is the last term written would be $1+2+3+4+...+n = \frac{n(n+1)}{2}$. At this point one can plug in values for n until one gets 5050 or more, which happens when $n = 100$
So 100 is the 5050th term.

6. Note that the radius is $OX = OY = r$. Furthermore, $\angle XZO = \angle YZ0$ as inscribed angles overlooking equal arcs. Then Law of Cosines on $\triangle XZO$ and $\triangle YZO$ give a pair of equations which can be combined to eliminate $\cos\angle$ and leave an equation of only r.
$r = \sqrt{30}$