AwesomeMath Academy Placement Test

1. Find the number of 6 digit numbers such that each digit occurs at most twice.

2. Let $\triangle ABC$ be a triangle such that $\angle BAC = 78^\circ$. Let E and F be a point inside triangle ABC such that

   $$\angle ABE = \angle EBF = \angle FBC \quad \text{and} \quad \angle ACF = \angle FCE = \angle ECB.$$

   Let D be the point of the intersection of EC and BF. Find the value $\angle BEC + \angle CFB + \angle EDF$.

3. Find, with proof, the value of $(i - \sqrt{3})^{2017}$.

4. Let a and b be positive integers such that $\gcd(a, b) = 1$. Find all possible values of the $\gcd(5a+2b, 5b+2a)$.

5. Find the remainder after $x^{2017} + 7x^{2016} + 7^2x^{2015} + \cdots + 7^{2016}x + 7^{2017}$ is divided by $(x-7)^2$.

6. Find the number of nonnegative integer solutions to the equation

   $$2x + 3y + 5z = 2017.$$

7. Let ω be a circle with radius 3. Point P is located outside the circle. A tangent to ω through P intersects ω at A, and a secant through P intersects ω at two points B and C with B closer to P than C is. Suppose $AP = 5$ and that the distance from the center of ω to BC is 2. Compute BP.

8. Let $a_{n+1} = a_n + a_{n-1}$. Find the largest integer a_1 such that, for some positive integer a_2 we have $a_{10} = 2017$.

9. Find the value $\dfrac{1}{2^{2018}} \displaystyle\sum_{n=0}^{\infty} (-3)^n \binom{2018}{2n}$

10. Let a and b the be the real roots of the equation

    $$x^2 - (k-2)x + (k^2 + 3k + 5) = 0$$

    What is the largest value of $a^2 + b^2$?

11. Find the smallest positive integer n such that 217 divides $3^n - 1$.

12. Consider a regular hexagonal pool table $ABCDEF$. Initially, the ball is placed at the midpoint P of AB. When the ball is hit, it bounces off of side BC at point Q, then it bounces off CD, DE, EF, and FA in that order before it hits side AB again Determine all possible values of $\angle BPQ$.

Answers

1. Find the number of 6 digit numbers such that each digit occurs at most twice.
ANSWER: $\boxed{758160}$

2. Let $\triangle ABC$ be a triangle such that $\angle BAC = 78^\circ$. Let E and F be a point inside triangle ABC such that

$$\angle ABE = \angle EBF = \angle FBC \quad \text{and} \quad \angle ACF = \angle FCE = \angle ECB.$$

Let D be the point of the intersection of EC and BF. Find the value $\angle BEC + \angle CFB + \angle EDF$.
ANSWER: $\boxed{404}$

3. Find, with proof, the value of $(i - \sqrt{3})^{2017}$.
ANSWER: $\boxed{2^{2016}(i - \sqrt{3})}$

4. Let a and b be positive integers such that $\gcd(a, b) = 1$. Find all possible values of the $\gcd(5a+2b, 5b+2a)$.
ANSWER: $\boxed{1, 3, 7, 21}$

5. Find the remainder after $x^{2017} + 7x^{2016} + 7^2x^{2015} + \cdots + 7^{2016}x + 7^{2017}$ is divided by $(x - 7)^2$.
ANSWER: $\boxed{\binom{2018}{2}(x-7)7^{2016} + \binom{2018}{1} \cdot 7^{2017}}$

6. Find the number of nonnegative integer solution to the equation

$$2x + 3y + 5z = 2017.$$

ANSWER: $\boxed{68141}$

7. Let ω be a circle with radius 3. Point P is located outside the circle. A tangent to ω through P intersects ω at A, and a secant through P intersects ω at two points B and C with B closer to P than C is. Suppose $AP = 5$ and that the distance from the center of ω to BC is 2. Compute BP.
ANSWER: $\boxed{\sqrt{30} - \sqrt{5}}$

8. Let $a_{n+1} = a_n + a_{n-1}$. Find the largest integer a_1 such that, for some positive integer a_2 we have $a_{10} = 2017$.
ANSWER: $\boxed{75}$

9. Find the value $\dfrac{1}{2^{2018}} \displaystyle\sum_{n=0}^{\infty} (-3)^n \binom{2018}{2n}$
ANSWER: $\boxed{-1/2}$

10. Let a and b the be the real roots of the equation

$$x^2 - (k-2)x + (k^2 + 3k + 5) = 0$$

What is the largest value of $a^2 + b^2$?
ANSWER: $\boxed{18}$

11. Find the smallest positive integer n such that 217 divides $3^n - 1$.
ANSWER: $\boxed{30}$

12. Consider a regular hexagonal pool table $ABCDEF$. Initially, the ball is placed at the midpoint P of AB. When the ball is hit, it bounces off of side BC at point Q, then it bounces off CD, DE, EF, and FA in that order before it hits side AB again Determine all possible values of $\angle BPQ$.

ANSWER: $\boxed{\arctan \frac{3\sqrt{3}}{10} < \theta < \arctan \frac{3\sqrt{3}}{8}}$