Introduction to Combinatorics Readiness Guide

Introduction

This class is intended for students who have already had some exposure to competition math and/or older students who have taken algebra in school and are comfortable working with variables and exponents. In addition to the mathematical background, the following are expected:

- Students must be able to listen attentively during lessons and be respectful to the instructor and their peers.
- Students must be driven and persistent in working on challenging material and problems that will take time and tenacity to understand.
- Students must be curious about the reasons why patterns exist or why solutions work, not just care about the final answer.
- Students must be willing to ask for help if they find themselves struggling.

On the next page are some problems testing basic skills. A student considering this course should be able to complete these with relative ease. If your child stuggles with these exercises but is still interested in the topics covered in this course, consider looking at Math that Counts.

In addition to the fundamental skills problems, there is a logic puzzle along with some questions to guide your child towards discovering and understanding an underlying pattern. It is recommended that you read this exercise to your child and watch them work through it to help gauge their readiness for the class. A child who gets very involved in and enjoys solving the problem as well as being able to effectively explain their thought process behind their answers is a good fit for the class.

Note that while there is significant overlap in the topics covered in Math that Counts and this course, this class will explore those topics in more depth and with more generality. There will be more of an emphasis on rigor (including mathematical proofs) and notation, and students will be expected to already have experience and some amount of skill in connecting ideas and breaking down more complicated problems as needed. In this respect, observe the differences in the logic puzzle below versus the puzzle in the readiness guide for Math that Counts. While there is still scaffolding provided, students are expected to explore and test ideas on their own with less direct guidance.

Fundamental Skills

1. Expand the following:

   (a) $(x+y)^4$

   (b) $(x-2y)^3$

   (c) $(x+y+z)^2$

2. Simplify the following:

   (a) $\dfrac{x^7(x-1)(2x+4)(x^2-y^2)}{5(1-x)(x+2)(x+3)(x+y)}$

   (b) $\dfrac{4^{11}}{8^5}$

A Puzzle

Suppose we have n mathematicians standing in a line all facing the same direction. A test administrator enters the room and explains that they have a total $n-1$ red hats and n blue hats. The administrator places one hat on each person's head. Each person in line can see the hats of everyone in front of them (but not their own hat nor the hats of anyone behind them). Thus, the person at the front of the line cannot see anyone's hat, the second person in line can see only the first person's hat, and so on through the nth person in line who can see everyone's hats but their own.

The test administrator moves from the back of the line to the front of the line, asking each person in turn whether they know the color of their own hat. If a person knows the color of their hat with 100% certainty, they will answer with that color. Otherwise, they will admit that they do not know their hat color. All of the people in line are perfectly rational, so if there is a logical conclusion they can reach, they will figure it out.

Ultimately, we want to give a general rule for who will be the first person who answers that they know the color of their hat. This rule should account for the case where no one knows the color of their hat.

1. Let's start with a specific number for n. Suppose we have four people in line and start with a pool of three red hats and four blue hats. Give an example of an assignment of hats such that the fourth person in line knows their hat color. If no such example exists, explain why.

2. Again, suppose we have four people in line and a pool of three red hats and four blue hats. Give an example of an assignment of hats such that *no one* in line knows their hat color. If no such example exists, explain why.

3. Now we'll look at a much bigger number. Suppose we have 99 people in line, 98 red hats, and 99 blue hats. The 50th person in line is the first person to know what color hat they have (person 51 to person 99 all answered they did not know). What color is the 50th person's hat? How do you know?

4. Now we return to the original goal. Give a general rule for who will be the first person who answers that they know the color of their hat. This rule should account for the case where no one knows the color of their hat.

Fundamental Skills Answers

1. (a) $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
   (b) $x^3 - 6x^2y + 12xy^2 - 8y^3$
   (c) $x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$
2. (a) $\dfrac{-2x^7(x-y)}{5(x+3)}$
   (b) $2^7 = 128$

Puzzle Solutions

1. If the first three people in line all have red hats, the fourth person in line will know they have a blue hat since there are only three red hats total.

2. There is no example of an assignment where no one knows their hat color. If a child comes to this conclusion by simply drawing all possible cases, they may be better suited for Math that Counts. A good reasoning would be to think about what it means for the fourth person not to know their hat color (they should know this from their answer to the first part of the puzzle), then the third, then the second, then the first rather than resorting to looking at every individual case.

3. The 50th person's hat is blue. In this case, the numbers are too big for a child to reasonably write down every possibility. It is possible a child faced with this problem will say that they do not know how to solve it; this is actually preferable to simply writing down random cases. If a child cannot make any progress towards solving this problem, however, they may be better suited for Math that Counts.

   A child should try to make connections with the previous parts of the problem and use what they have learned to solve this larger problem. They may decide to think about some other smaller cases to help them figure out a pattern before jumping straight to 99 people; this is a good sign of mathematical maturity. The pattern a child should notice from the past parts and any further exploration they do is that the first person who knows their hat color always has a blue hat. They might then guess that this person also has a blue hat.

The child should next observe that the 99th person not knowing their hat color means they see at least one blue hat. Since the 98th person knows that the 99th person will only answer "I don't know" if they see a blue hat, they will know their hat is blue unless they see another blue hat in front of them. This pattern continues by inductive logic for the 97th person and so on. In this case, the 50th person will know that the 51st person saw a blue hat in front of them. If there were a blue hat in front of the 50th person, they would not be sure which color hat they have. So all hats in front of the 50th person must be red, and the 50th person will conclude that their hat is blue.

4. At this point after working through the logic of the larger numbers in part 3, the child should realize that the first person who knows their hat color is the first person to see no blue hats in front of them, and their hat will always be blue. If a child is able to come to this conclusion or at least mostly articulate it, they will fit well in this course.