Level 2 Placement Quiz Solutions

1. By Vieta’s formulas, \( m + n = -9/7 \) and \( mn = 21/7 = 3 \), so

\[
(m + 7)(n + 7) = mn + 7(m + n) + 49 = 3 + 7 \cdot \left(-\frac{9}{7}\right) + 49 = 43.
\]

2. Let \( r \) be the radius of the large semicircle; then its arc length is \( \frac{1}{2} \cdot 2\pi r = \pi r = 12 \), so \( r = 12 \). Then the radius of both small semicircles is \( \frac{12}{2} = 6 \), so the area of the shaded region is \( \frac{1}{2} \cdot \pi \cdot 12^2 - 2 \cdot \frac{1}{2} \cdot \pi \cdot 6^2 = 36\pi \).

3. The smallest 3-digit number that is divisible by 7 is \( 105 = 7 \cdot 15 \), and the largest 3-digit number that is divisible by 7 is \( 994 = 7 \cdot 142 \), so the number of 3-digit numbers that are divisible by 7 is \( 142 - 15 + 1 = 128 \).

4. The prime factorization of 720 is \( 2^4 \cdot 3^2 \cdot 5 \). A number is a perfect cube if and only if every exponent in the prime factorization is divisible by 3, so the smallest such \( n \) is \( 2^2 \cdot 3^2 \cdot 5^2 = 300 \).

5. If we join any two vertices of the 12-gon, then we obtain either a side or a diagonal. There are \( \binom{12}{2} = 66 \) ways to choose a pair of vertices, and there are 12 sides, so there are \( 66 - 12 = 54 \) diagonals.

6. We can split up the given sum into two infinite geometric series. The first series is

\[
\sum_{n=0}^{\infty} \frac{3^n}{6^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1 - 1/2} = 2.
\]

The second series is

\[
\sum_{n=0}^{\infty} \frac{(-2)^n}{6^n} = \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n = \frac{1}{1 - (-1/3)} = \frac{3}{4}.
\]

Therefore, the given sum is \( 2 + 3/4 = 11/4 \).

7. If no two people chose the same number, then all three people chose different numbers. After Jacob chooses his number, the probability that Karina chooses a different number is \( 4/5 \), and the probability that Mark chooses a number different from both Jacob and Karina is \( 3/5 \), so the probability that all three people chose different numbers is \( 4/5 \cdot 3/5 = 12/25 \). Therefore, the probability that at least two people chose the same number is \( 1 - 12/25 = 13/25 \).

8. Each angle of the regular pentagon is \( 108^\circ \), and each angle of the regular hexagon is \( 120^\circ \), so \( \angle BAC = 360^\circ - 108^\circ - 120^\circ = 132^\circ \). Triangle \( ABC \) is isosceles with \( AB = AC \), so \( \angle ABC = (180^\circ - 132^\circ)/2 = 24^\circ \).
9. Let \( n - 50 = a^2 \) and \( n + 50 = b^2 \). Then \( b^2 - a^2 = 100 \), which is the same as \( (b-a)(b+a) = 2^2 \cdot 5^2 \). Since \( b - a \) and \( b + a \) are positive integers of the same parity, the only possibility is that \( b - a = 2 \), \( b + a = 50 \), which gives \( b = 26 \), \( a = 24 \) and thus \( n = 626 \).

10. Expanding \( (x + \frac{1}{x})^2 = 3 \), we get \( x^2 + 2 + \frac{1}{x^2} = 3 \), so \( x^2 + \frac{1}{x^2} = 1 \). Then by the sum-of-cubes factorization,

\[
x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right) \left(x^2 - 1 + \frac{1}{x^2}\right) = 0.
\]