Level 2 Placement Quiz Solutions

1. By Vieta's formulas, $m+n=-9/7$ and $mn=21/7=3$, so
$$(m+7)(n+7)=mn+7(m+n)+49=3+7\cdot\left(-\frac{9}{7}\right)+49=43.$$

2. Let r be the radius of the large semicircle; then its arc length is $\frac{1}{2}\cdot 2\pi r=\pi r=12$, so $r=12$. Then the radius of both small semicircles is $12/2=6$, so the area of the shaded region is $\frac{1}{2}\cdot\pi\cdot 12^2-2\cdot\frac{1}{2}\cdot\pi\cdot 6^2=36\pi$.

3. The smallest 3-digit number that is divisible by 7 is $105=7\cdot 15$, and the largest 3-digit number that is divisible by 7 is $994=7\cdot 142$, so the number of 3-digit numbers that are divisible by 7 is $142-15+1=128$.

4. The prime factorization of 720 is $2^4\cdot 3^2\cdot 5$. A number is a perfect cube if and only if every exponent in the prime factorization is divisible by 3, so the smallest such n is $2^2\cdot 3\cdot 5^2=300$.

5. If we join any two vertices of the 12-gon, then we obtain either a side or a diagonal. There are $\binom{12}{2}=66$ ways to choose a pair of vertices, and there are 12 sides, so there are $66-12=54$ diagonals.

6. We can split up the given sum into two infinite geometric series. The first series is
$$\sum_{n=0}^{\infty}\frac{3^n}{6^n}=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n=\frac{1}{1-1/2}=2.$$
The second series is
$$\sum_{n=0}^{\infty}\frac{(-2)^n}{6^n}=\sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n=\frac{1}{1-(-1/3)}=\frac{3}{4}.$$
Therefore, the given sum is $2+3/4=11/4$.

7. If no two people chose the same number, then all three people chose different numbers. After Jacob chooses his number, the probability that Karina chooses a different number is $4/5$, and the probability that Mark chooses a number different from both Jacob and Karina is $3/5$, so the probability that all three people chose different numbers is $4/5\cdot 3/5=12/25$. Therefore, the probability that at least two people chose the same number is $1-12/25=13/25$.

8. Each angle of the regular pentagon is 108°, and each angle of the regular hexagon is 120°, so $\angle BAC=360^\circ-108^\circ-120^\circ=132^\circ$. Triangle ABC is isosceles with $AB=AC$, so $\angle ABC=(180^\circ-132^\circ)/2=24^\circ$.

9. Let $n - 50 = a^2$ and $n + 50 = b^2$. Then $b^2 - a^2 = 100$, which is the same as $(b-a)(b+a) = 2^2 \cdot 5^2$. Since $b - a$ and $b + a$ are positive integers of the same parity, the only possibility is that $b - a = 2$, $b + a = 50$, which gives $b = 26$, $a = 24$ and thus $n = 626$.

10. Expanding $(x + \frac{1}{x})^2 = 3$, we get $x^2 + 2 + \frac{1}{x^2} = 3$, so $x^2 + \frac{1}{x^2} = 1$. Then by the sum-of-cubes factorization,

$$x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)\left(x^2 - 1 + \frac{1}{x^2}\right) = 0.$$