Math that Counts Readiness Guide

Introduction

This class is intended for kids who don't have much specialized mathematical background but who love puzzles and are interested in learning new kinds of math separate from school curriculum. While familiarity with variables, powers of integers, and fractions will be helpful, this is not strictly required for a student to join the class. What is more important is the student's level of maturity. In particular:

- Students must be able to listen attentively during lessons and be respectful to the instructor and their peers.
- Students must be driven and persistent in working on challenging material and problems that will take time and tenacity to understand.
- Students must be curious about the reasons why patterns exist or why solutions work, not just care about the final answer.
- Students must be willing to ask for help if they find themselves struggling.

Students who are frustrated if they cannot immediately grasp a concept or who are focused solely on finding a final answer without justifying it will be able to pick up some techniques and tricks from this course, but they will not get the full value and depth that the class seeks to deliver. On the next page is a logic puzzle along with some questions to guide your child towards discovering and understanding an underlying pattern. It is recommended that you read this exercise to your child and watch them work it out to help gauge their readiness for the class. A child who gets very involved in and enjoys solving the problem as well as being able to make a solid attempt at explaining the thought process behind their answers is a good fit for the class.

A Puzzle

Alice, Bob, and Carol line up, and Doug places a hat on each of them. Alice is in the front of the line and cannot see anyone else's hat. Bob is second in line can only see the hat of the person in front of him. Carol sees the hats of the other two, but not her own. All four know that there are a total of three blue hats and two red hats. Alice, Bob, and Carol are perfectly rational; if there is a logical conclusion they can reach, they will figure it out.

Doug asks each person if they know the color of their hat from back to front (Carol then Bob then Alice). If the person asked knows the the color of their hat with 100% certainty, they will say the color. Otherwise, they will say they don't know.

1. Give an example of an assignment of hats such that Carol knows what color her hat is.

2. Suppose when Doug asks Carol her hat color, she replies she does not know. Bob also answers that he does not know his hat color. Does Alice know her hat color, and if so what color is it? How do you know?

3. Suppose we add a fourth person to the back of the line and start with a pool of three red hats and four blue hats. Give an example of an assignment of hats such that no one in line knows their hat color. If no such example exists, explain why not.

4. Suppose we have 99 people in line, 98 red hats, and 99 blue hats. The 50th person in line is the first person to know what color hat they have (person 51 to person 99 all answered they did not know). What color is the 50th person's hat? How do you know?

Puzzle Solutions

1. If Alice and Bob both have red hats and Carol has a blue hat, Carol will know that her hat is blue. This is due to the fact that there are only two red hats, so if Carol sees them both, she knows her hat cannot be red.

2. Alice knows that her hat is blue. From the previous question, your child should realize that since Carol did not know her hat color, either Alice or Bob has to have a blue hat. A good observation for your child to make at this point is that it does not matter what color Carol's hat is. Your child might then focus on just Alice and Bob's hats, knowing that they are not both red. A child who draws out the remaining cases and figures out in which cases Bob does not know his hat color is in a good spot. A child who reasons out the fact that Bob knows that Carol must have seen a blue hat, and he will only not know his hat color if he also sees a blue hat is a great fit. If your child simply attempts to draw out all possible hat assignments without taking what they already figured out into account, this shows a lack of mathematical maturity and indicates it might be better to wait for a later session to ensure your child gets the most they can out of the class.

3. There is no example of an assignment where no one knows their hat color. Again, if your child simply starts drawing out cases before thinking about anything else, they lack mathematical maturity. A more mature response would be to take into account what was learned from the case of three people, then trying to apply this to help solve this new case with four people. A good observation would be to realize that in the previous problem we saw that if Carol and Bob did not know their hat color, Alice knew her hat color. This means when we had three people, there was no way to have no one knowing their hat color. A good guess to make would be that this means there is no example we can find for this problem. A very good reasoning would be able to explain why this is true by thinking about what it means for the fourth person not to know their hat color, then the third, then the second, then the first rather than resorting to looking at individual cases.

4. The 50th person's hat is blue. In this case, the numbers are too big for a child to reasonably write down every possibility. It is possible a child

faced with this problem will say that they do not know how to solve it; this is actually preferable to simply writing down random cases. If a child does not quite know how to solve this final part, but they have confidently solved the previous problems without completely resorting to exhausting all cases, they could still succeed in the course with hard work.

Hopefully, a child will try to make connections with the previous parts of the problem and use what they have learned to solve this larger problem. They may decide to think about some other smaller cases to help them figure out a pattern before jumping straight to 99 people; this is a very good sign for their mathematical maturity in regards to this course. The pattern a child should notice from the past three parts is that the first person who knows their hat color always has a blue hat.

The child should next observe that the 99th person not knowing their hat color means they see at least one blue hat. Since the 98th person knows that the 99th person will only answer "I don't know" if they see a blue hat, they will know their hat is blue unless they see another blue hat in front of them. This pattern continues by inductive logic for the 97th person and so on. In this case, the 50th person will know that the 51st person saw a blue hat in front of them. If there were a blue hat in front of the 50th person, they would not be sure which color hat they have. So all hats in front of the 50th person must be red, and the 50th person will conclude that their hat is blue.

The most general pattern is that the first person who knows their hat color is the first person to see no blue hats in front of them, and that person's hat will always be blue. If a child makes this observation without prompting, they may be better suited for Introduction to Combinatorics.