AwesomeMath Academy

Problem Solving Level 4

Algebra: (Korea 2000) Find all real number x satisfying the equation

$$2^x + 3^x - 4^x + 6^x - 9^x = 1.$$

Number Theory: Let a be a positive integer, and let $p, q > 2$ be primes with $a^p \equiv 1 \pmod{q}$. Prove that either $q \mid a - 1$ or $q = 1 + 2kp$ for some positive integer k.

Combinatorics: Let S be a set of 50 points in the plane, so that no three points are collinear. Every pair of points in S is joined by a line segment. Every line segment is colored either red or white, so that every triangle formed by three points in S has at least one red edge. Find the smallest possible number of red edges.

Geometry: Let $\triangle ABC$ be an acute-angled triangle with $AB \neq AC$. Let H be the orthocenter of triangle ABC, and let M be the midpoint of the side BC. Let D be a point on the side AB and E a point on the side AC such that $AE = AD$ and the points D, H, E are on the same line. Prove that the line HM is perpendicular to the common chord of the circumscribed circles of triangle $\triangle ABC$ and triangle $\triangle ADE$.