# AwesomeMath Admission Test – Sample

## Solutions

## Part I

Levels 1 & 2

**1.** What is the greatest integer $n$ for which $n^2 + 2009n$ is the square of an integer?

**Solution:**
Let $k$ be a positive integer such that $n^2 + 2009n = k^2$. Then, $4n^2 + 8036n = 4k^2$, i.e. $(2n + 2009)^2 - 4k^2 = 2009^2$, which gives

$$(2n + 2009 - 2k)(2n + 2009 + 2k) = 2009^2.$$

Clearly, $2n + 2009 - 2k < 2n + 2009 + 2k$ and since we want the greatest $n$, then we have to choose $n$ so that the sum

$$(2n + 2009 - 2k) + (2n + 2009 + 2k) = 2(2n + 2009)$$

is maximal. It follows that

$$\begin{aligned} 2n + 2009 - 2k &= 1 \\ 2n + 2009 + 2k &= 2009^2, \end{aligned}$$

and adding side by side these equations, we get $2(2n + 2009) = 2009^2 + 1$, i.e. $n = \dfrac{2009^2 + 1 - 2 \cdot 2009}{4} = \left(\dfrac{(2009 - 1)}{2}\right)^2 = 1004^2$.

**2.** In the table

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | | | | | | |
| 1 | 2 | 3 | | | | |
| 4 | 5 | 6 | 7 | 8 | | |
| 9 | 10 | 11 | 12 | 13 | 14 | 15 |

what is the number directly below 2010?

**Solution:**
Observe that each row starts with a perfect square and $44^2 < 2010 < 45^2$. Furthermore, since the difference of the numbers one above the other in consecutive rows is constant and it is $(n + 1)^2 - n^2$, where $n$ represents the $n$-th row, then the final answer is

$$2010 + 45^2 - 44^2 = 2010 + 89 = 2099.$$

**3.** What is the product of the real zeros of the polynomial

$$p(x) = x^4 + 4x^3 + 6x^2 + 4x - 2011?$$

**Solution:**
We have $p(x) = (x+1)^4 - 2012$. So, if $r$ is a real root of $p(x)$, then

$$r + 1 = \pm\sqrt[4]{2012} \implies r = \pm\sqrt[4]{2012} - 1.$$

So, the product of the two real roots is

$$(-1 - \sqrt[4]{2012})(-1 + \sqrt[4]{2012}) = 1 - \sqrt{2012}.$$

**4.** The sum of some consecutive integers is 2012. Find the smallest of these integers.

**Solution:**
Let the the sum of integers be

$$S = (l+1) + (l+2) + \ldots + (l+n).$$

Using Gauss' trick we obtain

$$S = \frac{n(2l+n+1)}{2}.$$

As $S = 2012$, then

$$n(2l+n+1) = 4024.$$

Now, $4024 = n^2 + 2ln + n > n^2$, whence $n \le \lfloor\sqrt{4024}\rfloor = 63$. Moreover, $n$ and $2l+n+1$ are divisors of 4024 of opposite parity. Since $4024 = 2^3 \cdot 503$, we have

$$\begin{aligned} n &= 1 \\ (2l+n+1) &= 4024, \end{aligned}$$

$$\begin{aligned} n &= 8 \\ (2l+n+1) &= 503. \end{aligned}$$

We get $(n, l) \in \{(1, 2011), (8, 247)\}$. So, the smallest of these integers is $l + 1 = 248$.

**5.** Find all pairs $(m,\, n)$ of positive integers such that

$$m(n+1) + n(m-1) = 2013.$$

**Solution:**
We rewrite the original equation as $2mn + m - n = 2013$, i.e.

$$(2m-1)(2n+1) = 4025.$$

$$\begin{cases} 2m-1 = 1 \\ 2n+1 = 4025 \end{cases} \qquad \begin{cases} 2m-1 = 5 \\ 2n+1 = 805 \end{cases}$$

$$\begin{cases} 2m-1 &= 7 \\ 2n+1 &= 575 \end{cases} \qquad \begin{cases} 2m-1 &= 23 \\ 2n+1 &= 175 \end{cases}$$

$$\begin{cases} 2m-1 &= 25 \\ 2n+1 &= 161 \end{cases} \qquad \begin{cases} 2m-1 &= 35 \\ 2n+1 &= 115 \end{cases}$$

$$\begin{cases} 2m-1 &= 115 \\ 2n+1 &= 35 \end{cases} \qquad \begin{cases} 2m-1 &= 161 \\ 2n+1 &= 25 \end{cases}$$

$$\begin{cases} 2m-1 &= 175 \\ 2n+1 &= 23 \end{cases} \qquad \begin{cases} 2m-1 &= 575 \\ 2n+1 &= 7 \end{cases}$$

$$\begin{cases} 2m-1 &= 805 \\ 2n+1 &= 5. \end{cases}$$

Solving each system of equations, we get

$$(m,n) \in \{(1,2012),(3,402),(4,287),(12,87),(13,80),(18,57),$$
$$(58,17),(81,12),(88,11),(288,3),(403,2)\}.$$

**6.** Find all four-digit numbers $n$ whose sum of digits is equal to $2014-n$.

**Solution:**
Since $1 \le 2014-n \le 36$, then $1978 \le n \le 2013$. We have two cases.

**(i)** If $n=\overline{19cd}$, then $2014-\overline{19cd}=10+c+d$, i.e. $114-10c-d=10+c+d$, which gives

$$11c+2d=104.$$

As $2d \le 18$, then $c \ge 8$ and $c$ is even, so $c=8$ and $d=8$, i.e. $n=1988$.

**(ii)** If $n=\overline{20cd}$, then $2014-\overline{20cd}=2+c+d$, i.e. $14-10c-d=2+c+d$, which gives

$$11c+2d=12.$$

We get $c=0$ and $d=6$, so $n=2006$.

In conclusion, $n \in \{1988, 2006\}$.

**7.** Find the least positive integer that is divisible by precisely 2015 perfect squares.

**Solution:**
Let $n$ be the positive integer with the desired property. Consider the prime factorization of $n$, i.e. $n=p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdot\ldots\cdot p_k^{\alpha_k}$ ($\alpha_i \in \mathbb{Z}_+$ for $i=1,2,\ldots,k$). Let $m=p_1^{\beta_1}\cdot p_2^{\beta_2}\cdot\ldots\cdot p_k^{\beta_k}$ ($\beta_i \in \mathbb{Z}_+$ for $i=1,2,\ldots,k$) such that $m^2 \mid n$. Then, $2\beta_i \le \alpha_i$ for all $i=1,2,\ldots,k$, i.e. $\beta_i \le \lfloor \alpha_i/2 \rfloor$ for all $i=1,2,\ldots,k$. Then,

$$\left(1+\left\lfloor\frac{\alpha_1}{2}\right\rfloor\right)\left(1+\left\lfloor\frac{\alpha_2}{2}\right\rfloor\right)\cdot\ldots\cdot\left(1+\left\lfloor\frac{\alpha_k}{2}\right\rfloor\right)=2015.$$

In order to minimize $n$ we want the greatest number of prime factors, where the prime factors are taken as small as possible. Since $2015 = 5 \cdot 13 \cdot 31$, then $k = 3$, $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $\alpha_3 \leq \alpha_2 \leq \alpha_1$ and

$$\left(1 + \left\lfloor \frac{\alpha_1}{2} \right\rfloor\right)\left(1 + \left\lfloor \frac{\alpha_2}{2} \right\rfloor\right)\left(1 + \left\lfloor \frac{\alpha_3}{2} \right\rfloor\right) = 5 \cdot 13 \cdot 31.$$

We get

$$\begin{aligned} \left\lfloor \frac{\alpha_1}{2} \right\rfloor &= 30 \\ \left\lfloor \frac{\alpha_2}{2} \right\rfloor &= 12 \\ \left\lfloor \frac{\alpha_3}{2} \right\rfloor &= 4, \end{aligned}$$

which gives $\alpha_1 \in \{60, 61\}$, $\alpha_2 \in \{24, 25\}$, $\alpha_3 \in \{8, 9\}$. So, the least positive integer is $n = 2^8 \cdot 3^{24} \cdot 5^{60}$.

**8.** Solve in positive integers the equation $x^2y^2z^2 - \min(x^2, y^2, z^2) = 2016$.

**Solution:**
Assume without loss of generality that $x \leq y \leq z$. Then the equation becomes

$$x^2(y^2z^2 - 1) = 2016.$$

Observe that $x^2(x^4 - 1) \leq 2016$, so $x \leq 3$. Since $2016 = 2^5 \cdot 3^2 \cdot 7$, then $x \in \{1, 2, 3\}$. If $x = 1$, then $y^2z^2 = 2017$, contradiction. If $x = 2$, then $y^2z^2 = 505$, contradiction. If $x = 3$, then $y^2z^2 = 225$, which gives $y = 3$ and $z = 5$. So, $(x, y, z) \in \{(3, 3, 5), (3, 5, 3), (5, 3, 3)\}$.

**9.** Let $a$ be a positive real number such that $a^2 + \dfrac{16}{a^2} = 2017$. Evaluate

$$\sqrt{a} + \frac{2}{\sqrt{a}}.$$

**Solution:**
From the given identity, we have

$$\left(a + \frac{4}{a}\right)^2 = 2025 \implies a + \frac{4}{a} = 45.$$

Hence,

$$\left(\sqrt{a} + \frac{2}{\sqrt{a}}\right)^2 = 49 \implies \sqrt{a} + \frac{2}{\sqrt{a}} = 7.$$

**10.** In the addition $\overline{AWE} + \overline{SOME} = 2018$, each letter represents a different nonzero digit of the decimal system. Find the minimum possible value of $W \cdot M$.

**Solution:**
Clearly, either $E = 4$ or $E = 9$. If $E = 4$, then $W + M = 11$ and the minimum possible value of $W \cdot M$ would be $2 \cdot 9 = 18$. If $E = 9$, then $W + M = 10$ and, since 9 is already taken, the minimum possible value of $W \cdot M$ would be $2 \cdot 8 = 16$. The addition $329 + 1689 = 2018$ shows that this is possible, hence the answer is 16.

# Part II

**1.** Let $P(x) = 2009x^9 + a_1x^8 + \cdots + a_9$ such that

$$P\left(\frac{1}{n}\right) = \frac{1}{n^3},\ n = 1, 2, \ldots, 9.$$

Find $P\left(\frac{1}{10}\right)$.

**Solution:**
Consider the polynomial $Q(x) = P(x) - x^3$. We have that $Q(1) = Q(1/2) = \ldots = Q(1/9) = 0$. Since $\deg Q(x) = 9$, then

$$Q(x) = 2009\,(x-1)\left(x - \frac{1}{2}\right) \cdot \ldots \cdot \left(x - \frac{1}{9}\right).$$

So,

$$\begin{aligned} Q\left(\frac{1}{10}\right) &= 2009\left(\frac{1}{10} - 1\right)\left(\frac{1}{10} - \frac{1}{2}\right) \cdot \ldots \cdot \left(\frac{1}{10} - \frac{1}{9}\right) \\ &= -\frac{2009}{10^9}. \end{aligned}$$

So,

$$P\left(\frac{1}{10}\right) = \frac{1}{10^3} - \frac{2009}{10^9} = \frac{10^6 - 2009}{10^9}.$$

**2.** In quadrilateral $ABCD$, $\angle B = \angle C = 120^\circ$ and

$$AD^2 = AB^2 + BC^2 + CD^2.$$

Prove that $ABCD$ has an inscribed circle.

**Solution:**
Let $\{T\} = AB \cap CD$. Note that if we want to choose a good $BC$ that satisfies the length condition, since it is of degree 2, there are at most two distinct solutions (the coefficient of $x^2$ written in terms of $TB$ is not 0). Thus, it suffices to prove that the tangent to the incircle at the two intersections of the angle bisector satisfies the length condition. Let $F$ be the intersection closer to $A$. We want to prove that the condition holds for $F$. The other case is similar. In triangle $ATD$, denote the lengths of the sides by $a$, $t$, $d$, respectively. Since $BCT$ is equilateral, the following equations are true:

$$BM = BF = \frac{BC}{2} = \frac{TB}{2}, \quad BM = \frac{p-t}{3}, \quad BC = \frac{2}{3}TM = \frac{2}{3}(p-t).$$

$$AB = p - a + \frac{p-t}{3}, \quad CD = p - d + \frac{p-t}{3}, \quad BC = \frac{2}{3}(p-t).$$

Let $p - d = z$, $p - a = y$, $p - t = x$. We want to prove that

$$AD^2 = AB^2 + BC^2 + CD^2,$$

which is equivalent to prove that

$$\frac{4}{9}x^2 + z^2 + \frac{x^2}{9} + \frac{2xz}{3} + y^2 + \frac{x^2}{9} + \frac{2xy}{3} = (y+z)^2 = y^2 + z^2 + 2yz,$$

i.e.

$$x^2 + xy + xz = 3yz.$$

Since $\angle ATD = 60^\circ$, we can write the Law of Cosines for the side $AD$ in triangle $ATD$, which gives

$$(x+y)^2 + (x+z)^2 - 2(x+y)(x+z)\cos 60^\circ = (y+z)^2,$$

i.e.

$$x^2 + xy + xz = 3yz.$$

**3.** Evaluate

$$\left(1 - \frac{2011}{2}\right)\left(1 - \frac{2011}{3}\right)\cdots\left(1 - \frac{2011}{2010}\right).$$

**Solution:**
Let $P = \left(1 - \frac{2011}{2}\right)\left(1 - \frac{2011}{3}\right)\cdot\ldots\cdot\left(1 - \frac{2011}{2010}\right)$. Then,

$$\begin{aligned} P &= \frac{-2009}{2}\cdot\frac{-2008}{3}\cdot\ldots\cdot\frac{-1}{2010} \\ &= (-1)^{2009}\frac{2009!}{2010!} \\ &= -\frac{1}{2010}. \end{aligned}$$

**4.** For a positive integer $N$, let $r(N)$ be the number obtained by reversing the digits of $N$. For example, $r(2013) = 3102$. Find all 3-digit numbers $N$ for which $r^2(N) - N^2$ is the cube of a positive integer.

**Solution:**
Let $N = 100a + 10b + c$ and $r(N) = 100c + 10b + c$ with $0 \le a, b, c \le 9$, $ac \ne 0$. Then

$$\begin{aligned} r^2(N) - N^2 &= (100c + 10b + a)^2 - (100a + 10b + c)^2 \\ &= 99(c-a)\left(101(c+a) + 20b\right). \end{aligned}$$

Since $r^2(N) - N^2 > 0$, it must be $c > a$. If $r^2(N) - N^2$ is a cube, as $c - a \le 8$, then $101(c+a) + 20b \equiv 0 \pmod{121}$, i.e. $b - (a+c) \equiv 0 \pmod{121}$. Since $-17 \le b - (a+c) \le 6$, then $b = a + c$. So,

$$r^2(N) - N^2 = 11^3 \cdot 3^2 (c-a)(c+a).$$

If $c - a = 3$ then $c + a = 2a + 3$ and $5 \le 2a + 3 \le 19$ and $2a + 3$ must be an odd cube, contradiction. Likewise, $c - a = 6$ yields $c + a = 2(a+3)$, so that $a + 3 = 2k^3$ where $k$ is a positive integer, contradiction. So, $c + a = 3, 6, 9$.

**(i)** If $c + a = 3$, then $a = 1, c = 2$ and it's easy to see that $r^2(N) - N^2 = 11^3 \cdot 3^3 = 33^3$.

**(ii)** If $c + a = 6$, then $c - a = 2(3-a)$ and $r^2(N) - N^2 = 11^3 \cdot 3^3 \cdot 2^2(3-a)$, so $a = 1$ and $c = 5$.

**(iii)** If $c+a=9$, then $c-a=9-2a$ and $r^2(N)-N^2=11^3\cdot 3^4(9-2a)$, but $9-2a=9k^3$ has no positive integer solution.

In conclusion, all the 3-digit numbers which satisfy the required conditions are 132 and 165.

**5.** If $a, b, c$ are positive real numbers such that $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}=\dfrac{2013}{a+b+c}$, evaluate

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right).$$

**Solution:**
Observe that $(a+b+c)\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)=2013$, i.e.

$$\frac{(a+b+c)(ab+bc+ca)}{abc}=2013.$$

We have

$$\begin{aligned}
&\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right)+1\\
=\ &\frac{(a+b)(b+c)(c+a)}{abc}+1\\
=\ &\frac{(a+b+c)(b+c)(c+a)-c(b+c)(c+a)+abc}{abc}\\
=\ &\frac{(a+b+c)(b+c)(c+a)-c((b+c)(c+a)-ab)}{abc}\\
=\ &\frac{(a+b+c)(b+c)(c+a)-c^2(a+b+c)}{abc}\\
=\ &\frac{(a+b+c)((b+c)(c+a)-c^2)}{abc}\\
=\ &\frac{(a+b+c)(ab+bc+ca)}{abc}\\
=\ &2013.
\end{aligned}$$

So, $\left(1+\dfrac{a}{b}\right)\left(1+\dfrac{b}{c}\right)\left(1+\dfrac{c}{a}\right)=2012.$

**6.** Find all integers $n$ such that $n-2014$ and $n+2014$ are both triangular numbers.

**Solution:**
Let $n-2014=\dfrac{k(k+1)}{2}$ and $n+2014=\dfrac{m(m+1)}{2}$, where $k, m\in\mathbb{N}$. Then,

$$\frac{m(m+1)}{2}-\frac{k(k+1)}{2}=4028 \implies m(m+1)-k(k+1)=8056,$$

i.e.

$$(m-k)(m+k+1)=8056.$$

Observe that $m-k$ and $m+k+1$ are positive divisors of 8056, they have different parity and $m-k<m+k+1$, so

$$\begin{cases} m-k &= 1\\ m+k+1 &= 8056\end{cases}\qquad \begin{cases} m-k &= 8\\ m+k+1 &= 1007\end{cases}$$

$$\begin{cases} m-k &= 19 \\ m+k+1 &= 424 \end{cases} \qquad \begin{cases} m-k &= 53 \\ m+k+1 &= 152. \end{cases}$$

Solving each system of equations, we get

$$(k, m) \in \{(4027, 4028), (499, 507), (202, 221), (49, 102)\}.$$

So, $n - 2014 \in \{1225, 20503, 124750, 8110378\}$, i.e.

$$n \in \{3239, 22517, 126764, 8112392\}.$$

**7.** Let $P(x) = x^{2015} + x + 5$. Find the remainder when $P(x)$ is divided by $x^5 - x$.

**Solution:**
By performing long division of the two polynomials, we get

$$P(x) = (x^5 - x)(x^{2010} + x^{2006} + x^{2002} + \ldots + x^2) + (x^3 + x + 5),$$

so the remainder is $x^3 + x + 5$.

**8.** All but one of the squares of an $n \times n$ chessboard are labeled with numbers from the set $\{8, 16, \ldots, 8n^2\}$, none of which being used more than once, such that the sum of the numbers on each row and each column is 2016. Find the number that has not been used.

**Solution:**
The sum of all the numbers in the table is

$$\frac{8n^2(n^2+1)}{2} - 8k^2 = 4n^2(n^2+1) - 8k^2$$

for some $k = 1, 2, \ldots, n^2$. Now, the sum of all the numbers in the table is also $2016n - 8k^2$, so $4n^2(n^2+1) = 2016n$, which gives $n(n^2+1) = 504$. However this equation has no positive integer solutions.

**9.** Find the least real number $a$ such that

$$x^4 + ax^3 + 2017x^2 - 360x + 16 \geq 0$$

for all positive real numbers $x$.

**Solution:**
We have

$$x^4 + ax^3 + 2017x^2 - 360x + 16 = (x^2 + 45x - 4)^2 + (a - 90)x^3.$$

So, $x^4 + ax^3 + 2017x^2 - 360x + 16 \geq 0$ for all positive real numbers $x$ if and only if $a \geq 90$. Observe that if $x_0$ is a positive root of $x^2 + 45x - 4$ and $a < 90$, then the given expression is negative. So, the minimal $a$ is 90.

**10.** Let $a, b, c$ be real numbers such that

$$(3a + 28b + 35c)(20a + 23b + 33c) = 1.$$

Prove that

$$a^2 + b^2 + c^2 > \frac{1}{2018}.$$

**Solution:**
By Cauchy-Schwarz Inequality,

$$(3a + 28b + 35c)^2 \leq (3^2 + 28^2 + 35^2)(a^2 + b^2 + c^2)$$

and

$$(20a + 23b + 33c)^2 \leq (20^2 + 23^2 + 33^2)(a^2 + b^2 + c^2).$$

Hence

$$1 = (3a + 28b + 35c)^2(20a + 23b + 33c)^2 \leq (2018)(2018)(a^2 + b^2 + c^2)^2$$

and the conclusion follows.