AwesomeMath Admission Test - Sample

1. Is there an equiangular hexagon whose side lengths are (in some order) 2006, 2007, 2008, 2009, 2010, and 2011? (AwesomeMath Admission Test-B 2006)

Solution. We construct such a hexagon. Begin with an equilateral triangle of side length 6027. Remove from the corners of this triangle three noncongruent smaller equilateral triangles: one each of side length 2009, 2010, and 2011. The remaining hexagon is equiangular, and has side lengths (in order) 2006, 2011, 2007, 2009, 2008, and 2010.

2. Find the least positive integer n such that each n-element subset of the set $\{1, 2, \ldots, 2007\}$ contains two elements, not necessarily distinct, such that their sum is a power of 2. (AwesomeMath Admission Test-A 2007)

Solution. We prove that $n = 1002$. We will prove that if $n < 1002$, then the condition will not be satisfied. If $n = 1001$, let $A = \{1025, 1026, \ldots, 2007\}$, $B = \{33, 34, \ldots, 40\}$, $C = \{17, 18, \ldots, 23\}$, $D = \{5, 6, 7\}$. Then, $|A| = 2007 - 1025 + 1 = 983$, $|B| = 40 - 33 + 1 = 8$, $|C| = 23 - 17 + 1$, $|D| = 7 - 5 + 1 = 3$. If $S = A \cup B \cup C \cup D$ is the subset, then the condition is not satisfied and $|S| = 983 + 8 + 7 + 3 = 1001$. For values of n less than 1001, we just keep removing elements from S. To prove that the conditions are satisfied when $n = 1002$, we partition the set $\{1, 2, \ldots, 2007\}$ into 1001 two-element sets

$$\{1, 7\}, \quad \{9, 23\}, \quad \{24, 40\}, \quad \{41, 2007\}$$

$$\{2, 6\}, \quad \{10, 22\}, \quad \{25, 39\}, \quad \{42, 2006\}$$

$$\{3, 5\}, \quad \ldots, \quad \ldots, \quad , \ldots,$$

$\{15, 17\}, \{31, 33\}, \{1023, 1025\}$ and $\{4, 8, 16, 32, 1024\}$. The five-element set only contains powers of two. We cannot have a power of two in our chosen subset because we can just choose the power of two twice and their sum would be $2^x + 2^x = 2 \cdot 2^x = 2^{x+1}$, which is also a power of two. Thus, our chosen subset must contain elements from the 1001 two-element sets. Therefore, by the Pigeonhole Principle, it must contain two elements from the same set, which add up to a power of two.

3. There is a pile of 2009 chips on a table. You are allowed to perform repeatedly the following operation: choose a pile containing more than two chips, throw away a chip from the pile, then divide it into two smaller (not necessarily equal) piles. Is it possible that eventually all the piles on the table consist of exactly three chips? (AwesomeMath Admission Test-B 2009)

Solution. Observe that at the k-th step there are $2009 - k$ chips and $2k + 1$ piles, where $k \in \mathbb{N}$. Assume that it is possible that at some moment all the piles on the table consist of exactly three chips. Let n be the number of piles. Then, there are $3n$ piles on the tables. So,

$$n = 2k + 1, \qquad 3n = 2009 - k.$$

We get $3(2k + 1) = 2009 - k$, which gives $7k = 2006$, i.e. $7 \mid 2006$, contradiction.

4. Find all integers n such that $n^2 + 2010n$ is a perfect square. (AwesomeMath Admission Test-B 2010)

Solution. Let k be a positive integer such that $n^2 + 2010n = k^2$. Then, $(n + 1005)^2 = k^2 + 1005^2$, i.e.

$$(n + k + 1005)(n - k + 1005) = 1005^2.$$

So, both $n - k + 1005$ and $n + k + 1005$ are divisors of 1005^2. Since $n - k + 1005 < n + k + 1005$ and $1005^2 = 3^2 \cdot 5^2 \cdot 67^2$, then

$$\begin{aligned} n - k + 1005 &= 1 \\ n + k + 1005 &= 1005^2 \end{aligned} \qquad \begin{aligned} n - k + 1005 &= 3 \\ n + k + 1005 &= 336675 \end{aligned}$$

$$\begin{aligned} n - k + 1005 &= 5 \\ n + k + 1005 &= 202005 \end{aligned} \qquad \begin{aligned} n - k + 1005 &= 9 \\ n + k + 1005 &= 112225 \end{aligned}$$

$$\begin{aligned} n - k + 1005 &= 15 \\ n + k + 1005 &= 67335 \end{aligned} \qquad \begin{aligned} n - k + 1005 &= 25 \\ n + k + 1005 &= 40401 \end{aligned}$$

$$\begin{aligned} n - k + 1005 &= 45 \\ n + k + 1005 &= 22445 \end{aligned} \qquad \begin{aligned} n - k + 1005 &= 67 \\ n + k + 1005 &= 15075 \end{aligned}$$

$$\begin{aligned} n - k + 1005 &= 75 \\ n + k + 1005 &= 13467 \end{aligned} \qquad \begin{aligned} n - k + 1005 &= 201 \\ n + k + 1005 &= 5025 \end{aligned}$$

$$\begin{aligned} n - k + 1005 &= 225 \\ n + k + 1005 &= 4489 \end{aligned} \qquad \begin{aligned} n - k + 1005 &= 335 \\ n + k + 1005 &= 3015 \end{aligned}$$

$$\begin{aligned} n - k + 1005 &= 603 \\ n + k + 1005 &= 1675 \end{aligned}$$

Solving each system of equations, we get $n \in \{134, 670, 1352, 1608, 5766, 6566, 10240, 19208, 32670, 55112, 100000, 167334, 504008\}$. Solving also the following systems of equations

$$\begin{aligned} n - k + 1005 &= -1005^2 \\ n + k + 1005 &= -1 \end{aligned} \qquad \begin{aligned} n - k + 1005 &= -336675 \\ n + k + 1005 &= -3 \end{aligned}$$

$$\begin{aligned} n-k+1005 &= -202005 \\ n+k+1005 &= -5 \end{aligned} \qquad \begin{aligned} n-k+1005 &= -112225 \\ n+k+1005 &= -9 \end{aligned}$$

$$\begin{aligned} n-k+1005 &= -67335 \\ n+k+1005 &= -15 \end{aligned} \qquad \begin{aligned} n-k+1005 &= -40401 \\ n+k+1005 &= -25 \end{aligned}$$

$$\begin{aligned} n-k+1005 &= -22445 \\ n+k+1005 &= -45 \end{aligned} \qquad \begin{aligned} n-k+1005 &= -15075 \\ n+k+1005 &= -67 \end{aligned}$$

$$\begin{aligned} n-k+1005 &= -13467 \\ n+k+1005 &= -75 \end{aligned} \qquad \begin{aligned} n-k+1005 &= -5025 \\ n+k+1005 &= -201 \end{aligned}$$

$$\begin{aligned} n-k+1005 &= -4489 \\ n+k+1005 &= -225 \end{aligned} \qquad \begin{aligned} n-k+1005 &= -3015 \\ n+k+1005 &= -335 \end{aligned}$$

$$\begin{aligned} n-k+1005 &= -1675 \\ n+k+1005 &= -603 \end{aligned}$$

we get also $n \in \{-506018, -169344, -102010, -57122, -34680, -21218, -12250, -8756, -7776, -3618, -3362, -2680, -2144\}$.

5. Prove that the diameter of the incircle of a triangle ABC is equal to $\frac{1}{\sqrt{3}}(AB-BC+CA)$ if and only if $\angle A = 60°$. (AwesomeMath Admission Test-C 2012)

First Solution. Let r be the radius of the incircle of $\triangle ABC$ and let $\alpha = \angle BAC$. Clearly $0° < \alpha < 180°$ and

$$2r = \frac{2AB \cdot CA \sin\alpha}{AB+BC+CA}.$$

Then

$$2r = \frac{AB-BC+CA}{\sqrt{3}} \iff AB^2+CA^2+2AB\cdot CA-BC^2 = 2\sqrt{3}AB\cdot CA\sin\alpha.$$

Since $BC^2 = AB^2+CA^2-2AB\cdot CA\cos\alpha$, we obtain

$$2r = \frac{AB-BC+CA}{\sqrt{3}} \iff 1+\cos\alpha = \sqrt{3}\sin\alpha,$$

i.e. if and only if $\sin(\alpha-30°) = \frac{1}{2}$, which gives $\alpha = 60°$.

Second Solution. Let a, b, c be the side-lengths, s the semiperimeter, and r the inradius of triangle ABC. Accordingly, we rewrite the condition

$$2r = \frac{AB-BC+CA}{\sqrt{3}}$$

as $\dfrac{r}{s-a} = \dfrac{1}{\sqrt{3}}$, which, by to the well-known formula $\tan\dfrac{A}{2} = \dfrac{r}{s-a}$, becomes $\tan\dfrac{A}{2} = \dfrac{1}{\sqrt{3}}$. This is equivalent with $\angle A = 60^\circ$, as claimed.

6. If a, b, c are positive real numbers such that $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} = \dfrac{2013}{a+b+c}$, evaluate

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right).$$

(AwesomeMath Admission Test-C 2013)

Solution. Observe that $(a+b+c)\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right) = 2013$, i.e. $\dfrac{(a+b+c)(ab+bc+ca)}{abc} = 2013$. We have

$$\begin{aligned}
& \left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right)+1 \\
= \ & \frac{(a+b)(b+c)(c+a)}{abc}+1 \\
= \ & \frac{(a+b+c)(b+c)(c+a)-c(b+c)(c+a)+abc}{abc} \\
= \ & \frac{(a+b+c)(b+c)(c+a)-c((b+c)(c+a)-ab)}{abc} \\
= \ & \frac{(a+b+c)(b+c)(c+a)-c^2(a+b+c)}{abc} \\
= \ & \frac{(a+b+c)((b+c)(c+a)-c^2)}{abc} \\
= \ & \frac{(a+b+c)(ab+bc+ca)}{abc} \\
= \ & 2013.
\end{aligned}$$

So, $\left(1+\dfrac{a}{b}\right)\left(1+\dfrac{b}{c}\right)\left(1+\dfrac{c}{a}\right) = 2012$.

7. Find all integers n such that $n-2014$ and $n+2014$ are both triangular numbers. (AwesomeMath Admission Test-C 2014)

Solution. Let $n - 2014 = \dfrac{k(k+1)}{2}$ and $n + 2014 = \dfrac{m(m+1)}{2}$, where $k, m \in \mathbb{N}$. Then,

$$\frac{m(m+1)}{2} - \frac{k(k+1)}{2} = 4028 \implies m(m+1) - k(k+1) = 8056,$$

i.e.

$$(m-k)(m+k+1) = 8056.$$

Observe that $m-k$ and $m+k+1$ are positive divisors of 8056, they have different parity and $m-k < m+k+1$, so

$$\begin{aligned} m-k &= 1 \\ m+k+1 &= 8056 \end{aligned} \qquad \begin{aligned} m-k &= 8 \\ m+k+1 &= 1007 \end{aligned}$$

$$\begin{aligned} m-k &= 19 \\ m+k+1 &= 424 \end{aligned} \qquad \begin{aligned} m-k &= 53 \\ m+k+1 &= 152. \end{aligned}$$

Solving each system of equations, we get $(k, m) \in \{(4027, 4028), (499, 507), (202, 221), (49, 102)\}$. So, $n - 2014 \in \{1225, 20503, 124750, 8110378\}$, i.e. $n \in \{3239, 22517, 126764, 8112392\}$.

8. Find all functions $f : \mathbb{R}^* \to \mathbb{R}$ such that

$$f\left(\frac{2016}{x}\right) = 1 - xf(x), \text{ for all } x \in \mathbb{R}^*.$$

(AwesomeMath Admission Test-B 2016)

Solution. Using the substitution $x \mapsto 2016/x$, we get

$$\begin{aligned} f(x) &= 1 - \frac{2016}{x} f\left(\frac{2016}{x}\right) \\ &= 1 - \frac{2016}{x}(1 - xf(x)) \\ &= 1 - \frac{2016}{x} + 2016f(x), \end{aligned}$$

so $f(x) = \dfrac{1}{2015}\left(\dfrac{2016}{x} - 1\right)$.

9. Let a, b, c be real numbers such that

$$(3a + 28b + 35c)(20a + 23b + 33c) = 1.$$

Prove that

$$a^2 + b^2 + c^2 > \frac{1}{2018}.$$

(AwesomeMath Admission Test-A 2018)

Solution. By Cauchy-Schwarz Inequality,

$$(3a + 28b + 35c)^2 \le (3^2 + 28^2 + 35^2)(a^2 + b^2 + c^2)$$

and

$$(20a + 23b + 33c)^2 \le (20^2 + 23^2 + 33^2)(a^2 + b^2 + c^2).$$

Hence

$$1 = (3a + 28b + 35c)^2(20a + 23b + 33c)^2 \le (2018)(2018)(a^2 + b^2 + c^2)^2$$

and the conclusion follows.

10. An $n \times n$ magic square is filled with the numbers $1, 2, \ldots, n^2$ such that the sum of the entries on each row, each column, and each of the two diagonals is the same. If, for some $n > 6$, we remove the number 37 from this square, the sum of all other entries in its row is 2019. Find n. (AwesomeMath Admission Test-A 2019)

Solution. The sum of all the numbers in the table is $\frac{n^2(n^2+1)}{2}$ and so we get that the sum of all the numbers in each row is $\frac{1}{n} \cdot \frac{n^2(n^2+1)}{2} = \frac{n(n^2+1)}{2}$. Removing 37 from a row, we get

$$\frac{n(n^2+1)}{2} - 37 = 2019 \iff n(n^2+1) = 4112.$$

Since $n^3 < n(n^2+1) < (n+1)^3$, we get $n^3 < 4112 < (n+1)^3$, which gives $n = 16$.