Sample Problems

Algebra

Algebra 1.5

1. If $a+b=1$ and $a^2+b^2=2$, evaluate a^4+b^4.
2. Simplify
$$\frac{(1+ax)^2-(a+x)^2}{(1+bx)^2-(b+x)^2} \div \frac{(1+ay)^2-(a+y)^2}{(1+by)^2-(b+y)^2}.$$
3. Let a,b,c be distinct real numbers such that
$$a+\frac{1}{b}=b+\frac{1}{c}=c+\frac{1}{a}.$$
Prove that $|abc|=1$.
4. Evaluate
$$\sum_{n=0}^{\infty}\frac{n}{n^4+n^2+1}.$$

Algebra 2.5

1. Let $a_1,a_2,\ldots,a_n,b_1,b_2,\ldots,b_n$ be real numbers. Prove that
$$(a_1^2+a_2^2+\cdots+a_n^2)(b_1^2+b_2^2+\cdots+b_n^2)\geq(a_1b_1+a_2b_2+\cdots+a_nb_n)^2.$$
2. Find all positive integers a,b,c such that the equations
$$x^2-ax+b=0,\quad x^2-bx+c=0,\quad x^2-cx+a=0$$
have integer roots.
3. Let $f(x)=ax^2+bx+c$ be a quadratic function with integer coe!cients with the property that for every positive integer n there is an integer c_n such that n divides $f(c_n)$. Prove that f has rational zeros.
4. Let a,b integer numbers. Solve the equation
$$(ax-b)^2+(bx-a)^2=x$$
when it is known that it has an integer root.

Algebra 3.5

1. Find all polynomials with complex coe!cients such that $P(x^2)=P^2(x)$ is identically true.
2. Let $a,b,c\geq 0$. Prove that
$$\frac{a}{a+1}+\frac{b}{b+1}+\frac{c}{c+1}+\frac{1}{a+b+c+1}\geq 1.$$
3. Let a,b,c,d,e,f be positive integers such that $S=a+b+c+d+e+f$ is a divisor of $ab+bc+ca-(de+ef+fd)$ and $abc+def$. Show that S is a composite number.
4. Find all real polynomials with real coeficients $P(x)$ which satisfy the equality
$$P(a-b)+P(b-c)+P(c-a)=2P(a+b+c)$$
for all triples a,b,c of real numbers such that $ab+bc+ca=0$.

Combinatorics

Math Counts with Proofs

1. How many even integers between 4000 and 7000 have four different digits?
2. How many ordered triples (x, y, z) of non-negative integers have the property that $x + y + z = 8$?
3. There are three men and eleven women taking a dance class. In how many different ways can each man be paired with a woman partner and then have the eight remaining women be paired into four pairs of two?
4. We want to paint some identically-sized cubes so that each face of each cube is painted a solid color and each cube is painted with six different colors. If we have seven different colors to choose from, how many distinguishable cubes can we produce?

Counting Strategies

1. How many positive integers less than 5000 are multiples or 3, 5 or 7, but not multiples of 35?
2. Let m be a positive integer, and let $n = 2^m$ Prove that the numbers
$$\binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n-1}$$
are all even. How many of these numbers are divisible by 4?
3. A number of n tennis players take part in a tournament in which each of them plays exactly one game with each of the others. If x_i and y_i denote the number of wins and losses, respectively, of the ith player, prove that
$$x_1^2 + x_2^2 + \cdots + x_n^2 = y_1^2 + y_2^2 + \cdots + y_n^2.$$
4. Let S be a set with 9 elements and let $A_1, A_2, \dots, A_{13}$ be distinct subsets of S, each having 3 elements. Prove that among these subsets there exist two, A_i and A_j, such that $|A_i \cap A_j| = 2$.

Combinatorial Arguments

1. The numbers $1, 2, \dots, 49$ are placed in a 7×7 table. We then add the numbers in each row and each column. Among these 14 sums we have a even numbers and b odd numbers. Is it possible that $a = b$?
2. The numbers $a_1, a_2, \dots, a_{108}$ are written on a circle such that the sum of any 20 consecutive numbers equals 1000. If $a_1 = 1, a_{19} = 19$, and $a_{50} = 50$, find a_{100}.
3. An even number, $2n$, of knights arrive at King Arthurs court, each one of them having at most $n - 1$ enemies. Prove that Merlin the wizard can assign places for them at a round table in such a way that every knight is sitting only next to friends.
4. On an 8×8 chessboard whose squares are colored black and white in an arbitrary way we are allowed to simultaneously switch the colors of all squares in any 3×3 and 4×4 region. Can we transform any coloring of the board into one where all the squares are black?

Geometry

Elements of Geometry

1. Let $ABCD$ be a parallelogram, and let M and N be the midpoints of sides BC and CD, respectively. Suppose $AM = 2, AN = 1$, and $\angle MAN = 60°$. Computer AB.

2. How large an equilateral triangle can one fit inside a square with side length 2?

3. Charlyn walks completely around the boundary of a square whose sides are each 5 km long. From any point on her path, she can see exactly 1 km horizontally in all directions. What is the area of the region consisting of all points Charlyn can see during her walk?

4. Points A and B lie on a circle centered at O, and $\angle AOB = 60°$ A second circle is internally tangent to the first and tangent to both OA and OB. What is the ratio of the area of the smaller circle to that of the larger circle?

Computational Geometry

1. In quadrilateral $ABCD$, $BC = 8$, $CD = 12$, $AD = 10$, and $\angle A = \angle B = 60°$. Given that $AB = p + \sqrt{q}$, where p and q are positive integers, find $p + q$.

2. (a) Let G be the centroid of triangle ABC. Prove that for any point M,
$$MA^2 + MB^2 + MC^2 = 3MG^2 + AG^2 + BG^2 + CG^2.$$
   (b) Let I be the incenter of triangle ABC. Prove that for any point X,
$$a \cdot AX^2 + b \cdot BX^2 + c \cdot CX^2 = (a + b + c) \cdot IX^2 + a \cdot IA^2 + b \cdot IB^2 + c \cdot IC^2.$$

3. (a) Prove that in any triangle
$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{1}{8}.$$
   (b) Prove that in any triangle
$$\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} \le \frac{9}{4}.$$

4. Let M, N, P, Q, R, S be the midpoints of the sides AB, BC, CD, DE, EF, FA of a hexagon. Prove that
$$RN^2 = MQ^2 + PS^2$$
   if and only if $MQ \perp PS$.

Geometric Proofs

1. Prove that for any point M inside parallelogram $ABCD$, the following relation holds:
$$MA \cdot MC + MB \cdot MD \ge AB \cdot BC.$$

2. Let AD be an altitude in $\triangle ABC$. Point P is on segment AD. Let E bet the intersection of BP and AC. Let F be the intersection of CP and AB. Prove that $\angle ADE = \angle ADF$.

3. Let w_1 be a circle smaller than and internally tangent to the circle w_2 at T. A tangent to w_1 (at T') intersects w_2 at A and B. If A, T', and B are fixed, what is the locus of T?

4. Let $ABCD$ be a quadrilateral, and let E and F be points on sides AD and BC, respectively, such that $AE : ED = BF : FC$. Ray FE meets rays BA and CD at S and T, respectively. Prove that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through a common point.

Number Theory

Number Sense

1. Show that the number 101010 cannot be a difference of two squares of integers.

2. Let a unit step be the diagonal of a unit square. Starting from the origin, go one step to $(1, 1)$. The turn $90°$ counterclockwise (to the left) and go two steps to $(-1, 3)$. Then turn $90°$ counterclockwise (to the left) and go three steps to $(-4, 0)$. At each step you continue to turn $90°$ counterclockwise and increase the length of the movement by one at each step. What is the final position after 100 moves?

3. Find all positive integers a and b such that $a^2 + b^2 = \text{lcm}(a, b) + 7 \gcd(a, b)$.

4. Compute the sum of the greatest odd divisor of each of the numbers $2006, 2007, \ldots, 4012$.

Modular Arithmetic

1. Show that $\frac{1}{9}(10^n + 3 \cdot 4^n + 5)$ is an integer for all $n \geq 1$.

2. Show that if $a^5 \pm 2b^5$ is divisible by 11, then both a and b are divisible by 11.

3. If $\{a_1, a_2, \ldots, a_{p-1}\}$ and $\{b_1, b_2, \ldots, b_{p-1}\}$ are complete sets of nonzero residue classes modulo some odd prime p, show that $\{a_1b_1, a_2b_2, \ldots, a_{p-1}b_{p-1}\}$ is not a set of complete residue classes modulo p.

4. Given that $a + b\sqrt[3]{2} + c\sqrt[3]{4} = 0$, where a, b, c are integers, show that $a = b = c = 0$.

Number Theory

1. Find all solutions to $2^k = 9^m + 7^n$.

2. Let p be a prime, and let k be a nonnegative integer. Calculate

$$\sum_{n=1}^{p-1} n^k \pmod{p}.$$

3. Prove that the equation $x^2 + 7xy - y^2 = 401$ has no integer solutions.

4. Determine all positive integers n for which there is an integer m such that $2^n - 1$ is a divisor of $m^2 + 9$.