AwesomeMath
January 2006

Sample Admission Test

1. Each entry of a 4×4 square table of numbers is either 1 or 2. Suppose that the sum of 9 entries in each of the four 3×3 sub-square tables is divisible by 4, while the sum of all the 16 entries in the table is not divisible by 4. Determine the greatest and least possible values of the sum of all the entries.

2. Determine the least positive integer n for which the following result holds: No matter how the elements of the set $\{1, 2, \ldots, n\}$ are colored in red or blue, there are integers x, y, z, and w in the set (not necessarily distinct) of the same color such that $x + y + z = w$.

3. Let $ABCD$ be a trapezoid with $AB \parallel CD$, $AB = 7$, and $CD = 17$.
   (a) The diagonals of the trapezoid cut the trapezoid into four triangular regions. If all the areas of the triangular regions are integers, what is minimum value of the area of the trapezoid?
   (b) Points F and E lie on sides AD and BC, respectively such that $EF \parallel AB$. If the trapezoids $ABEF$ and $CDFE$ have the same area, compute EF.

4. Each of the numbers $1, 2, \ldots, 8$ is written at a distinct corner of a cube. Assume that the sum of any three numbers written on a face of the cube is no less than 10. Determine minimum value the sum of numbers written on a face of the cube?

5. How many ways can 8 mutually non-attacking rooks be placed on the 9×9 chessboard so that all 8 rooks are on squares of the same color. (Two rooks are said to be attacking each other if they are placed in the same row or column of the board.)

6. Let x, y, and z be complex numbers such that $x + y + z = 2$, $x^2 + y^2 + z^2 = 3$, and $xyz = 4$. Evaluate
$$\frac{1}{xy + z - 1} + \frac{1}{yz + x - 1} + \frac{1}{zx + y - 1}.$$

7. For any positive integer n, let $f(n)$ denote the index of highest power of 2 which divides $n!$. (For example, since $10! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7$, $f(10) = 8$.) Compute $f(1) + f(2) + \cdots + f(1023)$.

8. Telephone numbers in a certain country have 6 digits. How many telephones can be installed such that any two numbers differ in at least two digits?

9. In triangle ABC, $AB = AC$ and D is the midpoint of side BC. Point E lies on side AB with $DE \perp AB$, and F is the midpoint of segment DE. Prove that $AF \perp EC$.

10. For positive integer k, let $p(k)$ denote the greatest odd divisor of k. Prove that for every positive integer n,
$$\frac{2n}{3} < \frac{p(1)}{1} + \frac{p(2)}{2} + \cdots + \frac{p(n)}{n} < \frac{2(n+1)}{3}.$$