AMSP Sample Problems

These are the types of problems and their difficulty levels that students would encounter taking our classes.

Algebra

Algebra 1.5

1. (ASHME 1999) Find all pairs (m, n) of integers such that $m^3 + n^3 + 99mn = 33^3$
2. Let $P(x)$ be a degree 4 polynomial such that $P(1) = 1, P(2) = 5, P(3) = 9, P(4) = 13$, and $P(5) = 1$. Determine the sum of the roots of $P(x)$.
3. (IMO 1961) Solve the system of equations:
$$\begin{aligned} x + y + z &= a \\ x^2 + y^2 + z^2 &= b^2 \\ xy &= z^2, \end{aligned}$$
where a and b are constants. Give the conditions that a and b must satisfy so that x, y, z (the solutions of the system) are distinct positive real numbers.
4. Let a, b, c be a positive real numbers. Prove that
$$\frac{a^2+b^2}{a+b} + \frac{b^2+c^2}{b+c} + \frac{c^2+a^2}{c+a} \geq a + b + c$$

Algebra 2.5

1. Find all $a, b, c \in N$ such that all the equations
$$x^2 - ax + b = 0, x^2 - bx + c = 0, x^2 - cx + a = 0$$
have integer roots.
2. (Austria-Poland 1979) Find all polynomials of the form
$$P_n(x) = n!x^n + a_{n-1}x^{n-1} + \cdots + a_1x + (-1)^n n(n+1)$$
with integer coefficients, having n real roots $x_1, \ldots, x_n$ satisfying $k \leq x_k \leq k + 1$ for $k = 1, \ldots, n$.
3. (Titu Andreescu, IMOSL 1983) Let $F_1 = F_2 = 1$, $F_{n+2} = F_n + F_{n+1}$, and let f be a polynomial of degree 990 such that $f(k) = F_k$ for $k \in \{992, \ldots, 1982\}$. Show that $f(1983) = F_{1983} - 1$.
4. (USAMO/JMO 2016) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y,
$$(f(x) + xy) \cdot f(x - 3y) + (f(y) + xy) \cdot f(3x - y) = (f(x + y))^2.$$

Algebra 3.5

1. (IMO 1976) Let $P_1(x) = x^2 - 2$ and $P_j(x) = P_1(P_{j-1}(x))$ for $j = 2, \ldots$. Prove that for any positive integer n the roots of the equation $P_n(x) = x$ are all real and distinct.
2. Prove that the polynomial $(x^2 + x)^{2^n} + 1$ is irreducible in $\mathbb{Z}[x]$.
3. (Tiberiu Popoviciu) Let f be a convex function and x_1, x_2, x_3 are in domain of f. Prove that
$$\frac{f(x_1) + f(x_2) + f(x_3)}{3} + f\left(\frac{x_1 + x_2 + x_3}{3}\right) \geq \frac{2}{3}\left[f\left(\frac{x_1+x_2}{2}\right) + f\left(\frac{x_2+x_3}{2}\right) + f\left(\frac{x_3+x_1}{2}\right)\right]$$
4. (IMO 2017) Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all real numbers x and y,
$$f(f(x)f(y)) + f(x + y) = f(xy).$$

Abstract Algebra

1. (Putnam 2018 A2) Let $Q_0(x) = 1$, $Q_1(x) = x$, and

$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$$

for all $n \geq 2$. Show that, whenever n is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.

2. Let A_1, A_2, A_3 be three pairwise unequal 2×2 matrices with entries in $\mathbb{C}$ (the complex numbers). Let tr denote the trace of a matrix (so $\operatorname{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d$). Suppose $\{A_1, A_2, A_3\}$ is closed under matrix multiplication (i.e. given i, j, there exists k such that $A_iA_j = A_k$), and $\operatorname{tr}(A_1 + A_2 + A_3) \neq 3$. Prove that there exists i such that $A_iA_j = A_jA_i$ for all j (here i, j are 1, 2 or 3).

3. (Putnam 2016 A5) Suppose that G is a finite group generated by the two elements g and h, where the order of g is odd. Show that every element of G can be written in the form

$$g^{m_1}h^{n_1}g^{m_2}h^{n_2}\cdots g^{m_r}h^{n_r}$$

with $1 \leq r \leq |G|$ and $m_1, n_1, m_2, n_2, \ldots, m_r, n_r \in \{-1, 1\}$. (Here $|G|$ is the number of elements of G.)

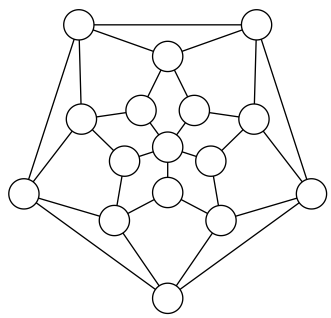
4. A Gaussian integer is a complex number $z = a + bi$ such that a, b are integers. Show that $x^3 - 2$ is irreducible over Gaussian integer.

Combinatorics

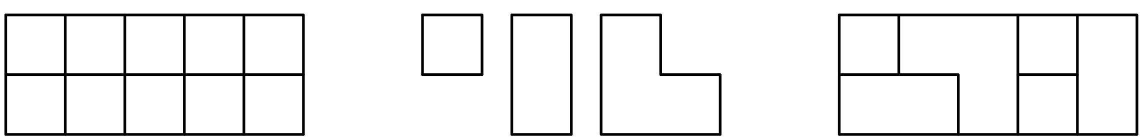
Math Counts with Proofs

1. We have 20 marbles, each of which is either yellow, blue, green, or red. Assuming marbles of the same color are indistinguishable, at most how many marbles can be blue, if the number of ways in which we can arrange the marbles into a straight line is 1140?

2. At each of the sixteen circles in the network below stands a student. A total of 3360 coins are distributed among the sixteen students. All at once, all students give away all their coins by passing an equal number of coins to each of their neighbors in the network. After the trade, all students have the same number of coins as they started with. Find the number of coins the student standing at the center circle had originally.



3. (Purple Comet 2013) You can tile a 2×5 grid of squares using any combination of three types of tiles: single unit squares, two side by side unit squares, and three unit squares in the shape of an L. The diagram below shows the grid, the available tile shapes, and one way to tile the grid. In how many ways can the grid be tiled?



4. In a chess tournament, each player played a match with each other. In each game, if there is a tie, each player earns 1/2 point, but if one of them wins, he wins 1 point and the loser earns 0 points. Men and women participate and each participant earns the same number of points against men as against women. Show that the total number of participants is a perfect square.

Counting Strategies

1. (Generalize AMC12 2007) Call a set of integers spacy if it contains no more than one out of any three consecutive integers. How many subsets of $\{1, 2, 3, \cdots, n\}$ including the empty set, are spacy?

2. We have a deck of n cards numbered $1, 2, \ldots, n$. If the card at the top has number k, we reverse the order of the first k cards. Prove that, eventually, we will have the card numbered 1 at the top.

3. Prove that any positive integer k has a multiple in the Fibonacci sequence

4. An $n \times n$ table is filled with 0 and 1 so that if we choose randomly n cells, no two of them on the same row or column, then at least one contains 1. Prove that we can find i rows and j columns so that $i + j \geq n + 1$ and their intersection contains only 1's.

Combinatorial Arguments

1. (USAMO 1994) Let $|U|$, $\sigma(U)$ and $\pi(U)$ denote the number of elements, the sum, and the product, respectively, of a finite set U of positive integers. (If U is the empty set, $|U| = 0$, $\sigma(U) = 0$, $\pi(U) = 1$.) Let S be a finite set of positive integers. As usual, let $\binom{n}{k}$ denote $\frac{n!}{k!(n-k)!}$. Prove that

$$\sum_{U \subseteq S} (-1)^{|U|} \binom{m - \sigma(U)}{|S|} = \pi(S)$$

for all integers $m \geq \sigma(S)$.

2. A $n \times m$ array is filled with the numbers 1, 2, ..., n each used exactly m times. Show that one can always permute the numbers within columns to arrange that each row contains every number 1, 2, ..., n exactly once.

3. (Romania TST 2005) Let $n \geq 1$ be an integer and let X be a set of $n^2 + 1$ positive integers such that in any subset of X with $n + 1$ elements there exist two elements $x \neq y$ such that $x \mid y$. Prove that there exists a subset $\{x_1, x_2, \ldots, x_n\} \in X$ such that $x_i \mid x_{i+1}$ for all $i = 1, 2, \ldots, n$.

4. Let $k < m$ be positive integers and let M be a set of cardinality m. Denote by p_{max} the maximal value of an integer p such that there exist distinct sets $A_1, \ldots, A_p$ with $|A_i \cap A_j| \leq k$ for $1 \leq i \leq j \leq p$. Show that

$$p_{max} = \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \cdots + \binom{m}{k+1}.$$

Combinatorics Level 4

1. Let $A = \{1, 4, \ldots, n^2\}$ be the set of the first n perfect squares of nonzero integers. Suppose that $A \subset B + B$ for some $B \subset \mathbb{Z}$. Here $B + B$ stands for the set $\{b_1 + b_2 : b_1, b_2 \in B\}$. Prove that $|B| \geq |A|^{2/3-\epsilon}$ holds for every $\epsilon > 0$.

2. Let n be a positive integer and let G be a graph on n vertices and with no cycles of length 6. Prove that G contains at most $cn^{4/3}$ edges, for some absolute constant $c > 0$ (independent of n). Moreover, prove that this is optimal (up to the choice of c) for infinitely many values of n.

3. Let $\mathcal{P}$ be a set of n points in the plane. Prove that there are at most $100n^{4/3}$ (unordered) pairs of points (A, B) with $A, B \in \mathcal{P}$ and such that the length of the segment AB equals 2020.

4. Let $A = \{a_1 < \cdots < a_n\}$ be a set of reals with the property that $a_i - a_{i-1} < a_{i+1} - a_i$ for all $i \in \{2, \ldots, n-1\}$. Prove that there exists two absolute constants $c > 0$ and $C > 0$ such that

$$|A + A| \geq c\,|A|^{3/2} \quad \text{and} \quad |A - A + A| \geq C\,|A|^2 .$$

Like above, $A + A$ denotes the set $\{a_1 + a_2 : a_1, a_2 \in A\}$, whereas $A - A + A$ stands for the set $\{a_1 - a_2 + a_3 : a_1, a_2, a_3 \in A\}$.

Geometry

Elements of Geometry

1. Let $ABCD$ be a square and let P be such a point inside the square, that CDP is an isosceles triangle with $\angle CPD = 150°$. Show that $\triangle ABP$ is equilateral.

2. (Poland 2010) In the convex pentagon $ABCDE$ all interior angles have the same measure. Prove that the perpendicular bisector of segment EA, the perpendicular bisector of segment BC and the angle bisector of $\angle CDE$ intersect in one point.

3. (AIME 2011) In triangle $ABC, AB = \frac{20}{11}AC$. The angle bisector of $\angle A$ intersects BC at point D, and point M is the midpoint of AD. Let P be the point of intersection of AC and BM. Find $\frac{CP}{PA}$

4. (USATST 2010) Let ABC be a triangle. Point M and N lie on sides AC and BC respectively such that $MN \mid\mid AB$. Points P and Q lie on sides AB and CB respectively such that $PQ \mid\mid AC$. The incircle of triangle CMN touches segment AC at E. The incircle of triangle BPQ touches segment AB at F. Line EN and AB meet at R, and lines FQ and AC meet at S. Given that $AE = AF$, prove that the incenter of triangle AEF lies on the incircle of triangle ARS.

Computational Geometry

1. Let D be a point from the interior of $\triangle ABC$, and denote by R, S, T the medicenters of $\triangle BCD, \triangle CAD$, and $\triangle ABD$ respectively. Let furthermore P be the medicenter of $\triangle RST$, and G be the medicenter of $\triangle ABC$. Is it true that D, P and G are collinear? If yes, find $DG : GP$.

2. A convex quadrilateral $ABCD$ is orthodiagonal if and only if the circumcenters of the triangles ABP, BCP, CDP and DAP are the midpoints of the sides of the quadrilateral where P is the intersection of the diagonals.

3. (All-Russian Olympiad 2006) Let ABC be a triangle. The A- and B- angle bisectors intersect opposite sides at K, L, respectively, and intersect each other at I. Line KL intersects the circumcircle ω of $\triangle ABC$ at X and Y. Prove that the circumcircle of $\triangle IXY$ passes through the A- and B-excenters of $\triangle ABC$.

4. (USATST 2002) Let $ABCD$ be a cyclic quadrilateral and let E and F be the feet of perpendiculars from the intersection of diagonals AC and BD to AB and CD, respectively. Prove that EF is perpendicular to the line through the midpoints of AD and BC.

Geometric Proofs

1. Let ABC be an acute triangle with circumcircle Ω. Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G and X are collinear.

2. Let ABC be a triangle and let E and F be the feet of the angle bisectors of B and C, respectively. Denote by O the circumcenter of triangle ABC and by I_a the center of the excircle corresponding to vertex A. Prove that $OI_a \perp EF$.

3. Points A, B, C, D, E, F lie on the common circle in this order. The intersection of the interiors of triangles ACE and BDF is a convex hexagon. Prove that the main diagonals of this hexagon pass through a common point.

4. Let ABC be a triangle. Points D, E, F lie on the sides BC, CA, AB, respectively. Point P lies inside triangle DEF. Lines DP, EP, FP intersect line segments EF, FD, DE at K, L, M, respectively. Prove that lines AK, BL and CM intersect at a common point if and only if lines AD, BE and CF intersect at a common point.

Geometry Level 4

1. Point P lies inside a given circle. Two perpendicular rays starting at P meet the circle at points A, B. The tangent lines to the circle at points A, B intersect at point X. Prove that all points X, corresponding to different choices of the rays, lie on a fixed circle.

2. Given are line m and a circle ω not intersecting m. Find the locus of all points that are centers of circles tangent to m and orthogonal to ω.

3. Given are disjoint circles ω_1 and ω_2. Consider all lines m that intersect ω_1 at points A and B, and ω_2 at points C and D, such that $AB = CD$. Prove that all lines m are tangent to a fixed parabola.

4. Let $A_1A_2 \dots A_6$ be a convex hexagon inscribed in a circle. Let X be any point inside the hexagon. Denote by O_i the circumcenter of triangle XA_iA_{i+1} for $i = 1, 2, \dots, 6$ (as usual, we set $A_7 = A_1$). Prove that lines O_1O_4, O_2O_5, and O_3O_6 intersect at a common point.

Number Theory

Number Sense

1. (ASHME 1994) Label one disc "1", two discs "2", three discs "3", . . . , fifty discs "50". Put these $1 + 2 + 3 + \ + 50 = 1275$ labeled discs in a box. Discs are then drawn from the box at random without replacement. What is the minimum number of discs that must me drawn in order to guarantee drawing at least ten discs with the same label?

2. For how many positive integers k the equation

$$6x + 5y = k$$

has no non-negative integer solutions?

3. (AIME 1983) Let $a_n = 6^n + 8^n$. Determine the remainder when a_{83} is divided by 49.

4. (Polish 1956) Prove that the equation $2x^2 - 215y^2 = 1$ has no integer solutions.

Modular Arithmetic

1. Determine all positive integers n such that n^2 divides $2^n + 1$.

2. Find all triples (x, y, z) of positive integers such that

$$\sqrt{\frac{2005}{x+y}} + \sqrt{\frac{2005}{y+z}} + \sqrt{\frac{2005}{z+x}}$$

is a positive integer.

3. Prove that there is a constant $c > 0$ with the following property: If a, b, n are positive integers such that $\gcd(a + i, b + j) > 1$ for all $i, j \in \{0, 1, \ldots n\}$, then

$$\min\{a, b\} > c^n \cdot n^{\frac{n}{2}}.$$

4. Prove that there exists a constant c such that for any positive integers a, b, n for which $a!b! \mid n!$, we have $a + b < n + c \ln n$

Number Theory

1. Prove that

$$v_2\left(\binom{4k}{2k} - (-1)^k \binom{2k}{k}\right) = s_2(k) + 2 + 3v_2(k),$$

where $s_2(k)$ is the sum of the digits in the base 2 expansion of k.

2. Show that there are only finitely many pairs of positive integers (n, m) such that $d(m!) = n!$, where $d(n)$ denote the number of positive divisors of n.

3. (USATST) Let p be a prime. We say that a sequence of integers $\{z_n\}_{n=0}^{\infty}$ is a p-pod if for each $e \geq 0$, there is an $N \geq 0$ such that whenever $m \geq N$, p^e divides the sum

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} z_k.$$

Prove that if both sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ are p-pods, then the sequence $\{x_n y_n\}_{n=0}^{\infty}$ is a p-pod.

4. Prove that for any positive integers a, b

$$\lim_{n\to\infty} \frac{|\{a \cdot b | 1 \leq a, b \leq n\}|}{n^2} = 0.$$

Number Theory Level 4

1. Find all polynomials with integers coefficients f such that $f(p) \mid (p-1)! + 1$ for all primes p.

2. The Fibonacci numbers are $1, 2, 3, 5, 8, 13, \ldots$ and the twin primes are $3, 5, 7, 11, 13, 17, 19, 29, 31, \ldots$. Which numbers are in both sequences?

3. (2015 Putnam) For each positive integer k, let $A(k)$ be the number of odd divisors of k in the interval $[1, \sqrt{2k})$. Evaluate
$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{A(k)}{k}.$$

4. (2017 Putnam) Evaluate the sum
$$\sum_{k=0}^{\infty} \left(3 \cdot \frac{\ln(4k+2)}{4k+2} - \frac{\ln(4k+3)}{4k+3} - \frac{\ln(4k+4)}{4k+4} - \frac{\ln(4k+5)}{4k+5} \right)$$
$$= 3 \cdot \frac{\ln 2}{2} - \frac{\ln 3}{3} - \frac{\ln 4}{4} - \frac{\ln 5}{5} + 3 \cdot \frac{\ln 6}{6} - \frac{\ln 7}{7} - \frac{\ln 8}{8} - \frac{\ln 9}{9} + 3 \cdot \frac{\ln 10}{10} - \cdots .$$
(As usual, $\ln x$ denotes the natural logarithm of x.)