AwesomeMath
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Sample Admission Test Solutions

1. Each entry of a 4×4 square table of numbers is either 1 or 2. Suppose that the sum of 9 entries in each of the four 3×3 sub-square tables is divisible by 4, while the sum of all the 16 entries in the table is not divisible by 4. Determine the least and greatest possible values of the sum of all the entries.

   Solution: The answers are $\boxed{19}$ and $\boxed{30}$, respectively.

   Since the sum of the entries in the top left 3×3 table is divisible by 4, the minimum of this sum is 12; that is, there are at least three 2's as the entries in the table. Hence the minimum of the sum of the entries in the whole table is at least $13 \times 1 + 3 \times 1 = 19$. This minimum can be easily achieved by making any of three entries of the middle 2×2 table as 2's and rest of the entries as 1's.

   In exactly the same way, we can show that the greatest sum is 30. We leave the details to the reader.

2. Determine the least positive integer n for which the following result holds: No matter how the elements of the set $\{1, 2, \ldots, n\}$ are colored in red or blue, there are integers x, y, z, and w in the set (not necessarily distinct) of the same color such that $x + y + z = w$.

   Solution: The answer is $\boxed{11}$.

   First, we note that $n \geq 10$ because we can color 1 and 2 red, 3, 4, 5, 6, 7, and 8 blue, and 9 and 10 red. It is not difficult to check that this coloring does not meet the conditions of the problem.

   Next we show that $n = 11$ suffices. We approach this indirectly by assuming that there is a coloring of the numbers in set $\{1, 2, \ldots, 11\}$ such that there is no quadruple (x, y, z, w) of numbers of the same color such that $x + y + z = w$. Without loss of generality, we assume that 1 is colored red. Then 3 must be blue and 9 must be red. This implies that 4 must be blue (otherwise $9 = 4 + 4 + 1$ violates our assumption). Then 11 cannot be colored because $11 = 1 + 1 + 9 = 3 + 4 + 4$, a contradiction! Hence our assumption was wrong and $n = 11$ suffices.

3. Let $ABCD$ be a trapezoid with $AB \parallel CD$, $AB = 7$, and $CD = 17$.

   (a) The diagonals of the trapezoid cut the trapezoid into four triangular regions. If all the areas of the triangular regions are integers, what is minimum value of the area of the trapezoid?

   (b) Points F and E lie on sides AD and BC, respectively such that $EF \parallel AB$. If the trapezoids $ABEF$ and $CDFE$ have the same area, compute EF.

   Solution: The answer are $\boxed{576}$ and $\boxed{13}$, respectively.

   Assume that diagonals AC and BD meet at P, and lines AD and BC meet Q. Let $[\mathcal{R}]$ denote the area of region $\mathcal{R}$. In general, we assume that $AB = a$ and $CD = b$.

(a) Triangles ABP and CDP are similar with side ratio $a : b$. Hence we assume $[ABP] = a^2x$ and $[CDP] = b^2x$. By similarity, we know that $\frac{BP}{PD} = \frac{a}{b}$. Triangles ABP and ADP share the same altitude (from A to line BC). Hence $\frac{[ABP]}{[ADP]} = \frac{a}{b}$, and so $[ADP] = abx$. Likewise, $[BPC] = abx$. It follows that $[ABCD] = (a+b)^2x$.

In this problem, $a = 7$ and $b = 17$, and a^2x, b^2x, and abx are all integers. Thus x must be an integer, and the minimum value of $[ABCD]$ is 576.

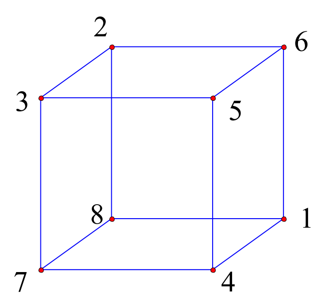
(b) We set $EF = c$. Note that triangles QAB, QFE, and QDC are similar. We may assume that $[QAB] = a^2y$, $[QFE] = c^2y$, and $[QDC] = b^2y$. Since $[ABEF] = [CDFE]$, $[QAB], [QFE]$, and $[QDC]$ form an arithmetic progression; that is, $c^2y - a^2y = b^2y - c^2y$, or $2c^2 = a^2 + b^2$.

In this problem, we have $2c^2 = 7^2 + 17^2$, implying that $c = 13$.

4. Each of the numbers $1, 2, \ldots, 8$ is written at a distinct corner of a cube. Assume that the sum of any three numbers written on a face of the cube is no less than 10. Determine minimum value the sum of numbers written on a face of the cube?

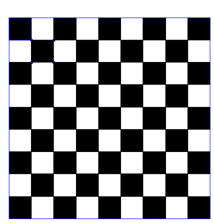
Solution: The answer is $\boxed{16}$.

Let m denote the minimum, and let a, b, c, d be the numbers written on a faces with $a + b + c + d = m$. Without loss of generality, we may assume that $a < b < c < d$. Then $a + b + c \geq 10$. Because $2 + 3 + 4 = 9$, it follows that $c \geq 5$, and so $d \geq 6$. Hence $m = (a + b + c) + d \geq 10 + 6 = 16$. The following example shows that 16 is also obtainable.



Note: This was problem 1 in the 2003 Chinese Western Mathematics Olympiad.

5. How many ways can 8 mutually non-attacking rooks be placed on the 9×9 chessboard so that all 8 rooks are on squares of the same color. (Two rooks are said to be attacking each other if they are placed in the same row or column of the board. Two placements are considered *different* if one can be obtained from the other by via reflections or rotations.)



Solution: The answer is $\boxed{40320}$.

We first assume that all the rooks are placed in the black fields. Note that a rook placed on a black square in an odd numbered row cannot attack a rook on a black square in an even row. This effectively partition the black squares into a 5×5 and a 4×4 sub-board. Exactly one of the rows must be empty, and each other row contains exactly one rook. If the empty row consists of 4 black squares. There are 4 ways to choose such a row. Then 5 rooks must be placed in each of the rows with 5 black squares. There are 5! ways to do so. We then have to place 3 rooks on a $3 \cdot 4$ black sub-board. There are 4! ways to do so (by considering the choices for columns). Thus there are $4 \cdot 5! \cdot 4!$ possible arrangements under our assumptions. In exactly the same way, we can show that there are $5 \cdot 5! \cdot 4!$ ways to arrange the rooks by assuming that they are placed in the black fields and the row without a rook contains 5 black squares. It follows that there are $4 \cdot 5! \cdot 4! + 5 \cdot 5! \cdot 4! = 9 \cdot 5! \cdot 4!$ ways to place the rooks in the black fields.

Likewise, we can show that there are $5 \cdot 5! \cdot 4!$ ways to place the rooks in the black fields. (There is only one case here, since we can not have a row with 5 white square empty.)

Hence the answer is $9 \cdot 5! \cdot 4! + 5 \cdot 5! \cdot 4! = 14 \cdot 5! \cdot 4! = 40320$.

Note: This was problem 2 in the 2004 Canadian Mathematics Olympiad.

We can also count the number ways of placing 8 mutually non-attacking rooks on black squares as following: Thinking about placing 9 mutually non-attacking rooks in stead. Then the answer would be $5! \cdot 4!$. Then there are 9 ways to remove one of the rook. Hence the answer is $9 \cdot 5!4!$. (It is important that there is no repetition by removing 1 rook from two distinct arrangements of 9 rooks, because two distinct 9-rook arrangements have to be different in at least two places.)

6. Let x, y, and z be complex numbers such that $x + y + z = 2$, $x^2 + y^2 + z^2 = 3$, and $xyz = 4$. Evaluate

$$\frac{1}{xy + z - 1} + \frac{1}{yz + x - 1} + \frac{1}{zx + y - 1}.$$

Solution: The answer is $\boxed{-\frac{2}{9}}$.

Let S be the desired value. Note that

$$xy + z - 1 = xy + 1 - x - y = (x - 1)(y - 1).$$

Likewise, $yz + x - 1 = (y - 1)(x - 1)$ and $zx + y - 1 = (z - 1)(x - 1)$. Hence

$$\begin{aligned} S &= \frac{1}{(x-1)(y-1)} + \frac{1}{(y-1)(z-1)} + \frac{1}{(z-1)(x-1)} \\ &= \frac{x + y + z - 3}{(x-1)(y-1)(z-1)} = \frac{-1}{(x-1)(y-1)(z-1)} \\ &= \frac{-1}{xyz - (xy + yz + zx) + x + y + z - 1} \\ &= \frac{-1}{5 - (xy + yz + zx)}. \end{aligned}$$

But

$$2(xy+yz+zx) = (x+y+z)^2 - (x^2+y^2+z^2) = 1.$$

Therefore $S = -\frac{2}{9}$.

7. For any positive integer n, let $f(n)$ denote the index of highest power of 2 which divides $n!$. (For example, since $10! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7$, $f(10) = 8$.) Compute $f(1) + f(2) + \cdots + f(1023)$.

Solution: The answer is $\boxed{518656}$.

Let p be a prime. For any positive integer n, let $e_p(n)$ be the exponent of p in the prime factorization of $n!$. We have

$$e_p(n) = \sum_{i\ge 1} \left\lfloor \frac{n}{p^i} \right\rfloor = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots.$$

In this problem, $f(n) = e_2(n)$, and we compute

$$\begin{aligned} S &= \sum_{n=1}^{1023} f(n) = \sum_{n=1}^{1023} e_2(n) = \sum_{n=1}^{1023}\sum_{i\ge 1} \left\lfloor \frac{n}{2^i} \right\rfloor = \sum_{i\ge 1}\sum_{n=1}^{1023} \left\lfloor \frac{n}{2^i} \right\rfloor \\ &= 2[1+2+\cdots+(2^9-1)] + 2^2[1+2+\cdots+(2^8-1)] + \cdots + 2^8[1+2+(2^2-1)] + 2^9\cdot 1 \\ &= 2^9(2^9-1) + 2^9(2^8-1) + \cdots + 2^9(2^2-1) + 2^9(2-1) \\ &= 2^9(2^9+2^8+\cdots+2) - 9\cdot 2^9 = 2^{19} - 11\cdot 2^9 = 518656. \end{aligned}$$

8. Telephone numbers in a certain country have 6 digits. How many telephones can be installed such that any two numbers differ in at least two places?

Solution: The answer is $\boxed{10^5}$; that is, 10^5 phone numbers can be assigned.

One method for doing so is to use a *check digit*, as follows. For each of the 10^5 5-digit strings $x_1x_2x_3x_4x_5$, define digit x_6 so that

$$x_6 \equiv x_1 + x_2 + x_3 + x_4 + x_5 \pmod{10}$$

and produce 6-digit phone number $x_1x_2x_3x_4x_5x_6$. For any pair of distinct 6-digit strings $x_1x_2x_3x_4x_5x_6$ and $y_1y_2y_3y_4y_5y_6$ constructed in this way, there must be at least one position j with $1 \le j \le 5$ such that $a_j \ne b_j$. If there are two such j's, these two strings differ at those two places. If there is only one such j, then

$$y_6 - x_6 \equiv (y_1+y_2+y_3+y_4+y_5) - (x_1+x_2+x_3+x_4+x_5) \equiv y_j - x_j \not\equiv 0 \pmod{10},$$

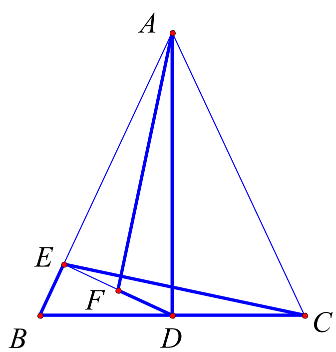
implying that $y_6 \ne x_6$. Hence $x_1x_2x_3x_4x_5x_6$ and $y_1y_2y_3y_4y_5y_6$ are differ at the j^{th} and the 6^{th} places. In any case, $x_1x_2x_3x_4x_5x_6$ and $y_1y_2y_3y_4y_5y_6$ must differ in at least two places.

To show that no method can produce a greater number of acceptable phone numbers, observe that among 10^5+1 distinct phone numbers, two would have agree in their first five places, where only 10^5 distinct 5-digit combinations are possible. These two phone numbers would differ in only in the 6^{th} place.

Note: The following problem was problem 1 in 1990 USAMO:

A certain state issue license plates consisting of six digits (from 0 through 9) The state requires that any two plates differ in at least two please. (Thus the plates $\boxed{027592}$ and $\boxed{020592}$ cannot both be used.) Determine, with proof, the maximum number of distinct license plates that the state can issue.

9. In triangle ABC, $AB = AC$ and D is the midpoint of side BC. Point E lies on side AB with $DE \perp AB$, and F is the midpoint of segment DE. Prove that $AF \perp EC$.



Proof: It is clear that $\angle ADB = 90°$. It is then not difficult to see that triangles AED and DEB are similar by AAA. It follows that $\frac{AD}{ED} = \frac{BD}{BE}$, implying that

$$\frac{AD}{FD} = \frac{2AD}{ED} = \frac{2BD}{BE} = \frac{BC}{BE}.$$

Combining the above relation with $\angle EBC = \angle ABC = \angle ADE = \angle ADF$, we conclude that triangle AFD is similar to triangle CEB (by SAS). It follows that the angles formed by the corresponding sides of the two triangles are equal to each other. (One can obtain one triangle by rotating and dilating the other.) Since $AD \perp CB$, we obtain $AF \perp CE$, as desired.

10. For positive integer k, let $p(k)$ denote the greatest odd divisor of k. Prove that for every positive integer n,

$$\frac{2n}{3} < \frac{p(1)}{1} + \frac{p(2)}{2} + \cdots + \frac{p(n)}{n} < \frac{2(n+1)}{3}.$$

Proof: Let

$$s(n) = \frac{p(1)}{1} + \frac{p(2)}{2} + \cdots + \frac{p(n)}{n}.$$

We need to show that

$$\frac{2n}{3} < s(n) < \frac{2(n+1)}{3}. \qquad (*)$$

We apply strong induction on n. The statement $(*)$ is true for $n = 1$ and $n = 2$ since

$$\frac{2 \cdot 1}{3} = \frac{2}{3} < s(1) = 1 < \frac{2(1+1)}{3} = \frac{4}{3}$$

and

$$\frac{2 \cdot 2}{3} = \frac{4}{3} < s(2) = 1 + \frac{1}{2} = \frac{3}{2} < \frac{2(2+1)}{3} = 2.$$

Assume that the statement $(*)$ is true for all integers n less than k, where k is some positive integer. We will show that the statement $(*)$ is true for integers $n = k+1$. The key fact is that $p(2k) = p(k)$. We consider two cases.

In the first case, we assume that k is even. We write $k = 2m$, where m is a positive integer less than k. For $n = k+1 = 2m+1$, we note that

$$\begin{aligned} s(2m+1) &= \left(\frac{p(1)}{1} + \frac{p(3)}{3} + \cdots + \frac{p(2m+1)}{2m+1}\right) + \left(\frac{p(2)}{2} + \frac{p(4)}{4} + \cdots + \frac{p(2m)}{2m}\right) \\ &= (m+1) + \left(\frac{p(1)}{2} + \frac{p(2)}{4} + \cdots + \frac{p(m)}{2m}\right) \\ &= (m+1) + \frac{1}{2}\left(\frac{p(1)}{1} + \frac{p(2)}{2} + \cdots + \frac{p(m)}{m}\right) \\ &= (m+1) + \frac{s(m)}{2}. \end{aligned}$$

By the induction hypothesis, we have

$$(m+1) + \frac{m}{3} < (m+1) + \frac{s(m)}{2} = s(2m+1) < (m+1) + \frac{(m+1)}{3}.$$

Since $\frac{2(2m+1)}{3} = \frac{4m+2}{3} < \frac{4m+3}{3} = (m+1) + \frac{m}{3}$ and $(m+1) + \frac{(m+1)}{3} = \frac{4(m+1)}{3} = \frac{2(2m+1+1)}{3}$, it follows that

$$\frac{2(2m+1)}{3} < s(2m+1) < \frac{2(2m+1+1)}{3},$$

which is $(*)$ for $n = 2m+1$.

In the second case, we assume that k is odd. We write $k = 2m+1$ and $n = k+1 = 2m+2$. Similar to the first case, we can show that

$$s(2m+2) = (m+1) + \frac{s(m+1)}{2}.$$

By induction hypothesis, it is not difficult to show that the statement $(*)$ is also true for $n = 2m+2$, which completes our induction.