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Segment 4: Pigeonhole Principle

Contents

1 Theoretical Concepts

The Pigeonhole Principle (or Dirichlet's box principle) is usually applied to problems in combinatorial set theory, combinatorial geometry, and in number theory. In its intuitive form, it can be stated as follows:

Pigeonhole Principle. If $kn+1$ objects ($k \geq 1$ not necessary finite) are distributed among n boxes, one of the boxes will contain at least $k+1$ objects.

1.1 Proof without words (from Mathematics Magazine)

Seven pigeons in six boxes

The theorems we will present play an important role in number theory. They are also interesting from other points of view and help solve a lot of difficult problems. One of these problems is

Prove that there exist powers of 2 that begin with any arbitrary combination of digits, (for example 2006).

To prove this statement we need to show that we can find two integers m and n such that

$$10^n \cdot 2006 < 2^m < 10^n \cdot 2007,$$

which is equivalent to

$$n + \log 2006 < m \cdot \log 2 < n + \log 2007.$$

So the problem reduces to proving that the interval $(\log 2006, \log 2007)$ contains a number of the form $m \cdot \log 2 - n$, where m, n are positive integers.

Kronecker's Theorem gives the answer to our question. It states that if α is irrational, then every interval (x, y), where $x, y \in \mathbb{R}$ contains numbers of the form $m\alpha - n$, where m, n are positive integers.

This result is closely related to another famous theorem whose proof involves the Pigeonhole Principle.

1.2 Dirichlet's Theorem

Let α be irrational and let p be a positive integer. There exist positive integers m, n such that

$$|m\alpha - n| \leq \frac{1}{p}$$

Proof. Let us divide the interval $[0, 1]$ in p intervals of length $\frac{1}{p}$.

$$\Delta_1 = \left[0, \frac{1}{p}\right], \Delta_2 = \left[\frac{1}{p}, \frac{2}{p}\right], \ldots, \Delta_p = \left[\frac{p-1}{p}, 1\right]$$

and consider the numbers

$$x_1 = \alpha - \lfloor \alpha \rfloor, x_2 = 2\alpha - \lfloor 2\alpha \rfloor, \ldots, x_{p+1} = (p+1)\alpha - \lfloor (p+1)\alpha \rfloor$$

These $p+1$ numbers belong to the interval $[0, 1]$, but the numbers of the intervals in which we divided it is p. By the Pigeonhole Principle there exist two numbers which are situated in the same interval. Suppose they are $k\alpha - \lfloor k\alpha \rfloor$ and $l\alpha - \lfloor l\alpha \rfloor$ with $k, l \in \{1, 2, .., p+1\}$, $k > l$. The distance between them cannot be greater than the length of the interval $\frac{1}{p}$, thus

$$|k\alpha - \lfloor k\alpha \rfloor - (l\alpha - \lfloor l\alpha \rfloor) \leq \frac{1}{p}.$$

This is what we wanted, because we can take $m = k - l$ and $n = \lfloor k\alpha \rfloor - \lfloor l\alpha \rfloor$ and we are done.

Definition. The set A consisting of real numbers is called dense in $\mathbb{R}$ if in every interval $I \in \mathbb{R}$ with nonzero length there exist at least one number in A.

1.3 Kronecker's Theorem

If α is irrational, then the set $A = \{m\alpha + n | m, n \in \mathbb{Z}\}$ is dense in $\mathbb{R}$.

Proof. Let $x, y \in \mathbb{R}$, $x < y$. We need to prove that there exists $a \in A$ such that $x < a < y$. Denote by L the length of the interval (x, y).

From Dirichlet's Theorem there exist positive integers m, n such that $|m\alpha - n| < L$. Let $b = |m\alpha - n|$. From the fact that α is irrational it follows that $b \neq 0$, so we can suppose $b > 0$ and $x > 0$.

Consider the numbers $b, 2b, \ldots, kb, \ldots$. They are distinct and there exists one which will be greater than x. We will prove that the least number kb that is greater than x is situated in (x, y).

Indeed, choosing kb as the least such number we have $(k-1)b \leq x$. Then assuming that $kb \geq y$ we get $b = kb - (k-1)b \geq y - x = L$, which contradicts the fact that $|m\alpha - n| < L$. Because kb is also a number of the form $|m\alpha - n|$ with $m, n \in \mathbb{Z}$, the theorem is proved.

Next we want to present two very important theorems, without proof, that may be helpful in solving some Pigeonhole-type problems.

1.4 Van der Warden's theorem

There are two versions of this theorem: infinite and finite.

Infinite: Let r and k be positive integers. If you color each positive integer with one color among r possible colors, then you can find a monochromatic arithmetic progression of length k.

Finite: Let r and k be two positive integers. There is an integer $n_0 = n_0(k, r)$ such that for all $n \geq n_0$ and all r-coloring of the integers $1, \cdots, n$, there is a monochromatic arithmetic progression of length k.

The least integer n_0 which satisfies the above property is the Van der Waerden number $W(k, r)$.

1.5 Ramsey's theorem

Let $k, r \geq 1$ and let $m_1, \ldots, m_r \geq k$ be integers. There is a integer n_0 such that, for each $n \geq n_0$, no matter how k-element subsets of a set X with $|X|$ are colored with r colors (say $c_1, \ldots, c_r$) there exists a subset X_i of X such that $|X_i| = m_i$ and each k-elements subset of X_i has color c_i.

It may be useful to try to understand this theorem by checking small values of $k, r, \cdots$. In particular, for $k = 1$ it is just the Pigeonhole Principle.

On the another hand, for $k = 2$, the 2-elements subsets of a set X can be seen as the edges of the complete graph whose vertices are the elements of X. Thus, Ramsey's theorem states that, given $m_1, \ldots, m_r \geq 2$, for n sufficiently large, no matter how one colors the edges of K_n, each with one of r colors, there will be a complete monochromatic K_{m_i}.

The least integer n_0 with the property described in the theorem is called the Ramsey number $R(m_1, \ldots, m_r, k)$. Note that the exact value of the Ramsey numbers are unknown, except for some few and small values of r, k, m_i.

The following example shows how this theorem could be used.

Problem(Erdos-Szekeres). Let $n \geq 3$ be an integer. Prove that there is an integer $n_0 > 0$ such that in each set of at least n_0 points in the plane, no three collinear, we can find n that are the vertices of a convex n-gon.

Solution. We will use two lemmas:

Lemma 1. Among any set of 5 points in the plane, no three collinear, we can find 4 that are the vertices of a convex quadrilateral.

Proof. Consider three possible cases.

case 1: the 5 points are the vertices of a convex pentagon. We are done.

case 2: 4 points are the vertices of a convex quadrilateral with one point inside it. Again, we are done.

case 3: 3 points are the vertices of a triangle with two points inside it. Observe that these two points and one pair of triangle's vertices will form a convex quadrilateral. Lemma 1 is proved.

Lemma 2. Let $n \geq 4$ be an integer. If n points in the plane, no three collinear, are such that any 4 of them are the vertices of a convex quadrilateral, then these n points are the vertices of a convex n-gon.

Proof. Take the convex hull of the n points. Suppose by way of contradiction that we have a convex m-gon, $m < n$, and at least one point P lies inside it. Clearly, there exists a triangle with vertices among the ones of the m-gon that contains P. But this contradicts the hypothesis, so Lemma 2 is proved.

Let us choose $n_0 = R_4(5, n)$, where $R_4(5, n)$ denotes the Ramsey number such that for any two-coloring in red and blue of the 4-element sets of $\{1, \cdots, R_4(5, n)\}$ there exist either a 5-elements set with all its 4-element subsets blue or a red n-set with all its 4-element subsets red. Now, let us consider a set of at least n_0 points in the plane, no three collinear. Color in blue each group of 4 points which are not the vertices of a convex quadrilateral, and in red each group of 4 points which are the vertices of a convex quadrilateral. From lemma 1, there is no 5-element set with all its 4-element subsets blue. Thus, from Ramsey's theorem, there exists a n-element set with all its 4-element subsets red. According to lemma 2, these n points are the vertices of a convex n-gon.

2 Examples and Problems

2.1 Level: Easy

2.1.1 Examples

1. Twenty one boys have a total of two hundred dollars in notes. Prove that it is possible to find two boys who have the same amount of money.

   Solution. Assume the contrary, all boys have different amount of money. Thus the total amount of money is at least $0 + 1 + 2 + \cdots + 20 = 210$ dollars. Contradiction.

2. Prove that among any 7 perfect squares there exist two whose difference is divisible by 10.

   Solution. It is not difficult to observe that the last digit of a perfect square should be $\{0, 1, 4, 5, 6, 9\}$. Thus by the Pigeonhole Principle there will be two which will have the same last digit and their difference will be divisible by 10.

3. Inside a room of area 5, you place 9 rugs, each of area 1 and an arbitrary shape. Prove that there are two rugs which overlap by by an area of at least $\frac{1}{9}$.

   Solution. Suppose every pair of rugs overlaps by less than $\frac{1}{9}$. Place the rugs one by one on the floor. We note how much of the yet uncovered area each succeeding rug will cover area greater than $\frac{8}{9}, \frac{7}{9}, \ldots, \frac{1}{9}$. Since $\frac{9}{9} + \frac{8}{9} + \cdots + \frac{1}{9} = 5$, all nine rugs cover area greater than five. Contradiction.

4. There are n persons in a room. Prove that among them there are two who have the same number of acquaintances (having an acquaintance is a symmetric relation).

   Solution. Assign to a person a number i if he/she has i acquaintances. We have n persons and n possible numbers to be assigned: $0, 1, 2 \ldots, n-1$. However numbers 0 and $n-1$ cannot be assigned at the same time, because it is impossible to have both a person who knows everybody and a person who knows nobody. Thus we have $n-1$ numbers assigned to n persons. By the Pigeonhole Principle, there exist two numbers that are equal which means that two persons have the same number of acquaintances.

5. Let S be a set of n integers. Prove that S contains a subset such that sum of its elements is divisible by n.

   Solution. Let $S = \{a_1, a_2, \ldots, a_n\}$. Consider the following sums

$$s_1 = a_1,$$

$$s_2 = a_1 + a_2,$$

$$\ldots\ldots$$

$$s_n = a_1 + a_2 + \cdots + a_n.$$

There are in total n numbers. If s_k is divisible by n, then our subset is $\{a_1, a_2, \ldots, a_k\}$. If none of s_k, $1 \le k \le n$, is divisible by n, then the residues of s_k modulo n are $1, 2, \ldots, n-1$. We have n sums and $n-1$ possible residues. Thus by Pigeonhole Principle there exist two sums s_i and s_j, $1 \le i < j \le n$, which have the same residue. Then their difference $s_j - s_i$ is divisible by n, hence the subset $\{a_{i+1}, a_{i+2}, \ldots, a_j\}$ satisfies the desired condition.

6. Find the greatest number of distinct positive rational numbers with the property that from any seven of them we can find two whose product is 1.

   Solution. First of all we show that there exist 12 numbers which satisfy the conditions of the problem. Let them be $a_1, 1/a_1$, $a_2, 1/a_2$, $\ldots$, $a_6, 1/a_6$, where a_i be distinct positive rational numbers different from one. Then we divide them in pairs $(a_1, 1/a_1)$, $(a_2, 1/a_2)$,...,$(a_6, 1/a_6)$. We have 7 numbers and 6 pairs, thus by the Pigeonhole Principle there exist one pair in those 7. Hence there are two numbers whose product is 1.

   To prove that 12 is the greatest number we will show that from every 13 distinct positive rational numbers there exist 7 from whom we can find two whose product is 1. Let $S = \{a_1, a_2, \ldots, a_{13}\}$ be our set of numbers and let $S_1 = \{a | a \in A, a \ge 1\}$, $S_2 = \{a | a \in A, a < 1\}$. We have that S_1 and S_2 are nonintersecting subsets and $|S_1| + |S_2| = |S| = 13$. Thus by the Pigeonhole Principle one from the subsets has 7 elements. Obviously that the subset with these 7 numbers doesn't contain a pair whose product is 1.

7. Let $a_1, a_2, \ldots, a_n$ represent an arbitrary arrangement of the numbers $1, 2, 3, \ldots, n$. Prove that, if n is odd, the product

$$(a_1 - 1)(a_2 - 2)(a_3 - 3) \cdots (a_n - n)$$

   is an even number.

   First solution. We shall prove that at least one factor in this product is even. Since n is odd, we may write $n = 2k + 1$, where k is an integer, and observe that our product has $2k + 1$ factors. Moreover, of the numbers $1, 2, 3, \ldots, 2k + 1$, exactly $k + 1$ are odd because

$$\begin{aligned} 1 &= 2 \cdot 1 - 1 \\ 3 &= 2 \cdot 2 - 1 \\ &\vdots \\ 2k + 1 &= 2(k + 1) - 1. \end{aligned}$$

   Since the a's also consist of the numbers from 1 to $2k + 1$, exactly $k + 1$ of the a's are odd. Therefore, there are exactly $2(k + 1) = n + 1$ odd numbers among the $2n$ numbers

$a_1, a_2, \ldots, a_n$; $1, 2, 3, \ldots, n$ appearing in the above product. However, there are only n factors. Hence, at least one of the factors contains two odd numbers, say a_m and m so that $a_m - m$ is even. Therefore the entire product is divisible by 2.

Second solution. The number of factors is odd and their sum is clearly 0, which is even. This would be impossible if all the factors were odd. Consequently at least one of the factors is even, and so is their product.

8. A circular table has exactly 60 chairs around it. There are N people seated at this table in such a way that the next person to be seated must sit next to someone. Find the least possible value of N.

   Solution. [AHSME 1991]: Divide the chairs around the table into $60/3 = 20$ sets of three consecutive seats. If fewer than 20 people are seated at the table, then at least one of these sets of three seats will be unoccupied. If the next person sits in the center of this unoccupied set, then that person will not be seated next to anyone already seated. On the other hand, if 20 people are already seated, and each occupies the center seat in one of the sets of three, then the next person to be seated must sit next to one of these 20 people.

9. A subset of the integers $1, 2, \ldots, 100$ has the property that none of its members is 3 times another. What is the largest number of members such a subset have?

   Solution. [AHSME 1990]: For each positive integer b that is not divisible by 3, we must decide which of the numbers in the set

$$b, 3b, 9b, 27b, 81b \leq 100$$

   to place in the subset. Clearly, a maximal subset can be obtained by using alternate numbers from this list starting with b. Thus, it will contain $67 = 100 - 33$ members b that are not divisible by 3, $8 = 11 - 3$ members of the form $9b$ that are divisible by 9 but not by 27, and the number 81, for a total of $67 + 8 + 1 = 76$ elements.

10. Label one disk "1", two disks "2", three disks "3",..., fifty disks "50". Put these $1 + 2 + 3 + \cdots + 50 = 1275$ labeled disks in a box. Disks are then drawn from the box at random without replacement. What is the minimum number of disks that must be drawn to guarantee drawing at least ten disks with the same label?

    Solution.[AHSME 1993]: The largest set of disks that does not contain ten with the same label consist of all 45 disks labeled "1" through "9", and nine of each of the other 41 types. Hence, the maximum number of disks that can be drawn without having ten with the same label is $45 + 41 \cdot 9 = 414$. The 415th draw must result in ten disks with the same label, so 415 is the minimum number of draws that guarantees at least ten disks with the same label.

11. Prove that no matter how we choose $n+2$ numbers from the array $\{1, 2, \ldots, 3n\}$ there are among them two whose difference is in the interval $(n, 2n)$.

Solution: We will start with the following observation. If among the chosen numbers the number $3n$ does not appear we increase all chosen numbers by the same number such that the greatest of them becomes $3n$. In this way the difference between any two numbers remains the same and the problem does not change. Now, if a number from $\{n+1, \ldots, 2n-1\}$ is among the ones chosen then its difference to $3n$ gives us a number in $(n, 2n)$. Thus all remaining numbers are from the arrays $\{1, 2, \ldots, n\}$ and $\{2n, 2n+1, \ldots, 3n-1\}$. Divide those in pairs $(1, 2n), (2, 2n+1), \ldots, (n, 3n-1)$. We have $n+1$ numbers to choose from n pairs, so by the Pigeonhole Principle there exist a pair which will be chosen. Hence there will exist two numbers whose difference is in interval $(n, 2n)$.

12. Prove that among any three distinct integers we can find two, say a and b, such that the number $a^3b - ab^3$ is a multiple of 10.

    Solution. Denote
$$E(a, b) = a^3b - ab^3 = ab(a-b)(a+b)$$

    Since if a and b are both odd then $a+b$ is even, it follows that $E(a, b)$ is always even. Hence we only have to prove that among any three integers we can find two, a and b, with $E(a, b)$ divisible by 5. If one of the numbers is a multiple of 5, the property is true. If not, consider the pairs $\{1, 4\}$ and $\{2, 3\}$ of residues classes modulo 5. By the Pigeonhole Principle, the residues of two of the given numbers belong to the same pair. These will be a and b. If $a \equiv b (\text{mod } 5)$ then $a-b$ is divisible by 5, and so is $E(a, b)$. If not, then by the way we defined our pairs, $a+b$ is divisible by 5, and so again $E(a, b)$ is divisible by 5. The problem is solved.

2.1.2 Practice Problems

1. What is the least number of people that must be chosen to be sure that at least two have the same first initial ?

2. A box contains 10 French books, 20 Spanish books, 8 German books, 15 Russian books, and 25 Italian books. How many must I choose to be sure I have 12 books in the same language?

3. At the beginning of each year I tell my class three jokes. I have been teaching for 12 years and have not repeated the exact same triple if jokes. What is the smallest number of jokes that I could have for this to be possible?

4. There are 30 students in a class. While doing a typing test one student made 12 mistakes, while the rest made fewer mistakes. Show that at least 3 students made the same number of mistakes.

5. A consumer organizer selects eleven phone numbers from the phone book. Show that at least 2 have the same last digit.

6. How many times must I throw two dice in order to be sure I get the same score at least twice?

7. Prove that of any 5 points chosen in an equilateral triangle of side length 1, there are two points whose distance apart is at most $\frac{1}{2}$.

8. Show that in any set of 27 different positive odd numbers, all less than 100, there is at least one pair whose sum is 102.

9. If I arrange the numbers 1 to 10 in a circle in random order, show that one set of three consecutive numbers will have a sum of at least 17.

10. Six swimmers training together either swam in a race or watched the other swim. At least how many races must have been scheduled if every swimmer had opportunity to watch all of the others ?

11. There are 15 people at a party. Some of them exchange handshakes with some of the others. Prove that at least two people have shaken hands the same number of times.

12. A computer has been used for 99 hours over a period of 12 days a whole number of hours every day. Prove that on some pair of consecutive days, the computer was used at least 17 hours.

13. Show that given any 17 numbers it is possible to choose 5 whose sum is divisible by 5.

14. Prove that of any 10 points chosen within an equilateral triangle of side 1 there are two whose distance apart is at most $\frac{1}{3}$.

15. A circle is divided into 8 equal sectors. Half are colored red and half are colored blue. A smaller circle is also divided into 8 sectors and half are colored red and half are colored blue. The smaller circle is placed concentrically on the larger. Prove that no matter how the red and blue sectors are chosen it is always possible to rotate the smaller circle so that at least 4 colors matches are obtained.

16. I have 5 computers to be connected to 3 printers. How many connections are necessary between computers and printers in order to be certain that whenever any three computers each require a printer, the printers are available?

17. Prove that, of any 5 points chosen within a square of side length 2, there are two whose distance apart is at most $\sqrt{2}$.

18. A disk of radius 1 is completely covered by 7 identical smaller disks. (They may overlap.) Show that the radius of each of the smaller disks must be not less than $\frac{1}{2}$.

19. Prove that among any group of 10 consecutive two-digit numbers there must be at least one number which is divisible by the sum of its digits.

20. The decimal representation of $\frac{a}{b}$ with coprime a, b has at most period $(b-1)$.

21. Let $a_1, a_2, \ldots, a_{99}$ be a permutation of $1, 2, 3, \ldots, 99$. Prove that there exist two equal numbers from

$$|a_1 - 1|, |a_2 - 2|, \ldots, |a_{99} - 99|.$$

22. If none of the numbers $a, a+d, \ldots, a+(n-1)d$ is divisible by n, then d and n are coprime.

23. Each of ten segments is longer than 1 cm but shorter than 55 cm. Prove that you can select from the ten three segments that form a triangle.

24. In each of 3 classes there are n students. Every student has exactly $n+1$ friends altogether in the two other classes (not in his class). Prove that in each class you can find one student, so that all three of them, taken in pairs, are friends.

25. What is the size of the largest subset, S, of $\{1, 2, 3, \ldots, 50\}$ such that no pair of distinct elements of S has a sum divisible by 7?

26. Twenty five teams participate in a tournament; every team must play one game with each other team. Prove that at any moment you can find either a team which has played at least five games, or five teams which have not played any game among themselves.

27. Let p be a prime greater than 5. Prove that among the numbers 11, 111, $\ldots$, $11\ldots11$ (p ones) there is a multiple of p.

28. Given three real numbers a, b and c. Prove that there exist a real root for at least one of the equations

$$ax^2 + 2bx + c = 0, bx^2 + 2cx + a = 0, cx^2 + 2ax + b = 0$$

29. Find the greatest number n such that any subset with $1984 - n$ elements of the set $\{1, 2, 3, \ldots, 1984\}$ contains a pair of coprime numbers.

30. If none of the numbers $a, a + d, \ldots, a + (n - 1)d$ is divisible by n, then d and n are coprime.

31. Suppose a is relatively prime to 2 and 5. Prove that for any n there is a power of a ending with $000\ldots01$ (n digits).

32. Divide the numbers $1, 2, \ldots, 200$ into 50 groups of four numbers. Prove that in at least one of these groups there are three numbers that are the side length of a triangle.

33. A square of dimensions 5×6 cells is cut into 8 rectangles: the cuts run only along the lines of the cells. Prove that there are 2 equal rectangles among the 8 cut ones.

34. Two of six points placed in a 3×4 rectangle will have distance less than or equal to $\sqrt{5}$,

2.2 Level: Medium

2.2.1 Examples

1. Prove that no matter how we pick $n+1$ members of the set $\{1, 2, \ldots, 2n\}$, there will be two among them, say a and b, such that $a + b = 2n + 1$.

   Solution. Let $x_1, x_2, \ldots, x_{n+1}$ be the picked numbers. Consider also the numbers $2n+1-x_1, 2n+1-x_2, \ldots, 2n+1-x_{n+1}$. Altogether there are $2n+2$ numbers, all in the interval $[1, 2n]$. By the Pigeonhole Principle, at least two of them are equal. Because x_k, $k = 1, 2, \ldots, n$ are all different and so are $2n+1-x_k$, $k = 1, 2, \ldots, n$, it follows that $x_i = 2n+1-x_j$ for some i and j, $i \neq j$. Hence $\{a, b\} = \{x_i, x_j\}$.

2. Prove that among any nine points situated in the interior of a square of side 1, there are three which form a triangle whose area is no more than $\frac{1}{8}$.

   Solution. The lines which connect the two parts of midpoints of the square's opposite sides divide the square into 4 equal parts. By the Pigeonhole Principle, in at least one of these there exist at least 3 of the 9 given points, let's call them A, B, C. At least one of the parallels through A, B, C to the sides of this square will intersect the segment line determined by the opposite side. Let's say this is AA' and let B', C' be the projections of B and C to AA', respectively. Then

   $$K_{ABC} = K_{ABA'} + K_{ACA'} = \frac{1}{2}AA'(BB' + CC') \leq \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}.$$

3. Given 69 distinct positive integers not exceeding 100, prove that one can choose four of them a, b, c, d such that $a < b < c$ and $a + b + c = d$. Is this statement true for 68 numbers?

   Solution. Let the numbers be $1 \leq a_1 < a_2 < \cdots < a_{69} \leq 100$. Clearly $a_1 \leq 32$. Consider the sequence

   $$a_3 + a_1 < a_4 + a_1 < \cdots < a_{69} + a_1$$

   $$a_3 - a_2 < a_4 - a_2 < \cdots < a_{69} - a_2.$$

   Each of their terms is a positive integer not exceeding $100 + 32 = 132$. Since the two sequences have jointly $67 + 67 = 134$ terms, there must exist indices $i, j \in \{3, 4, \ldots, 69\}$ such that $a_i + a_1 = a_j - a_2$. We have $a_1 < a_2 < a_i$, and since $a_1 + a_2 + a_i = a_j$, the first part is done.

   A counterexample for the second part is given by the set $\{33, 34, 35, \ldots, 100\}$.

4. Let the sum of a set of numbers be the sum of its elements. Let S be a set of positive integers, none greater than 15. Suppose no two disjoint subsets of S have the same sum. What is the largest sum a set S with these properties can have?

Solution. First of all we will prove that S contains at most 5 elements. Suppose otherwise, then S has at least $\binom{6}{1}+\binom{6}{2}+\binom{6}{3}+\binom{6}{4}$ or 56 subsets of 4 or fewer members. The sum of each of these subsets is at most 54, since $15+14+13+12=54$. Hence, by the Pigeonhole Principle, at least two of these sums are equal. If these subsets are disjoint, we are done; if not, then the removal of the common element(s) yields the desired contradiction.

It is not difficult to show that the set $S'=\{15,14,13,11,8\}$, on the other hand, satisfies the conditions of the problem. The sum of S' is 61. Hence the set S we seek is a 5-element set with a sum of at least 61. Let $S=\{a,b,c,d,e\}$ with $a<b<c<d<e$, and let s denote the sum of S. Then it is clear that $d+e\leq 29$ and $c\leq 13$. Since there are $\binom{5}{2}$ 2-element subsets of S, $a+b\leq d+e-\binom{5}{2}+1=d+e-10+1\leq 20$, because $a+b$ has the least sum from all pairs in our five element subset. Hence $s\leq 20+13+29=62$. If $c\leq 12$, then $S\leq 61$; if $c=13$, then $d=14$ and $e=15$. Then $s\leq a+b+42$. Since $12+15=13+14$, $b\leq 11$. If $b\leq 10$, then $a+b\leq 19$ and $s\leq 61$; if $b=11$, then $a\leq 8$ as $10+15=11+14$ and $9+15=11+13$, implying, that $s\leq 8+11+42=61$. In all cases, $s\leq 61$. It follows that the maximum we seek is 61.

5. Prove that among any $n+1$ numbers from the set $\{1,2\ldots,2n\}$ there one that is divisible by another.

   Solution. We select $(n+1)$ numbers $a_1,a_2,\ldots,a_{n+1}$ and write them in the form $a_i=2^kb_i$ with b_i odd. Then we have $(n+1)$ odd numbers $b_1,b_2,\ldots,b_{n+1}$ from the interval $[1,2n-1]$. But there are only n odd numbers in this interval. Thus two of them p,q are such that $b_p=b_q$. Then one of the numbers a_p,a_q is divisible by the other.

6. Consider the set $M=1,2,3,\ldots,2007$. Prove that in any way we choose the subset X with 15 elements of M there exist two disjoint subsets A and B in X such that the sum of the members of A is equal to the sum of the members of B.

   Solution. The number of all non-empty subsets of X is equal to $2^{15}-1=32767$. The fifteen greatest elements of M give us the sum equal to

$$1993+1994+\cdots+2007=\frac{15(1993+2007)}{2}=30000$$

   Thus for every $X\in M$ with $|X|=15$, the subsets of X have the sum of elements in the interval $[1,30000]$. Because X has 32767 non-empty subsets and sum of their elements can take at most 30000 different values, by the Pigeonhole Principle there exist two subsets $U,V\in X$ such that

$$\sum_{i\in U} i=\sum_{j\in V} j$$

   Let us take sets $A=U-(U\cap V)$ and $B=V-(U\cap V)$. Clearly they are disjoint and in plus

$$\sum_{i\in A}=\sum_{i\in U} i-\sum_{i\in U\cap V}=\sum_{j\in V} j-\sum_{j\in U\cap V} j=\sum_{j\in B} j$$

they satisfy our condition and we are done.

7. Find the greatest positive integer n for which there exist n nonnegative integers $x_1, x_2, \dots, x_n$, not all zero, such that for any sequence $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ of elements $\{-1, 0, 1\}$, not all zero, n^3 does not divide $\epsilon_1 x_1 + \epsilon_2 x_2 + \cdots + \epsilon_n x_n$.

   Solution. The statement holds for $n = 9$, by choosing $1, 2, 2^2, \dots, 2^8$, since in that case

$$|\epsilon_1 + \epsilon_2 \cdot 2 + \cdots + \epsilon_9 \cdot 2^8| \leq 1 + 2 + \cdots + 2^8 < 9^3$$

   However, if $n = 10$, then $2^{10} > 10^3$, so by the Pigeonhole Principle, there are two subsets A and B of $\{x_1, x_2, \dots, x_{10}\}$ whose sums are congruent modulo 10^3. Let $\epsilon_i = 1$ if x_i occurs in A but not in B, -1 if x_i occurs in B but not in A, and 0 otherwise; then $\sum \epsilon_i x_i$ is divisible by n^3.

8. 1981 points lie inside a cube of side 9. Prove that there are two points within a distance less than 1.

   Solution. Assume by way of contradiction that the distance between any two points is greater than or equal to 1. Then the spheres of radius $\frac{1}{2}$ with centers at these 1981 points have disjoint interiors and are included in the cube of side 10 determined by the six parallel planes to the given cube's faces and situated in the exterior at a distance of $\frac{1}{2}$. It follows that the sum of the volumes of the 1981 spheres is less than the volume of the cube of side 10, hence

$$1981 \cdot \frac{4\pi(\frac{1}{2})^3}{3} = 1981 \cdot \frac{\pi}{6} > 1000,$$

   a contradiction. The proof is complete.

   Remark. The Pigeonhole Principle does not help us here. Indeed, dividing each side of the cube in $\sqrt[3]{1981} = 12$ congruent segments we obtain $12^3 = 1728$ small cubes of side $\frac{9}{12} = \frac{3}{4}$. In such a cube there will be two points from the initial 1981 points. The distance between is less than $\frac{3}{4}\sqrt{3}$ which is not enough, since $\frac{3}{4}\sqrt{3} < 1$.

9. Prove that in any party attended by n persons, there are two people for which at least $[n/2] - 1$ of the remaining $n - 2$ persons attending the party know both or else neither of the two. Assume that "knowing" is a symmetric relation.

   Solution. To prove this, count(with multiplicity) the number of times a person knows both or else neither members of a pair. If the person in question knows exactly k persons at the party, then his contribution to the count is

$$\frac{(n-1)(n-2)}{2} - k(n-1-k) \geq \frac{(n-1)(n-2)}{2} - \frac{(n-1)^2}{4} \geq \frac{(n-1)(n-3)}{4}$$

   Thus, the total count is at least $n(n-1)(n-3)/4$. Since there are $n(n-1)/2$ pairs, the desired result follows from the Pigeonhole Principle and the fact that $[n/2] - 1$ is the smallest integer greater or equal to $(n-3)/2$.

10. Prove that every sequence of $mn+1$ real numbers contains either a nondecreasing subsequence of length of $m+1$ or a nonincreasing subsequence of length $n+1$.

    Solution. Let the sequence of $mn+1$ numbers be $a_1, a_2, \ldots, a_{mn+1}$. For every number a_i in the sequence we associate a pair of numbers (x_i, y_i), where x_i is the length of the largest nondecreasing sequence starting with a_i and y_i is the length of largest with nonincreasing subsequence starting with a_i. Assume for the sake of contradiction that there is no x_i greater than m and y_i greater that n, so that $1 \le x_i \le m$ and $1 \le y_i \le n$. Thus the cardinality of pairs (x_i, y_i) is at most mn. We have $mn+1$ numbers in our sequence and by the Pigeonhole Principle there exist two numbers a_k and a_l, $k < l$, which have exactly the same pairs (x_k, y_k) and (x_l, y_l) associated to them. If $a_k \ge a_l$, then $x_k \ge x_l + 1$, because we can increase our nondecreasing subsequence, which starts with a_l, by adding a_k in front of it. If $a_l \le a_l$, then $y_k \ge y_l + 1$ because we can increase our nonincreasing subsequence that starts with a_l , again by adding a_k in front of it. Thus there can be no equal pairs and we have a contradiction. Hence the statement of the problem is true.

11. We have 2^n prime numbers written on the blackboard in a line. We know that there are less than n different prime numbers on the blackboard. Prove that there is a compact subsequence of numbers in that line whose product is a perfect square.

    Solution. Suppose that $p_1, p_2, \ldots, p_m$ $(m < n)$ are primes which we met in the sequence $a_1, a_2, \ldots, a_{2^n}$ written on the blackboard. It is enough to prove that there is a compact subsequence, where each prime occurs even times. Denote c_{ij} the exponent of the prime p_i $(1 \le i \le m)$, in the product of the first j numbers $a_1 \cdot a_2 \cdots a_j$ from our sequence. Let d_{ij} be the residue modulo 2 of c_{ij}, then we can write $c_{ij} = 2t_{ij} + d_{ij}$, $d_{ij} \in \{0, 1\}$. Every system $(d_{1j}, d_{2j}, \ldots, d_{mj})$ is formed from m zeros and ones. Number of possible such systems is 2^m which is less than 2^n. Hence by Pigeonhole Principle there exist two identical systems.

    $$(d_{1k}, d_{2k}, \ldots, d_{mk}) = (d_{1l}, d_{2l}, \ldots, d_{ml}), 1 \le k < l \le 2^n$$

    We have $d_{ik} = d_{il}$ for $1 \le i \le m$ and from here $c_{il} - c_{ik} = 2(t_{il} - t_{ik}) + (d_{il} - d_{ik}) = 2(t_{il} - t_{ik})$ and $c_{il} - c_{ik}$ is divisible by 2 for $1 \le i \le m$.

    Thus the exponent of the p_i in the product $a_{k+1}a_{k+2} \cdots a_l = \dfrac{a_1 a_2 \cdots a_l}{a_1 a_2 \cdots a_k}$ is equal to $c_{il} - c_{ik}$, so every number p_i has an even exponent is the product $a_{k+1}a_{k+2} \cdots a_l$. Hence $a_{k+1}a_{k+2} \cdots a_l$ is the perfect square.

12. An international society has members from six different countries. The list of members contains 1957 names, numbered $1, 2, \ldots, 1957$. Prove that there is at least one member whose number is the sum of the numbers of two members from his own country or twice as large as as the number of one member from his own country.

    Solution. Assume that there is partitioning of $1, 2, \ldots, 1957$ into six sum-free subsets A, B, C, D, E, F. By the Pigeonhole principle one of these subsets, say A, has at least

327 elements

$$a_1 < a_2 < \cdots < a_{327}$$

The 326 differences $a_{327} - a_i = a_j$, $i = 1, \ldots, 326$ do not lie in A, since A is sum-free. Indeed, from $a_{327} - a_i = a_j$ follows $a_i + a_j = a_{327}$. So they must lie in B to F. By Pigeonhole Principle one of these subsets, say B, has at least 66 of these differences

$$b_1 < b_2 < \cdots < b_{66}.$$

The 65 differences $b_{66} - b_i$, $i = 1, \ldots, 65$ lie neither in A nor in B since both sets are sum free. Hence they lie in C to F. Again by Pigeonhole principle one of these subsets,say C, has at least 17 of these differences.

$$c_1 < c_2 < \cdots < c_{17}.$$

The 16 differences $c_{17} - c_i$, $i = 1, \ldots, 16$ do not lie in A to C, that is, in D to F. One of these subsets, say D, has at least 6 of these differences. $d_1 < d_2 < \cdots < d_6$. The 5 differences $d_6 - d_i$ do not lie in A to D, that is, in E or F. One of these, say E, has at least 3 elements $e_1 < e_2 < e_3$. The two differences $f_1 = e_3 - e_2, f_2 = e_3 - e_1$ do not in A to E. Hence they lie in F. The difference $g = f_2 - f_1$ does not lie not A to F. Contradiction.

2.2.2 Problems

1. Five lattice points are chosen in the plane lattice. Prove that you can choose two of these points such that the segment joining these points passes through another lattice point. (The plane lattice consists of all points of the plane with integral coordinates.)

2. A chessmaster has 77 days to prepare for a tournament. He wants to play at least one game per day, but not more then 132. Prove that there is a sequence of successive days on which he plays exactly 21 games.

3. Consider $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Prove that for any subset, A, of S, at least one of the sets A and S/A contains three elements in arithmetical progression.

4. Each of 17 scientists corresponds with all others. They correspond about only three topics and any two treat exactly one topic. Prove that there are at least three scientists, who correspond with each other about the same subject.

5. Twenty different positive integers are all less than 70. Prove that among their pairwise differences there are four equal numbers.

6. Fifty-one small insects are placed inside a square of side 1. Prove that at any moment there are at least three insects which can be covered by a single disk of radius $\frac{1}{7}$.

7. Let n be a positive integer that is not divisible by 2 or 5. Prove that there is a multiple of n consisting entirely of ones.

8. Let S be a set of 25 points such that, in any 3-subset of S, there are at least two points with distance less than 1. Prove that there exists a 13-subset of S which can be covered by a disk of radius 1.

9. From ten distinct two-digit numbers, one can always choose two disjoint nonempty subsets, so that their elements have the same sum.

10. Let $a_1, a_2, .., a_n$ $(n \geq 5)$ be any sequence of positive integers. Prove that it is always possible to select at a subsequence and add or subtract its elements such that the sum is a multiple of n^2.

11. Of k positive integers with $a_1 < a_2 < \cdots < a_k \leq n$ and $k > \lfloor \frac{n+1}{2} \rfloor$, there is at least one pair a_i, a_r such that $a_i + a_1 = a_r$.

12. Let $n \in \mathbb{Z}^*$ and $a \in \mathbb{R}$. Prove that one of $a, 2a, \ldots, (n-1)a$ has at most distance $\frac{1}{n}$ from an integer.

13. Prove that from six points in a 3×4 rectangle there are two that are more than $\sqrt{5}$ apart.

14. In any convex polygonal there is a diagonal not parallel to any side.

15. Prove that among 18 persons, there are four who know each other or four persons who do not each other.

16. In a circular arrangement of zeros and ones, with n terms altogether, prove that if the number of ones exceeds $n(1-\frac{1}{k})$, there must be string of k consecutive ones.

17. Let $a_1, \ldots, a_{100}$ and $b_1, \ldots, b_{100}$ be two permutations of the integers from 1 to 100. Prove that, among the products $a_1b_1, a_2b_2, \ldots, a_{100}b_{100}$, there are two with the same remainder upon division by 100.

18. Consider a 7×7 square grid where the 49 points are colored with two colors. Prove that there exist at least 21 rectangles with vertices of the same color and with sides parallel to the sides of the square.

19. We have $2n+1$ real numbers greater than 1 and less than 2^n. Prove that there exist three from those who can be side of a triangle.

20. Prove that among 70 distinct positive integers ≤ 200 there exist two whose differences is $4, 5$ or 9.

21. A sloppy tailor uses a machine to cut 120 square patches of area 1 from a 25×20 rectangle sheet. Prove that, no matter how he performs the cutting, he can still cut a circular patch of diameter 1 from the remaining fabric.

22. We have numbers $\{1, 2, \ldots, 16\}$ divided in 3 groups. Prove that it is possible to find numbers x, y, z (not necessarily distinct) in one of those groups such that $x + y = z$.

23. There are 10 participants in a math circle. On holidays each of them sent 5 greeting cards to the other participants of the circle. Prove that there are 2 participants who have sent greetings cards to each other.

24. The squares of an 8×8 checkerboard are filled with numbers $\{1, 2, \ldots, 64\}$. Prove that there exist 2 squares with a common side, such that the difference between the numbers in them is at least 5.

25. Prove that from any set of 105 pairwise distinct three-digit numbers, it is possible to select 4 pairwise disjoint subsets such that the sums of the numbers in each subset are equal.

26. In the sequence $1, 1, 2, 3, 5, 8, 3, 1, 4, \ldots$ each term starting with the third is the sum of the two preceding terms modulo 10. Prove that the sequence is purely periodic. What is the maximum possible length of the period?

27. Consider the Fibonacci sequence defined by $F_1 = F_2 = 1$,

$$F_{n+1} = F_{n-1} + F_n, n \geq 1$$

Prove that for any n, there is a Fibonacci number ending with n zeros.

28. Let $x_1, x_2, x_3, \ldots$ be a sequence of integers such that

$$1 = x_1 < x_2 < x_3 \cdots, \text{ and } x_{n+1} \leq 2n \text{ for } n = 1, 2, 3, \ldots$$

Show that every positive integer k is equal to $x_i - x_j$ for some i and j.

29. Prove that among four real numbers greater than or equal to 1, there are two, say a and b, such that

$$\frac{\sqrt{(a^2-1)(b^2-1)}+1}{ab} \geq \frac{\sqrt{3}}{2}$$

30. Determine the smallest integer $n \geq 4$ for which from any n distinct integers one can choose four, say a, b, c, and d, no two equal, such that $a + b - c - d$ is divisible by 20.

31. Prove that if $n \geq 3$ prime numbers form an arithmetic progression, then the ratio of the progression is divisible by any prime number $p < n$.

32. Prove that for any positive number n there is an n-digit number that is divisible by 5^n, all whose digits are odd.

33. Each point of the plane is painted white, black or red(the painting is entirely arbitrary). Prove that it is possible to find 2 points which are painted in the same color and the distance between them is 1m.

34. The sides of a regular triangle are bicolored. Do there exist on its perimeter three monochromatic vertices of a rectangular triangle?

2.3 Level: Hard

2.3.1 Example

1. Prove that among $\binom{m+n}{n}$ persons there exist either $n+1$ persons who pairwise know each other or $m+1$ persons who pairwise don't know each other.

   Solution. Let us call two people who know each other friends and two people who don't know each other enemies. We will use mathematical induction to solve this problem. Consider the base case $m=1, n=1$. The problem asks that from $\binom{2}{1}=2$ persons there exist either two friends or two enemies, which is obviously true.

   Suppose our statement is true for (m,n). We will prove that the statement is also true for $(m+1,n)$ and $(m,n+1)$. In this way we can say that the problem statement is true for all (m,n).

   Let us prove our statement for $(m+1,n)$. Take an arbitrary person P from $\binom{m+n+1}{n}$ people. Assuming the contrary, P cannot have more than $\binom{m+n}{m+1}-1$ friends. Otherwise by the inductive hypothesis there exist n friends whom P knows, so totally $n+1$ friends, or $m+2$ persons who are enemies. Also P cannot have more than $\binom{m+n}{m}-1$ enemies. Otherwise by the inductive hypothesis there exist $m+1$ enemies whom P doesn't know, thus there exist $m+2$ enemies, or $n+1$ friends from enemies of P. Hence our group has at most $1+(\binom{m+n}{m+1}-1)+(\binom{m+n}{m}-1)=\binom{m+n+1}{n}-1$ persons, contradiction.

   Analogously we can prove that our statement is true for $(m,n+1)$. Thus the problem is solved by mathematical induction.

2. Show that any convex polyhedron has two faces with the same number of edges.

   Solution. Choose a face with maximal number of edges, and let n be this number. The number of edges of each of the n adjacent faces ranges between 3 and n, so, by the Pigeonhole Principle, two of these faces have the same number of edges.

3. Consider $2n$ distinct positive integers $a_1, a_2, \ldots, a_{2n}$ not exceeding n^2 $(n>2)$. Prove that some three of the differences $a_i - a_j$ are equal.

   Solution. The natural approach is via the Pigeonhole Principle to show that

$$\frac{\binom{2n}{n}}{n^2-1} > 2$$

   However this approach fails. The solution proceeds as follows. We may assume that $a_1 < a_2 < \cdots < a_{2n}$. Consider the differences $a_2-a_1, a_3-a_2, \ldots, a_{2n}-a_{2n-1}$. If no three are equal, then

$$(a_2-a_1)+(a_3-a_2)+\cdots+(a_{2n}-a_{2n-1}) \geq 1+1+2+2+\cdots+(n-1)+(n-1)+n = n^2$$

   On the other hand, this sum is equal to $a_{2n}-a_1$, hence is less than or equal to n^2-1. This is a contradiction, and the problem is solved.

4. Let p be a prime number, $p \neq 2$, and $m_1, m_2, \ldots, m_p$ positive consecutive integers and σ is a permutation of the set $\{1, 2, \ldots, p\}$. Prove that there exist numbers $k, l \in \{1, 2, \ldots, p\}$, $k \neq l$ with the property

   $m_k \cdot m_{\sigma(k)} \equiv m_l \cdot m_{\sigma(l)} (\text{mod } p)$

   Solution. It is clear that numbers $m_1, m_2, \ldots, m_p$ give different residues modulo p. Without loss of generality suppose $m_1 = 0, m_2 = 1, \ldots, m_p = p-1$. Let σ be a permutation of the set $\{1, 2, \ldots, p\}$. If $m_{\sigma(1)} = 0$ then there exist $k \in \{2, 3, \ldots, p\}$ such that $m_{\sigma(k)} = 0$, thus $m_1 \cdot m_{\sigma(1)} \equiv m_k \cdot m_{\sigma(k)} \equiv 0 (\text{mod } p)$ and the problem is solved in this case.

   It follows that $m_1 = m_{\sigma(1)} = 0$, then using Wilson Theorem we get

   $$m_2 \cdot m_3 \cdots m_p = (p-1)! \equiv -1 (\mod p)$$

   $$m_{\sigma(2)} \cdot m_{\sigma(3)} \cdots m_{\sigma(p)} = (p-1)! \equiv -1 (\mod p)$$

   Multiplying them

   $$m_2 m_{\sigma(2)} \cdot m_3 m_{\sigma(3)} \cdots m_p m_{\sigma(p)} \equiv 1 (\mod p)$$

   Suppose that numbers $m_2 m_{\sigma(2)}, m_3 m_{\sigma(3)}, \ldots, m_p m_{\sigma(p)}$ are different modulo p and different from 0, then

   $$m_2 m_{\sigma(2)} \cdot m_3 m_{\sigma(3)} \cdots m_p m_{\sigma(p)} \equiv -1 (\mod p)$$

   Contradiction, thus there exist $k, l \in \{2, 3, \ldots, p\}$, $k \neq l$ such that

   $m_k \cdot m_{\sigma(k)} \equiv m_l \cdot m_{\sigma(l)} (\text{mod } p)$ and we are done.

5. Prove that among any 16 distinct positive integers not exceeding 100 there are four different ones a, b, c, d, such that $a + b = c + d$.

   Solution: Let $a_1 < a_2 < \cdots < a_{16}$ denote the 16 numbers. Consider the difference of each of those integers. There are $\binom{16}{2} = 120$ such pairs.

   Let (a_i, a_j) denote a pair of numbers with $a_i > a_j$. If we have two distinct pairs of numbers (a_{i_1}, a_{i_2}) and (a_{i_3}, a_{i_4}) such that $a_{i_1} - a_{i_2} = a_{i_3} - a_{i_4}$, then we get the desired quadruple $(a, b, c, d) = (a_{i_1}, a_{i_4}, a_{i_2}, a_{i_3})$ unless $a_{i_2} = a_{i_3}$. We say a is bad for the pair of pairs (a_{i_1}, a) and (a, a_{i_2}) if $a_{i_1} - a = a - a_{i_2}$ (or $2a = a_{i_1} + a_{i_2}$). Note that we are done if a number a is bad for two pairs of pairs of numbers. Indeed, if a is bad for (a_{i_1}, a), (a, a_{i_2}) and (a_{i_3}, a), (a, a_{i_4}), then $a_{i_1} + a_{i_2} = 2a = a_{i_3} + a_{i_4}$.

   Finally, we assume that each a_i is bad for at most one pair of numbers. For each such pair of numbers, we take one pair of numbers out of consideration. Hence there are no bad numbers anymore. Then we still have at least $120 - 16 = 104$ pairs of numbers left. The difference of the numbers in each remaining pair ranges from 1 to 99. By the Pigeonhole Principle, some of these differences have the same value. Assume that $a_{i_1} - a_{i_2} = a_{i_3} - a_{i_4}$, then $(a_{i_1}, a_{i_4}, a_{i_2}, a_{i_3})$ satisfies the condition of the problem.

6. Let $x_1 = x_2 = x_3 = 1$ and $x_{n+3} = x_n + x_{n+1}x_{n+2}$ for all positive integers n. Prove that for any positive integer m there is an integer $k > 0$ such that m divides x_k.

   Solution. Observe that setting $x_0 = 0$ the condition is satisfied for $n = 0$.

   We prove that there is integer $k \leq m^3$ such that x_k divides m. Let r_t be the remainder of x_t when divided by m for $t = 0, 1, \ldots, m^3 + 2$. Consider the triples $(r_0, r_1, r_2), (r_1, r_2, r_3), \ldots, (r_{m^3}, r_{m^3+1}, r_{m^3+2})$. Since r_t can take m values, it follows by the Pigeonhole Principle that at least two triples are equal. Let p be the smallest number such that triple $(r_p, r_{p+1}, r_{p+2}$ is equal to another triple (r_q, r_{q+1}, r_{q+2}), $p < q \leq m^3$. We claim that $p = 0$.

   Assume by way of contradiction that $p \geq 1$. Using the hypothesis we have

   $$r_p \equiv r_{p-1} + r_p r_{p+1} \pmod{m}$$

   and

   $$r_{q+2} \equiv r_{q-1} + r_q r_{q+1} \pmod{m}.$$

   Since $r_p = r_q, r_{p+1} = r_{q+1}$ and $r_{p+2} = r_{q+2}$, it follows that $r_{p-1} = r_{q-1}$, so

   $$(r_{p-1}, r_p, r_{p+1}) = (r_{q-1}, r_q, r_{q+1}),$$

   which is a contradiction with the minimality of p. Hence $p = 0$, so $r_q = r_0 = 0$, and therefore $x_q \equiv 0 (\text{mod } m)$.

7. Prove that among any ten consecutive positive integers, there exist at least one which is relatively prime with any of other nine.

   Solution. Our array of ten consecutive positive integers contains five odd numbers. Among them, there exist at most two which are divisible by 3, one multiple of 5, and at most a multiple of 7. Hence, at least one of these five odds is not divisible by $3, 5, 7$ (and of course, 2). Let's call this number m, and suppose it is nor relatively prime with any of the other nine. Then, the array contains a number, n, such that $(m, n) = d$, $d > 1$, so d divides $|m - n|$. Since $|m - n| \leq 9$, d can be $2, 3, 5$ or 7, which is absurd (none of these numbers divides m).

8. Consider a regular hexagon and k lines with the property that each of them divides the hexagon into two pentagons such that the ratio of their areas is equal to $\frac{1}{3}$ or $\frac{1}{2}$. Prove that if $k \geq 25$ then there exist three concurrent lines among the k considered.

   Solution. We need the following lemma:

   Lemma. Let $ABCD$ be a rectangle and let M, N be two points such that $M \in AB$ and $N \in CD$. If $Area[AMND]/Area[BCMN] = m$ (constant), then MN passes through a fix point, which depends upon m.

Let P be the midpoint of MN and draw the line $M'N'$ through P parallel to AD. Because $Area[AMND] = Area[AM'N'D]$, we have

$$m = \frac{Area[AMND]}{Area[BCMN]} = \frac{Area[AM'N'D]}{Area[BM'N'C]} = \frac{AM'}{M'B}$$

Thus the line $M'N'$ is fix and P is the midpoint of MN and is fix.

Now let d be one of those k lines. In order to divide the hexagon into two pentagons d should pass through two opposite sides, so through one of the pairs of sides (AB,DE), (BC,EF), (CD,AF). By the Pigeonhole Principle there exist 9 lines that pass through the same pair of opposite sides; suppose this is (AB,CD). We have the lines which dividing the hexagon into two pentagons such that the ratio of areas is equal to $\frac{1}{3}$ or $\frac{1}{2}$. Again by the Pigeonhole Principle there exist 5 lines that divide our hexagon in ratio $\frac{1}{3}$ or $\frac{1}{2}$. Consider line l, one of those 5 lines l passing through (AB, CD). Suppose it divides the hexagon in two pentagons P_1, P_2. We can have $\frac{Area[P_1]}{Area[P_2]} = p$ or $\frac{Area[P_2]}{Area[P_1]} = p$. By the Pigeonhole Principle there exist 3 lines l_1, l_2, l_3 satisfying only one of the above relations. For each of l_i we have $\frac{Area[P_1]}{Area[P_2]}$ =constant, thus $Area[AMNE]$ =constant and using our lemma we get that these three lines lines are passing through the same point, in other words are collinear. The problem is solved.

9. Consider $p_n = [en!] + 1$ points in space. Each pair of points is connected by a line, and each line is colored with one of n colors. Prove that there is at least one triangle with sides of the same color.

Suppose that p_n is the number of points such that coloring them arbitrarily in n colors we will always will find a triangle with sides of the same color. Clearly, $p_1 = 3$ and $p_2 = 6$. Let us estimate p_{n+1} in terms by p_n. Take a point P from our p_{n+1} points. Suppose the number of the remaining points is $(n+1)(p_n - 1) + 1$. Then by the Pigeonhole Principle there exist one color connecting P with p_n other points. If this color is used in connecting a pair from those p_n points, then we are done, otherwise these p_n points are colored with n colors and by the hypothesis should be a triangle with sides of the same color. In general, we have

$$p_{n+1} - 1 = (n+1)(p_n - 1) + 1,$$

hence

$$\frac{p_{n+1} - 1}{n+1} = (p_n - 1) + \frac{1}{n+1}$$

With $q_n = p_n - 1$, we get $q_1 = 2$,

$$\frac{q_{n+1}}{(n+1)!} = \frac{q_n}{n!} + \frac{1}{(n+1)!}$$

From this, we easily get by summing up that

$$q_n = n!(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}).$$

We recognize the truncated series for e in the parentheses, recall that

$$e = \lim_{n\to\infty}(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}).$$

Thus, $e = \frac{q_n}{n!} + r_n$,

$$r_n = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots = \frac{1}{n!}(\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \cdots) < \frac{1}{n\cdot n!}.$$

Hence $q_n < en! < q_n + \frac{1}{n}$, that is, $q_n = [en!]$, or

$$p_n = [en!] + 1.$$

10. Soldiers are in a rectangular formation of m rows and n columns, with the soldiers in each row arranged in order of increasing height and from the left to right. The commanding officer decides to rearrange the soldiers one column at time, so that each each column is in order of increasing height from front to back. Show that the rows are still arranged by increasing height.

    An alternative version of the problem may be helpful. A given $m \times n$ matrix A of real numbers satisfies $a_{i1} \le a_{i2} \le \cdots \le a_{in}$ for $i = 1, 2, \ldots, m$. The elements in each column are now rearranged to obtain a new matrix b satisfying $b_{1j} \le b_{2j} \le \cdots \le b_{mj}$ for $j = 1, 2, \ldots, n$. Prove that $b_{i1} \le b_{i2} \le \cdots \le b_{in}$ for $i = 1, 2, \ldots, m$.

    Solution. For $1 \le j \le n$, let

$$N_j(x) = |\{i | b_{ij} \ge x\}|, -\infty, x < +\infty$$

    Thus $N_j(x)$ counts the number of elements of the jth column of B that are equal or exceed x. Since the numbers in any given column of B are the same as those in the corresponding column A, only in a different order, and $a_{ij} \le a_{i,j+1}$ for $i = 1, 2, \ldots, m$, we see that for any real number x,

$$N_j(x) \le N_{j+1}(x), j = 1, 2, \ldots, n-1.$$

    Suppose the desired property of B fails. Then $b_{rc} > b_{r,c+1}$ for some pair (r, c). Let $x = b_{rc}$. Then $b_{rc} \le b_{r+1,c} \le \cdots \le b_{mc}$ implies $N_c(x) \ge m - r + 1$ and $b_{1,c+1} \le b_{2,c+1} \le .. \le b_{r,c+1} < b_{rc}$ implies $N_{c+1}(x) \le m - r$. Since $N_c(x) > N_{c+1}(x)$ is impossible, the desired property of B holds.

11. Suppose that the squares of $n \times n$ chessboard are labeled arbitrarily with the numbers 1 through n^2. Prove that there are two adjacent squares whose label differ (in absolute value) by at least n.

    Solution. With each pair of adjacent squares, associate a label (min,max) where min (max) is the minimum (maximum) of the labels of the square involved. Suppose there exist integers A, B with $A < B$ and n non-overlapping pairs of adjacent squares whose labels (a_i, b_i) satisfy

$\max\{a_1, a_2, \ldots, a_n\} \leq A$, and $B \leq \min\{b_1, b_2, \ldots, b_n\}$

Then

$$\sum_{i=1}^{n} b_i \geq B + (B+1) + \cdots + (B+n-1) = \left(B + \frac{n-1}{2}\right) n$$

and

$$\sum_{i=1}^{n} a_i \geq A + (A-1) + \cdots + (A-n+1) = \left(A - \frac{n-1}{2}\right) n$$

so $\sum_{i=1}^{n}(b_i - a_i) \geq n^2$, from which we have $b_j - a_j \geq n$ for some j.

To see that there exist integers A, B and n pairs of adjacent squares as desired, let k be the smallest integer such that the set of labels for set for every row (column) is a subset of $[k]$. The label set for every row(column) is a subset of $[n^2]$ so such a minimum value exists. We may assume that the square in row r and column c of the chessboard is labeled k and the labels of the squares in this row r are all from $[k-1]$. By the minimality of k, every column except possibly column c contains two adjacent squares, one with label $\leq k-1$ and the other with label $\geq k+1$, then the column c contains two adjacent squares, one with label $\leq k$ and the other with label $B = k+1$. Otherwise, the label set for column c is a subset of $[k]$ and we have the desired situation with $A = k-1$ and $B = k$.

This result is clearly best possible since the squares of the chessboard can be labeled $1, 2, \ldots, n^2$ with the square in row i and column j receiving the label $n(i-1)+j$, which results in every pair of adjacent squares having labels that differ by either 1 or n.

12. Given a positive integer n, prove that there exists $\epsilon > 0$ such that for any n positive integers real numbers $a_1, a_2, \ldots, a_n$, there exists $t > 0$ such that

$$\epsilon < \{ta_1\}, \{ta_2\}, \ldots, \{ta_n\} < \frac{1}{2}$$

Solution. More generally, we prove by induction on n that for any real number $0 < r < 1$, there exists $0 < \epsilon < r$ such that for $a_1, a_2, \ldots, a_n$ any positive real numbers, there exist $t > 0$ with

$$\{ta_1\}, \{ta_2\}, \ldots, \{ta_n\} \in (\epsilon, r)$$

The case $n = 1$ needs no further comment.

Assume without loss of generality that a_n is the largest of the a_i. By the hypothesis, for any $r' > 0$ (which we will specify later) there exist $\epsilon' > 0$ such that for any $a_1, a_2, \ldots, a_{n-1} > 0$ there exist $t' > 0$ such that

$$\{t'a_1\}, \{t'a_2\}, \ldots, \{t'a_n\} \in (\epsilon', r')$$

Let N be an integer also to be specified later, a standard argument using the Pigeonhole Principle shows that one of $t'a_n, 2t'a_n, \ldots, nt'a_n$ has fractional part in $(-1/N, 1/N)$. Let $st'a_n$ be one such term, and take $t = st' + c$ for $c = (r - 1/N)a_n$. Then

$$ta_n \in (r - 2/N, r)$$

So we choose N such that $0 < r - 2/N$, thus making $\{ta_n\} \in (r - 2/N, r)$. Note that this choice of N makes $c > 0$ and $t > 0$, as well.

As for the other ta_i, for each i we have $k_i + \epsilon' < t'a_i < k_i + r'$ for some integer k_i, so $sk_i + s\epsilon' < st'a_i < sk_i + sr_i$ and

$$sk_i + \epsilon' < (st' + c)a_i < sk_i + sr' + \frac{a_i(r - 1/N))}{a_n} \leq sk_i + Nr' + r - 1/N.$$

So we choose r' such that $Nr' - 1/N < 0$, thus making $\{ta_i\} \in (\epsilon', r)$. Therefore, letting $\epsilon = \min(r - 2/N, \epsilon')$, we have

$$0 < \epsilon < \{ta_1\}, \{ta_2\}, \ldots, \{ta_n\} < r$$

for any choices of a_i. This completes the inductive step, and the claim is true for all natural numbers n.

2.3.2 Problems

1. Let $P_1, P_2, \ldots, P_9$ be nine lattice points in space, no three collinear. Prove that there is a lattice point L lying on some segment P_i, P_k, $i \neq k$.

2. Let k be a positive integer and $n = 2^{k-1}$. Prove that, from $(2n-1)$ positive integers, one can select n integers, such that their sum is divisible by n.

3. After the elections of the parliament the deputies formed 12 factions(each deputy became member of exactly 1 faction). After the first plenary meeting the deputies' opinions changed, and they united in 16 new factions (still each deputy was member of exactly 1 faction). Prove that now at least 5 deputies are in smaller factions than immediately after the election of the parliament.

4. Prove that among 18 persons, there are four who know each other or four persons who do not each other.

5. Prove that the set $\{1, 2, 3, \ldots, \frac{3^n-1}{2}\}$ can be split into n sum-free subsets. A subset of A of positive integers is called sum-free, if the equation $x + y = z$ for $x, y, z \in A$ is not solvable.

6. Prove that the function $f(x) = \cos x + \cos(x\sqrt{2})$ is not periodic.

7. A closed disk of radius 1 contains seven points with each point at a distance of at least 1 from all others. Prove that the center of the disk is one of the seven points.

8. Prove that there exist squares which begin with any arbitrary combination of digits.

9. Thirty-three rooks are placed on an 8×8 chessboard, Prove that you can choose five of them which are not in the same row or column. Problem 10. Prove that from any set of 105 pairwise distinct three-digit numbers, it is possible to select 4 pairwise disjoint subsets such that the sums of the numbers in each subset are equal.

10. The positive integers $a_1, a_2, \ldots, a_n \leq 2n$ are such that the least common multiple of any two of them is greater than $2n$. Prove that $a_1 > \lfloor \frac{2n}{3} \rfloor$.

11. The length of each side of a convex quadrilateral $ABCD$ is less than 24. Let P be any point inside $ABCD$. Prove that there exists a vertex, say A, such that $|PA| < 17$.

12. Each of m cards is labeled by one of the numbers $1, 2, \ldots, m$. Prove that if the sum of the labels of any subset of the cards is not a multiple of $m+1$, then each card is labeled by the same number.

13. Find the least positive integer n with the property that among any n numbers from the interval $[0, 1]$ there are two, say a and b, such that

$$\frac{(a^2+1)(b^2+1)}{ab+1} > 4|a-b|$$

14. Prove that for every prime number p there exists Fibonacci number F_i such that $p|F_i$.

15. Let $P = A_1A_2 \cdots A_n$ be a convex polygon. For any point M in the interior, let B_i be the point where A_iM intersects the perimeter. We say that P is balanced if for some such M the points $B_1, B_2, \ldots, B_n$ are interior to distinct sides of P. Prove that if n is even, then P is not balanced.

16. Let k be a positive integer. Consider the sequence x_n, $(n > 0)$ with $x_0 = 0, x_1 = 1$ and $x_{n+1} = kx_n + x_{n-1}$ for all $n \geq 1$. Prove that among the numbers $x_1, x_2, \ldots, x_{1986}$ there are two whose product is divisible by $19 \cdot 86$.

17. A table consists of 17×17 squares. In each square there is written one natural number from 1 to 17; each such number is written in exactly 17 squares. Prove that it is possible to find such a row or such a column, where at least 5 different numbers are written.

18. Given a square grid S containing 49 points in 7 rows and 7 columns, a subset T consisting of k points is selected. The problem is to find the maximum value of k such that no 4 points of T determine a rectangle R having sides parallel to the sides of S.

19. Prove that among any seven distinct real numbers there are two, say a and b, such that $ab + 1 \geq \sqrt{3}|a - b|$.

20. Let n be a positive integer. Find the greatest integer m for which the set $M = \{2, 3, \ldots, m\}$ can be partitioned into n subsets with the property that none contains two members such that one of them is an integer power of the other.

21. Let m be a positive integer. Prove that among any $2m+1$ distinct integers of absolute value less than $2m - 1$ there exist three whose sum is equal to zero.

22. An equilateral triangle ABC is divided into 25 smaller equilateral triangles. In each of the small triangles there is written a natural number from 1 to 25 (different numbers in different triangles). Prove that it is possible to find 2 triangles with a common side where the written numbers differ at least by 4.

23. Let p be a prime number and a, b, c integers such that a and b are not divisible by p. Prove that the equation $ax^2 + by^2 \equiv c (\text{mod } p)$ has integer solutions.

24. Let n, k be positive integers such that $n^k > (k + 1)!$ and let M be a set $M = \{(x_1, x_2, \ldots, x_k), x_i \in \{1, 2, \ldots, n\}, i = 1, 2, \ldots, k\}$. Prove that if A is in M and has $(k+1)!+1$ elements, than there exist α, β in A, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$ such that $(\beta_1 - \alpha_1)(\beta_2 - \alpha_2) \cdots (\beta_k - \alpha_k)$ is divisible by $(k + 1)!$.

25. In each square of a 10×10 board is written a positive integer not exceeding 10. Any two numbers that appear in adjacent or diagonally adjacent spaces of the board are relatively prime. Prove that some number appears at least 17 times.

26. Draw the diagonals of a 21-gon. Prove that at least one angle less than 1 degree is formed.

27. Let $a_1, a_2, \ldots$ be an infinite increasing sequence of positive integers. Prove that an infinite number of terms of the sequence can be written as $a_m = xa_p + ya_q$ with x, y integers and $p \neq q$.

28. Let $P_1P_2 \cdots P_{2n}$ be a permutation of the vertices of a regular polygon. Prove that the closed polygonal line $P_1P_2 \cdots P_{2n}$ contains a pair of parallel segments.

29. The points of the plane are colored by finitely many colors. Prove that one can find a rectangle with vertices of the same color.

30. Inside the unit square lie several circles with the sum of the circumferences equal to 10. Prove that there exist infinitely many lines each of which intersects at least four circles.

31. Let S be a convex set in the plane that contains three noncollinear points. The points of S are colored by p colors, $p > 1$. Prove that for any $n \geq 3$ there exist infinitely many isometric n-gons whose vertices are all colored by the same color.

32. The positive integers 1 to 101 are written down in any order. Prove that you can strike 90 of these numbers, so that a monotonically increasing or decreasing sequence remains.

33. Prove that, among any 13 real numbers, there are two, x and y, such that

$$|x - y| \leq (2 - \sqrt{2})|1 + xy|.$$

34. Prove that among 70 positive integers less than or equal to 200 there are two whose difference is 4,5, or 9.