Let’s Talk About Symmedians!

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Abstract
We will introduce symmedians from scratch and prove an entire collection of interconnected results that characterize them.

Symmedians represent a very important topic in Olympiad Geometry since they have a lot of interesting properties that can be exploited in problems. But first, what are they?

Definition. In a triangle $ABC$, the reflection of the $A$-median in the $A$-internal angle bisector is called the $A$-symmedian of triangle $ABC$. Similarly, we can define the $B$-symmedian and the $C$-symmedian of the triangle.

Do we always have symmedians? Well, yes, only that we have some weird cases when for example $ABC$ is isosceles. Then, if, say $AB = AC$, then the $A$-median and the $A$-internal angle bisector coincide; thus, the $A$-symmedian has to coincide with them.

Now, symmedians are concurrent from the trigonometric form of Ceva’s theorem, since we can just cancel out the sines. This concurrency point is called the symmedian point or the Lemoine point of triangle $ABC$, and it is usually denoted by $K$.

//As a matter of fact, we have the more general result.

Theorem -1. Let $P$ be a point in the plane of triangle $ABC$. Then, the reflections of the lines $AP, BP, CP$ in the angle bisectors of triangle $ABC$ are concurrent. This concurrency point is called the isogonal conjugate of the point $P$ with respect to the triangle $ABC$.

We won’t dwell much on this more general notion here; we just prove a very simple property that will lead us immediately to our first characterization of symmedians.
**Theorem 0** (Steiner’s Theorem). If $D$ is a point on the sideline $BC$ of triangle $ABC$, and if the reflection of the line $AD$ in the internal angle bisector of the angle $A$ intersects the line $BC$ at a point $E$, then

$$\frac{BD}{CD} \cdot \frac{BE}{CE} = \frac{AB^2}{AC^2}$$

![Figure 2: Theorem 0](image)

**Proof.** From the Ratio Lemma, we write

$$\frac{DB}{DC} = \frac{AB}{AC} \cdot \frac{\sin DAB}{\sin DAC} \quad \text{and} \quad \frac{EB}{EC} = \frac{AB}{AC} \cdot \frac{\sin EAB}{EAC}.$$  

Thus, keeping in mind that $\angle DAB = \angle EAC$ and $\angle DAC = \angle EAB$, by multiplying, we obtain that

$$\frac{BD}{CD} \cdot \frac{BE}{CE} = \frac{AB^2}{AC^2},$$

as claimed.

As a corollary, we thus get the following result.

**Characterization 1.** In a triangle $ABC$ with $X$ on the side $BC$, we have that

$$\frac{XB}{XC} = \frac{AB^2}{AC^2}$$

if and only if $AX$ is the $A$-symmedian of triangle $ABC$.

This represents what is perhaps the most important characterization of the $A$-symmedian of the triangle and all the results that we will prove next will return to this, more or less.

**Characterization 2.** Let $ABC$ be a triangle and let $X$ be a point on the side $BC$. Obviously, for any point $P$ on the line $AX$, we have that

$$\frac{\delta(P, AB)}{\delta(P, AC)} = \frac{\delta(X, AB)}{\delta(X, AC)}$$

In other words, the ratio of the distances from $P$ to the sides is independent of the point $P$ chosen on $AX$. Now, the claim is the following. For any point $P$ on $AX$, we have that

$$\frac{\delta(P, AB)}{\delta(P, AC)} = \frac{\delta(X, AB)}{\delta(X, AC)} = \frac{AB}{AC}$$

if and only if $AX$ is the $A$-symmedian of triangle $ABC$. 
Proof. By Characterization 1, we know that $AX$ is the $A$-symmedian if and only if

$$\frac{XB}{XC} = \frac{AB^2}{AC^2}.$$  

Hence, by now using the Ratio Lemma, we get that $AX$ is the $A$-symmedian and only if

$$\frac{\sin XAB}{\sin XAC} = \frac{AB}{AC}.$$  

But for any point $P$ on $AX$, we have that

$$\frac{\delta(P, AB)}{\delta(P, AC)} = \frac{\delta(X, AB)}{\delta(X, AC)} = \frac{\sin XAB}{\sin XAC}.$$  

Hence, we immediately obtain the conclusion that

$$\frac{\delta(P, AB)}{\delta(P, AC)} = \frac{\delta(X, AB)}{\delta(X, AC)} = \frac{AB}{AC}$$  

if and only if $AX$ is the $A$-symmedian of triangle $ABC$. \hfill \Box
Characterization 3. Let $ABC$ be a triangle and let $ACUV$ and $ABST$ be the squares constructed on the sides which are directed towards the exterior of the triangle. Let $X$ be the circumcenter of triangle $ATV$. Then, the line $AX$ is the $A$-symmedian of triangle $ABC$.

Proof. The point $X$, being the circumcenter of triangle $ATV$, lies on the line bisectors of segments $AV$ and $AT$. Hence, we have that $\delta(X, AB) = \frac{1}{2}AT$ and $\delta(X, AC) = \frac{1}{2}AV$; thus, we get that

$$\frac{\delta(X, AB)}{\delta(X, AC)} = \frac{AT}{AV} = \frac{AB}{AC},$$

and so by Characterization 2, we get that $AX$ is the $A$-symmedian of triangle $ABC$. \hfill \Box

Characterization 4 (BMO 2009). Let $MN$ be a line parallel to the side $BC$ of a triangle $ABC$, with $M$ on the side $AB$ and $N$ on the side $AC$. The lines $BN$ and $CM$ meet at point $P$. The circumcircles of triangles $BMP$ and $CNP$ meet at two distinct points $P$ and $Q$. Then, the line $AQ$ is the $A$-symmedian of triangle $ABC$.

![Figure 5: Characterization 4](image)

Proof. Note that $\angle BQM = \angle BPM = \angle CPN = \angle CQN$ and $\angle MBQ = \angle CPQ = \angle CNQ$; thus triangles $BQM$ and $NQC$ are similar. So, we get that

$$\frac{\delta(Q, AB)}{\delta(Q, AC)} = \frac{\delta(Q, MB)}{\delta(Q, NC)} = \frac{BM}{CN} = \frac{AB}{AC},$$

hence $AQ$ is the $A$-symmedian of $ABC$, by Characterization 2. \hfill \Box

Characterization 5 (Lemoine’s Pedal Triangle Theorem). The symmedian point $K$ of triangle $ABC$ is the only point in the plane of $ABC$ which is the centroid of its own pedal triangle.

Proof. For the direct implication, let $D, E, F$ be the projections of $K$ on the sides $BC$, $CA$, $AB$ and take $X$ to be the intersection of $DK$ with $EF$. We would like to show that $X$ is the midpoint of $EF$, since after that we could just repeat the argument for $EY$ and $FZ$ and conclude that $K$ is the centroid of $DEF$. 
By the Ratio Lemma, we know that
\[ \frac{XE}{XF} = \frac{KE}{KF} \cdot \frac{\sin XKE}{\sin XKF}. \]
However, $K$ obviously lies on the $A$-symmedian, thus by Characterization 2,
\[ \frac{KE}{KF} = \frac{\delta(K, AC)}{\delta(K, AB)} = \frac{AC}{AB}. \]

Furthermore, $\angle XKE = \angle C$ and $\angle XKF = \angle B$ since the quadrilaterals $KDC E$ and $KFBD$ are cyclic; thus, we conclude that
\[ \frac{XE}{XF} = \frac{AC}{AB} \cdot \frac{\sin C}{\sin B} = \frac{AC}{AB} \cdot \frac{AB}{AC} = 1. \]
This proves that $X$ is the midpoint of $EF$ and settles the direct implication. As for the converse, things are essentially similar. Now, we know that $K$ is a point having projections $D, E, F$ so that
\[ 1 = \frac{XE}{XF} = \frac{KE}{KF} \cdot \frac{\sin XKE}{\sin XKF}. \]

The equalities $\angle XKE = \angle C$ and $\angle XKF = \angle B$ coming from the cyclic quadrilaterals $KDC E$ and $KFBD$ are independent of $K$ being the symmedian point; thus, we immediately get that
\[ \frac{KE}{KF} = \frac{AB}{AC}. \]

Hence, by Characterization 2, we conclude that $K$ needs to lie on the $A$-symmedian, and similarly we can do that for the vertices $B$ and $C$; thus we get that $K$ is the symmedian point of the triangle. This completes the proof. \qed

**Characterization 6.** Let the tangents at vertices $B$ and $C$ of triangle $ABC$ to the circumcircle meet at a point $X$. Then, the line $AX$ is the $A$-symmedian of triangle $ABC$.

**Proof.** Let $T$ be the intersection of $AX$ with the side $BC$. Since this point lies in the interior of the segment $BC$, we notice again that, by Characterization 1, it is enough to show that $\frac{TB}{TC} = \frac{AB^2}{AC^2}$. In this case, the line $AX$ would represent the $A$-symmedian and we
would be done. So, let’s prove that \( \frac{TB}{TC} = \frac{AB^2}{AC^2} \). This is where the Ratio Lemma comes in.

We have that

\[
\frac{TB}{TC} = \frac{XB}{XC} \cdot \frac{\sin TXB}{\sin TXC},
\]

but \( XB = XC \) as they are both tangents from the same point to the circumcircle of \( ABC \); hence \( \frac{TB}{TC} = \frac{\sin TXB}{\sin TXC} \).

Now, we apply the Law of Sines twice, in triangles \( XAB \) and \( XAC \). We get that

\[
\frac{AB}{\sin TXB} = \frac{AX}{\sin XBA} = \frac{AX}{\sin(B + XBC)}
\]

and

\[
\frac{AC}{\sin TXC} = \frac{AX}{\sin XCA} = \frac{AX}{\sin(C + XCB)}.
\]

But \( \angle XBC = \angle XCB = \angle A \), since the lines \( XB \) and \( XC \) are both tangent to the circumcircle of \( ABC \). Hence, it follows that

\[
\frac{AB}{\sin TXB} = \frac{AX}{\sin C} \quad \text{and} \quad \frac{AC}{\sin TXC} = \frac{AX}{\sin B}.
\]

Therefore, by dividing the two relations, we conclude that

\[
\frac{TB}{TC} = \frac{\sin TXB}{\sin TXC} = \frac{AB}{AC} \cdot \frac{\sin C}{\sin B} = \frac{AB^2}{AC^2},
\]

where the last equality holds because of the Law of Sines applied in triangle \( ABC \). This completes the proof. \( \Box \)
Corollary 6'. If $D$, $E$, $F$ denote the tangency points of the incircle with the sides $BC$, $CA$, $AB$ of triangle $ABC$, then the lines $DA$, $EB$, $FC$ are the symmedians of triangle $DEF$.

Characterization 7. Suppose $X$ is a point of the circumcircle of $ABC$, different from the vertex $A$, such that $\frac{XB}{XC} = \frac{AB}{AC}$. Then, the line $AX$ is the $A$-symmedian of triangle $ABC$.

![Figure 8: Characterization 7](image)

Note that this means nothing but that the $A$-symmedian is the radical axis of the circumcircle of $ABC$ and the $A$-Appolonius circle. This leads to a series of nice observations involving the Appolonius circles.

Proof. Let $T$ be the intersection of $AX$ with $BC$. Again, we hope to show that $\frac{TB}{TC} = \frac{AB^2}{AC^2}$, so that we can use Characterization 1. Well, the Ratio Lemma gives us that

$$\frac{TB}{TC} = \frac{XB}{XC} \cdot \frac{\sin AXB}{\sin AXC} = \frac{AB}{AC} \cdot \frac{\sin AXB}{\sin AXC}.$$ 

But $\angle AXB = \angle C$ and $\angle AXC = \angle B$; hence, it follows that $\frac{TB}{TC} = \frac{AB^2}{AC^2}$, as desired.

Characterization 8. The $A$-symmedian is the locus of the midpoints of the antiparallels to $BC$ bounded by the lines $AB$ and $AC$.

Proof. Let $YZ$ be an antiparallel to the line $BC$ with $Y$ on $AB$ and $Z$ on $AC$ and let $M$ be the midpoint of $YZ$. It suffices to show that $AM$ is the $A$-symmedian. Again, we use the same idea as above! Let $X$ be the intersection of $AM$ with $BC$ and let’s try to prove that $\frac{XB}{XC} = \frac{AB^2}{AC^2}$. According to Characterization 1, this will mean that $AX$ and thus $AM$ is the $A$-symmedian of $ABC$, as we desire.
We use again the Ratio Lemma. More precisely, we have that
\[
\frac{XB}{XC} = \frac{AB}{AC} \cdot \frac{\sin XAB}{\sin XAC} = \frac{AB}{AC} \cdot \frac{\sin MAY}{\sin MAZ}.
\]
And from the way we wrote the angles $\angle XAB$ and $\angle XAC$ in the last term, we already know what’s the next step. The Ratio Lemma applied again, only this time in triangle $AYZ$, gives us
\[
1 = \frac{MY}{MZ} = \frac{AY}{AZ} \cdot \frac{\sin MAY}{\sin MAZ},
\]
hence
\[
\frac{\sin MAY}{\sin MAZ} = \frac{AZ}{AY} = \frac{AB}{AC},
\]
where the last equality holds because of the similarity of triangles $ABC$ and $AZY$ - remember that $YZ$ is antiparallel to $BC$. Thus, we conclude that
\[
\frac{XB}{XC} = \frac{AB^2}{AC^2},
\]
which completes the proof. 

This also admits a nice converse which we will use. More precisely, if some segment bounded by the lines $AB$ and $AC$ is bisected by the $A$-symmedian, then it has to be antiparallel with the line $BC$.

Now, some very nice applications of these due to Lemoine.

**Theorem 9** (The First Lemoine Circle). Let $K$ be the symmedian point of triangle $ABC$ and let $x, y, z$ be the antiparallels drawn through $K$ to the lines $BC, CA$, and $AB$, respectively. Prove that the six points determined by $x, y, z$ on the sides of $ABC$ all lie on one circle. This is called the First Lemoine Circle of triangle $ABC$.

**Proof.** Let $X_b, X_c$ be the intersections of $x$ with $CA, AB$, respectively. Similarly, let $Y_c, Y_a$ be the intersections of $y$ with $AB, BC$, and $Z_a, Z_b$ the intersections of $z$ with $BC, CA$. By Characterization 8, we know that $KX_b = KX_c$, $KY_c = KY_a$, $KZ_a = KZ_b$. Moreover, since $y, z$ are antiparallels, we have that $\angle KZ_aY_a = \angle KY_aZ_a = \angle A$, thus triangle $KY_aZ_a$ is isosceles, i.e. $KY_a = KZ_a$. Hence, $KY_a = KZ_a = KY_c = KZ_b$. Moreover, we can do the
same thing for triangles $KX_bZ_b$, $KY_cX_c$ to argue that they are isosceles, so we also have that $KX_b = KZ_b$ and $KY_c = KX_c$. Therefore, we conclude that

$$KZ_a = KY_a = KX_b = KZ_b = KY_c = KX_c,$$

so we get that all six points $X_b, X_c, Y_c, Y_a, Z_a, Z_b$ lie on one circle that is centered at $K$. This completes the proof.

Of course, given this name, you expect to have a second Lemoine circle. Indeed, this is the case!

**Theorem 10** (The Second Lemoine Circle). Let $K$ be the symmedian point of the triangle $ABC$ and let $x, y, z$ this time be the parallels drawn through $K$ to $BC$, $CA$, and $AB$, respectively. Prove that the six points determined by $x, y, z$ on the sides of $ABC$ all lie on one circle.

**Proof.** Let the line $x$ meet $AC$ and $AB$ at $X_b$ and $X_c$, $y$ meet $BC$, $BA$ at $Y_a, Y_c$, and $z$ meet $CA$, $CB$ at $Z_b, Z_a$. First, note that $AY_cKZ_b$ is a parallelogram, and thus the line $AK$ bisects the segment $Y_cZ_b$. However, $AK$ is the $A$-symmedian of triangle $ABC$; hence the line supporting the segment $Y_cZ_b$ needs to be antiparallel to $BC$, according to the converse we gave for Characterization 8. Thus, $\angle AZ_bY_c = \angle B = \angle Y_cX_cX_b$; hence, we get that $Y_c, X_c, X_b, Z_b$ all need to lie on one circle, say $\Gamma_1$. Similarly, the points $Y_c, X_c, Z_a, Y_a$ need...
to lie on one circle $\Gamma_2$, and the points $Z_a, Y_a, X_b, Z_b$ need to lie on one circle $\Gamma_3$. However, these three circles need to be the same, for otherwise, the radical axis of the pairs are not concurrent (since they are just the sidelines of the triangle!) and that’s impossible. Thus, $\Gamma_1 = \Gamma_2 = \Gamma_3$ and so all six points $X_b, X_c, Y_c, Y_a, Z_a, Z_b$ are concyclic. This completes the proof.

**Theorem 11.** Let $ABC$ be a triangle and let $M$ be the midpoint of $BC$ and $X$ be the midpoint of the $A$-altitude of $ABC$. Prove that the symmedian point of $ABC$ lies on the line $MX$.

Note that this means you can draw the symmedian point of $ABC$ as the intersection point of the lines determined by the midpoints of the sides and the midpoints of the altitudes.

**First Proof.** The locus of the centers of the rectangles inscribed in triangle $ABC$ and having one side on $BC$ is precisely the line $MX$! Why? Well, in the first place, it is a line. The reason goes as follows. First, let us see how these rectangles are obtained. Take a
rectangle $X_1X_2Y_1Z_1$ inscribed in $ABC$ with $X_1$, $X_2$ on $BC$. Erect the perpendiculars to $BC$ at the vertices $B$ and $C$ and intersect these perpendiculars with the lines $AX_1$, $AX_2$ at two points $X'_1$, $X'_2$. Then the rectangle $X_1X_2Y_1Z_1$ is the image of the rectangle $BCX'_2X'_1$ under a homothety with center $A$; thus since the locus of the centers of the rectangles $BCX'_2X'_1$ is the perpendicular bisector of $BC$ (and thus a line), it follows that the locus of the centers of rectangles $X_1X_2Y_1Z_1$ is also a line (the image of the perpendicular bisector under a certain rotation composed with a certain homothety) - the reader is encouraged to fill in the details.

Now, it is clear that the midpoint of $BC$ and the midpoint of the $A$-altitude belong to this line, since they are the centers of the two degenerate rectangles inscribed in $ABC$ with one side on $BC$; hence the locus is precisely the line $MX$. Now, why does $K$ lie on this line $MX$? Well, because $K$ is the center of a rectangle inscribed in $ABC$ which has one side on $BC$! Indeed, recall from the proof of Theorem 9 that $KZ_b = KY_c = KZ_a = KY_a$, so $Z_aY_aZ_bY_c$ is a rectangle inscribed in $ABC$ with $Z_aY_a$ on $BC$ with center $K$.

Second Proof. Let $D$, $E$, $F$ be the projections of the symmedian point $K$ on the sides $BC$, $CA$, $AB$, respectively. Let $D'$ be the reflection of $D$ in $K$, so $K$ is the midpoint of segment $DD'$. Since $DD'$ is parallel to the $A$-altitude, showing that $K$ lies on the line $MX$ is the same thing as showing that $D'$ lies on the $A$-median $AM$. But this is now rather straightforward to see. Notice that since $K$ is the centroid of triangle $DEF$ (Characterization 5), $FKED'$ is a parallelogram, as its diagonals bisect one another; thus $D'E \parallel KE$ and $D'F \parallel KE$. However, $KF$ and $KE$ are perpendicular to $AB$ and $AC$, respectively; hence $D'E \parallel AF$ and $D'F \parallel AE$; hence $D'$ is the orthocenter of triangle $AEF$; so $AD'$ is perpendicular to $EF$. Hence, $AD'$ needs to be the $A$-median. \[\square\]

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