Let's Talk About Symmedians!

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Abstract

We will introduce symmedians from scratch and prove an entire collection of interconnected results that characterize them.

Symmedians represent a very important topic in Olympiad Geometry since they have a lot of interesting properties that can be exploited in problems. But first, what are they?

Definition. In a triangle ABC, the reflection of the A-median in the A-internal angle bisector is called the A-symmedian of triangle ABC. Similarly, we can define the B-symmedian and the C-symmedian of the triangle.

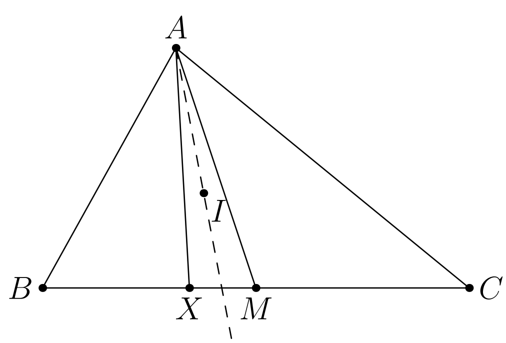

Figure 1: The A-symmedian AX

Do we always have symmedians? Well, yes, only that we have some weird cases when for example ABC is isosceles. Then, if, say $AB = AC$, then the A-median and the A-internal angle bisector coincide; thus, the A-symmedian has to coincide with them.

Now, symmedians are concurrent from the trigonometric form of Ceva's theorem, since we can just cancel out the sines. This concurrency point is called *the symmedian point* or *the Lemoine point* of triangle ABC, and it is usually denoted by K.

//As a matter of fact, we have the more general result.

Theorem -1. Let P be a point in the plane of triangle ABC. Then, the reflections of the lines AP, BP, CP in the angle bisectors of triangle ABC are concurrent. This concurrency point is called the *isogonal conjugate* of the point P with respect to the triangle ABC.

We won't dwell much on this more general notion here; we just prove a very simple property that will lead us immediately to our first characterization of symmedians.

Theorem 0 (Steiner's Theorem)**.** If D is a point on the sideline BC of triangle ABC, and if the reflection of the line AD in the internal angle bisector of the angle A intersects the line BC at a point E, then

$$\frac{BD}{CD} \cdot \frac{BE}{CE} = \frac{AB^2}{AC^2}$$

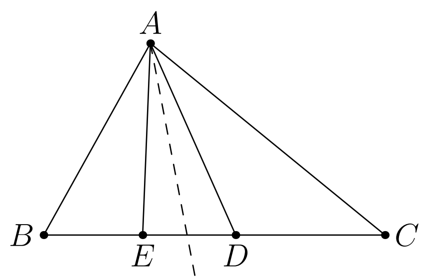

Figure 2: Theorem 0

Proof. From the Ratio Lemma, we write

$$\frac{DB}{DC} = \frac{AB}{AC} \cdot \frac{\sin DAB}{\sin DAC} \text{ and } \frac{EB}{EC} = \frac{AB}{AC} \cdot \frac{\sin EAB}{\sin EAC}.$$

Thus, keeping in mind that $\angle DAB = \angle EAC$ and $\angle DAC = \angle EAB$, by multiplying, we obtain that

$$\frac{BD}{CD} \cdot \frac{BE}{CE} = \frac{AB^2}{AC^2},$$

as claimed. □

As a corollary, we thus get the following result.

Characterization 1. In a triangle ABC with X on the side BC, we have that

$$\frac{XB}{XC} = \frac{AB^2}{AC^2}$$

if and only if AX is the A-symmedian of triangle ABC.

This represents what is perhaps the most important characterization of the A-symmedian of the triangle and all the results that we will prove next will return to this, more or less.

Characterization 2. Let ABC be a triangle and let X be a point on the side BC. Obviously, for any point P on the line AX, we have that

$$\frac{\delta(P, AB)}{\delta(P, AC)} = \frac{\delta(X, AB)}{\delta(X, AC)}$$

In other words, the ratio of the distances from P to the sides is independent of the point P chosen on AX. Now, the claim is the following. For any point P on AX, we have that

$$\frac{\delta(P, AB)}{\delta(P, AC)} = \frac{\delta(X, AB)}{\delta(X, AC)} = \frac{AB}{AC}$$

if and only if AX is the A-symmedian of triangle ABC.

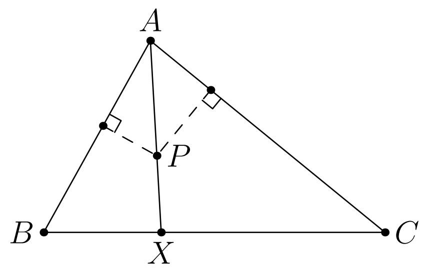

Figure 3: Characterization 2

Proof. By Characterization 1, we know that AX is the A-symmedian if and only if

$$\frac{XB}{XC} = \frac{AB^2}{AC^2}.$$

Hence, by now using the Ratio Lemma, we get that AX is the A-symmedian and only if

$$\frac{\sin XAB}{\sin XAC} = \frac{AB}{AC}.$$

But for any point P on AX, we have that

$$\frac{\delta(P, AB)}{\delta(P, AC)} = \frac{\delta(X, AB)}{\delta(X, AC)} = \frac{\sin XAB}{\sin XAC}.$$

Hence, we immediately obtain the conclusion that

$$\frac{\delta(P, AB)}{\delta(P, AC)} = \frac{\delta(X, AB)}{\delta(X, AC)} = \frac{AB}{AC}$$

if and only if AX is the A-symmedian of triangle ABC. $\square$

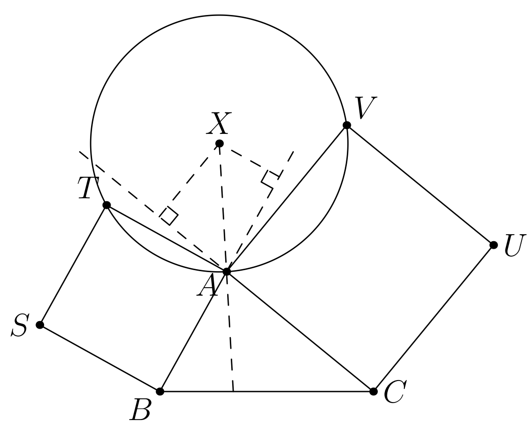

Figure 4: Characterization 3

Characterization 3. Let ABC be a triangle and let $ACUV$ and $ABST$ be the squares constructed on the sides which are directed towards the exterior of the triangle. Let X be the circumcenter of triangle ATV. Then, the line AX is the A-symmedian of triangle ABC.

Proof. The point X, being the circumcenter of triangle ATV, lies on the line bisectors of segments AV and AT. Hence, we have that $\delta(X, AB) = \frac{1}{2}AT$ and $\delta(X, AC) = \frac{1}{2}AV$; thus, we get that

$$\frac{\delta(X, AB)}{\delta(X, AC)} = \frac{AT}{AV} = \frac{AB}{AC},$$

and so by Characterization 2, we get that AX is the A-symmedian of triangle ABC. □

Characterization 4 (BMO 2009)**.** Let MN be a line parallel to the side BC of a triangle ABC, with M on the side AB and N on the side AC. The lines BN and CM meet at point P. The circumcircles of triangles BMP and CNP meet at two distinct points P and Q. Then, the line AQ is the A-symmedian of triangle ABC.

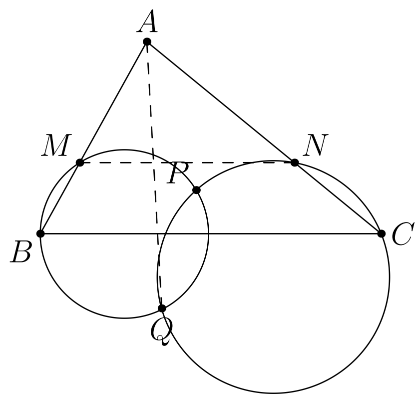

Figure 5: Characterization 4

Proof. Note that $\angle BQM = \angle BPM = \angle CPN = \angle CQN$ and $\angle MBQ = \angle CPQ = \angle CNQ$; thus triangles BQM and NQC are similar. So, we get that

$$\frac{\delta(Q, AB)}{\delta(Q, AC)} = \frac{\delta(Q, MB)}{\delta(Q, NC)} = \frac{BM}{CN} = \frac{AB}{AC};$$

hence AQ is the A-symmedian of ABC, by Characterization 2. □

Characterization 5 (Lemoine's Pedal Triangle Theorem)**.** The symmedian point K of triangle ABC is the only point in the plane of ABC which is the centroid of its own pedal triangle.

Proof. For the direct implication, let D, E, F be the projections of K on the sides BC, CA, AB and take X to be the intersection of DK with EF. We would like to show that X is the midpoint of EF, since after that we could just repeat the argument for EY and FZ and conclude that K is the centroid of DEF.

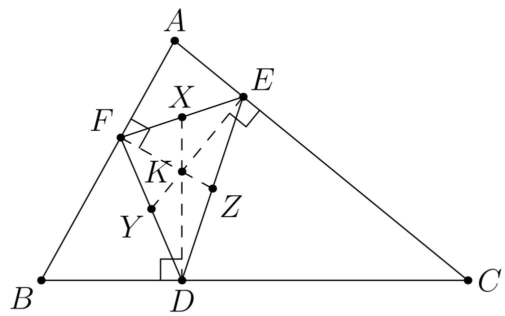

Figure 6: Characterization 5

By the Ratio Lemma, we know that

$$\frac{XE}{XF} = \frac{KE}{KF} \cdot \frac{\sin XKE}{\sin XKF}.$$

However, K obviously lies on the A-symmedian, thus by Characterization 2,

$$\frac{KE}{KF} = \frac{\delta(K, AC)}{\delta(K, AB)} = \frac{AC}{AB}.$$

Furthermore, $\angle XKE = \angle C$ and $\angle XKF = \angle B$ since the quadrilaterals $KDCE$ and $KFBD$ are cyclic; thus, we conclude that

$$\frac{XE}{XF} = \frac{AC}{AB} \cdot \frac{\sin C}{\sin B} = \frac{AC}{AB} \cdot \frac{AB}{AC} = 1.$$

This proves that X is the midpoint of EF and settles the direct implication. As for the converse, things are essentially similar. Now, we know that K is a point having projections D, E, F so that

$$1 = \frac{XE}{XF} = \frac{KE}{KF} \cdot \frac{\sin XKE}{\sin XKF}.$$

The equalities $\angle XKE = \angle C$ and $\angle XKF = \angle B$ coming from the cyclic quadrilaterals $KDCE$ and $KFBD$ are independent of K being the symmedian point; thus, we immediately get that

$$\frac{KE}{KF} = \frac{AB}{AC}.$$

Hence, by Characterization 2, we conclude that K needs to lie on the A-symmedian, and similarly we can do that for the vertices B and C; thus we get that K is the symmedian point of the triangle. This completes the proof. □

Characterization 6. Let the tangents at vertices B and C of triangle ABC to the circumcircle meet at a point X. Then, the line AX is the A-symmedian of triangle ABC.

Proof. Let T be the intersection of AX with the side BC. Since this point lies in the interior of the segment BC, we notice again that, by Characterization 1, it is enough to show that $\frac{TB}{TC} = \frac{AB^2}{AC^2}$. In this case, the line AX would represent the A-symmedian and we

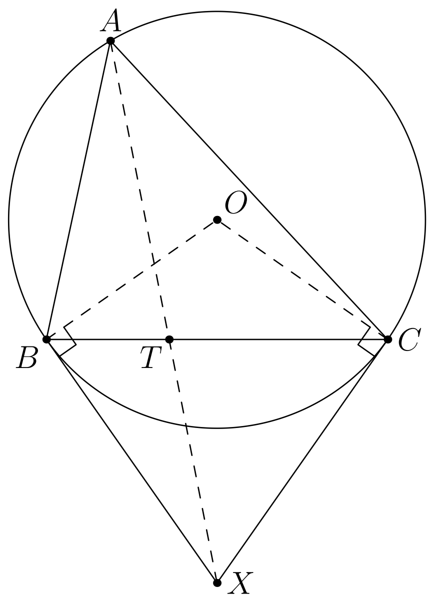

Figure 7: Characterization 6

would be done. So, let's prove that $\frac{TB}{TC} = \frac{AB^2}{AC^2}$. This is where the Ratio Lemma comes in. We have that

$$\frac{TB}{TC} = \frac{XB}{XC} \cdot \frac{\sin TXB}{\sin TXC};$$

but $XB = XC$ as they are both tangents from the same point to the circumcircle of ABC; hence $\frac{TB}{TC} = \frac{\sin TXB}{\sin TXC}$.

Now, we apply the Law of Sines twice, in triangles XAB and XAC. We get that

$$\frac{AB}{\sin TXB} = \frac{AX}{\sin XBA} = \frac{AX}{\sin(B + XBC)}$$

and

$$\frac{AC}{\sin TXC} = \frac{AX}{\sin XCA} = \frac{AX}{\sin(C + XCB)}.$$

But $\angle XBC = \angle XCB = \angle A$, since the lines XB and XC are both tangent to the circumcircle of ABC. Hence, it follows that

$$\frac{AB}{\sin TXB} = \frac{AX}{\sin C} \text{ and } \frac{AC}{\sin TXC} = \frac{AX}{\sin B}.$$

Therefore, by dividing the two relations, we conclude that

$$\frac{TB}{TC} = \frac{\sin TXB}{\sin TXC} = \frac{AB}{AC} \cdot \frac{\sin C}{\sin B} = \frac{AB^2}{AC^2},$$

where the last equality holds because of the Law of Sines applied in triangle ABC. This completes the proof. □

Corollary 6'. If D, E, F denote the tangency points of the incircle with the sides BC, CA, AB of triangle ABC, then the lines DA, EB, FC are the symmedians of triangle DEF.

Characterization 7. Suppose X is a point of the circumcircle of ABC, different from the vertex A, such that $\frac{XB}{XC} = \frac{AB}{AC}$. Then, the line AX is the A-symmedian of triangle ABC.

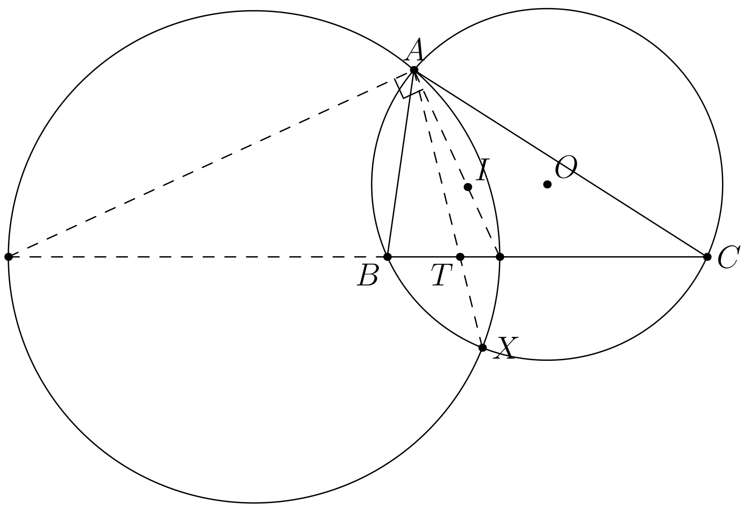

Figure 8: Characterization 7

Note that this means nothing but that the A-symmedian is the radical axis of the circumcircle of ABC and the A-Appolonius circle. This leads to a series of nice observations involving the Appolonius circles.

Proof. Let T be the intersection of AX with BC. Again, we hope to show that $\frac{TB}{TC} = \frac{AB^2}{AC^2}$, so that we can use Characterization 1. Well, the Ratio Lemma gives us that

$$\frac{TB}{TC} = \frac{XB}{XC} \cdot \frac{\sin AXB}{\sin AXC} = \frac{AB}{AC} \cdot \frac{\sin AXB}{\sin AXC}.$$

But $\angle AXB = \angle C$ and $\angle AXC = \angle B$; hence, it follows that $\frac{TB}{TC} = \frac{AB^2}{AC^2}$, as desired. $\square$

Characterization 8. The A-symmedian is the locus of the midpoints of the antiparallels to BC bounded by the lines AB and AC.

Proof. Let YZ be an antiparallel to the line BC with Y on AB and Z on AC and let M be the midpoint of YZ. It suffices to show that AM is the A-symmedian. Again, we use the same idea as above! Let X be the intersection of AM with BC and let's try to prove that $\frac{XB}{XC} = \frac{AB^2}{AC^2}$. According to Characterization 1, this will mean that AX and thus AM is the A-symmedian of ABC, as we desire.

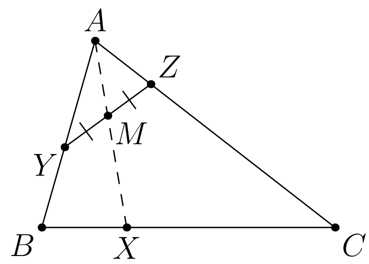

Figure 9: Characterization 8

We use again the Ratio Lemma. More precisely, we have that

$$\frac{XB}{XC} = \frac{AB}{AC} \cdot \frac{\sin XAB}{\sin XAC} = \frac{AB}{AC} \cdot \frac{\sin MAY}{\sin MAZ}.$$

And from the way we wrote the angles $\angle XAB$ and $\angle XAC$ in the last term, we already know what's the next step. The Ratio Lemma applied again, only this time in triangle AYZ, gives us

$$1 = \frac{MY}{MZ} = \frac{AY}{AZ} \cdot \frac{\sin MAY}{\sin MAZ},$$

hence

$$\frac{\sin MAY}{\sin MAZ} = \frac{AZ}{AY} = \frac{AB}{AC},$$

where the last equality holds because of the similarity of triangles ABC and AZY - remember that YZ is antiparallel to BC. Thus, we conclude that

$$\frac{XB}{XC} = \frac{AB^2}{AC^2},$$

which completes the proof. □

This also admits a nice converse which we will use. More precisely, if some segment bounded by the lines AB and AC is bisected by the A-symmedian, then it has to be antiparallel with the line BC.

Now, some very nice applications of these due to Lemoine.

Theorem 9 (The First Lemoine Circle)**.** Let K be the symmedian point of triangle ABC and let x, y, z be the antiparallels drawn through K to the lines BC, CA, and AB, respectively. Prove that the six points determined by x, y, z on the sides of ABC all lie on one circle. This is called the *First Lemoine Circle* of triangle ABC.

Proof. Let X_b, X_c be the intersections of x with CA, AB, respectively. Similarly, let Y_c, Y_a be the intersections of y with AB, BC, and Z_a, Z_b the intersections of z with BC, CA. By Characterization 8, we know that $KX_b = KX_c, KY_c = KY_a, KZ_a = KZ_b$. Moreover, since y, z are antiparallels, we have that $\angle KZ_aY_a = \angle KY_aZ_a = \angle A$, thus triangle KY_aZ_a is isosceles, i.e. $KY_a = KZ_a$. Hence, $KY_a = KZ_a = KY_c = KZ_b$. Moreover, we can do the

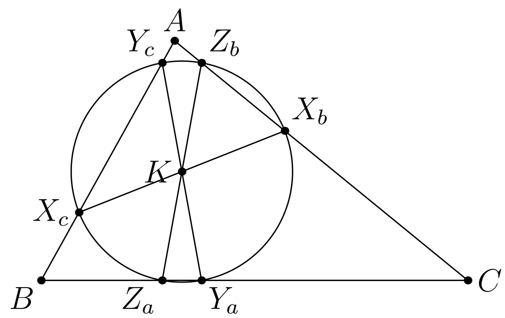

Figure 10: Theorem 9

same thing for triangles KX_bZ_b, KY_cX_c to argue that they are isosceles, so we also have that $KX_b = KZ_b$ and $KY_c = KX_c$. Therefore, we conclude that

$$KZ_a = KY_a = KX_b = KZ_b = KY_c = KX_c,$$

so we get that all six points $X_b, X_c, Y_c, Y_a, Z_a, Z_b$ lie on one circle that is centered at K. This completes the proof. □

Of course, given this name, you expect to have a second Lemoine circle. Indeed, this is the case!

Theorem 10 (The Second Lemoine Circle)**.** Let K be the symmedian point of the triangle ABC and let x, y, z this time be the *parallels* drawn through K to BC, CA, and AB, respectively. Prove that the six points determined by x, y, z on the sides of ABC all lie on one circle.

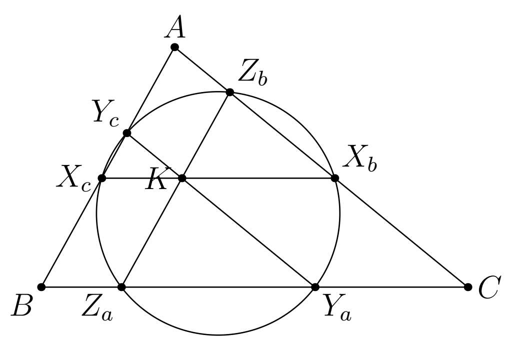

Figure 11: Theorem 10

Proof. Let the line x meet AC and AB at X_b and X_c, y meet BC, BA at Y_a, Y_c, and z meet CA, CB at Z_b, Z_a. First, note that AY_cKZ_b is a parallelogram, and thus the line AK bisects the segment Y_cZ_b. However, AK is the A-symmedian of triangle ABC; hence the line supporting the segment Y_cZ_b needs to be antiparallel to BC, according to the converse we gave for Characterization 8. Thus, $\angle AZ_bY_c = \angle B = \angle Y_cX_cX_b$; hence, we get that Y_c, X_c, X_b, Z_b all need to lie on one circle, say Γ_1. Similarly, the points Y_c, X_c, Z_a, Y_a need to lie on one circle Γ_2, and the points Z_a, Y_a, X_b, Z_b need to lie on one circle Γ_3. However, these three circles need to be the same, for otherwise, the radical axis of the pairs are not concurrent (since they are just the sidelines of the triangle!) and that's impossible. Thus, $\Gamma_1 = \Gamma_2 = \Gamma_3$ and so all six points $X_b, X_c, Y_c, Y_a, Z_a, Z_b$ are concyclic. This completes the proof. $\square$

Theorem 11. Let ABC be a triangle and let M be the midpoint of BC and X be the midpoint of the A-altitude of ABC. Prove that the symmedian point of ABC lies on the line MX.

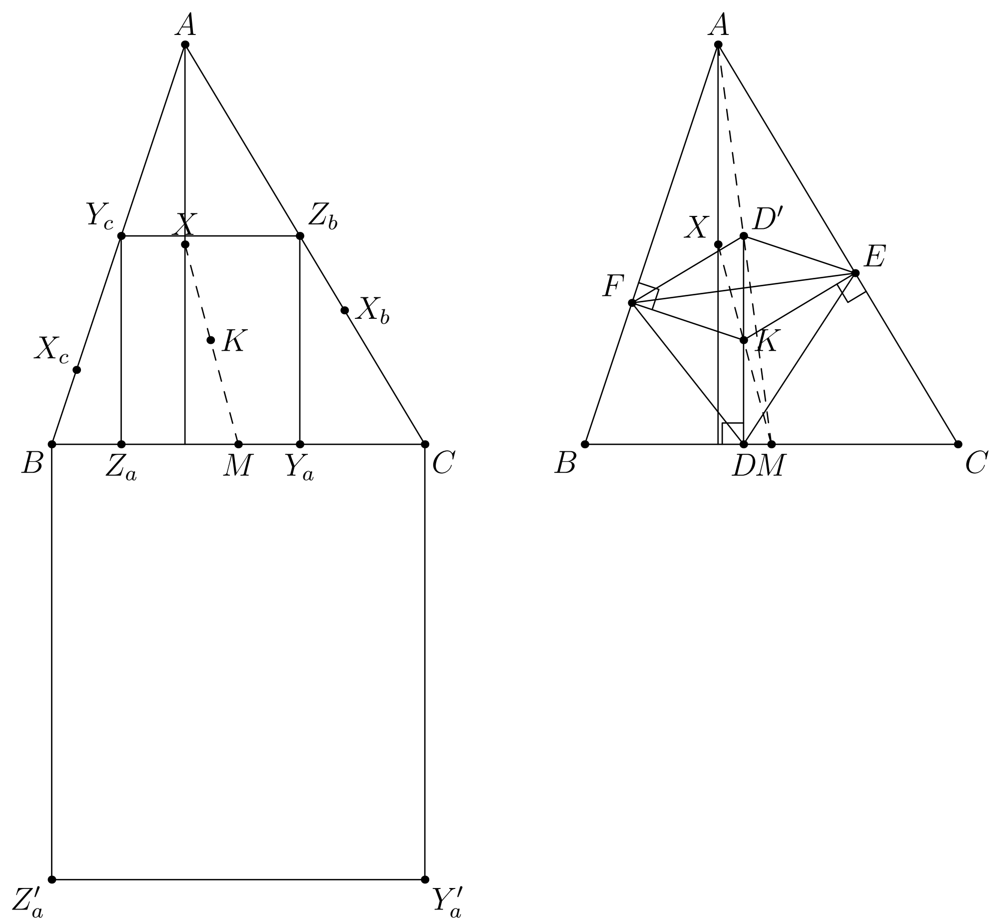

Figure 12: Two Proofs of Theorem 11

Note that this means you can draw the symmedian point of ABC as the intersection point of the lines determined by the midpoints of the sides and the midpoints of the altitudes.

First Proof. The locus of the centers of the rectangles inscribed in triangle ABC and having one side on BC is precisely the line MX! Why? Well, in the first place, it is a line. The reason goes as follows. First, let us see how these rectangles are obtained. Take a rectangle $X_1X_2Y_1Z_1$ inscribed in ABC with X_1, X_2 on BC. Erect the perpendiculars to BC at the vertices B and C and intersect these perpendiculars with the lines AX_1, AX_2 at two points X_1', X_2'. Then the rectangle $X_1X_2Y_1Z_1$ is the image of the rectangle $BCX_2'X_1'$ under a homothety with center A; thus since the locus of the centers of the rectangles $BCX_2'X_1'$ is the perpendicular bisector of BC (and thus a line), it follows that the locus of the centers of rectangles $X_1X_2Y_1Z_1$ is also a line (the image of the perpendicular bisector under a certain rotation composed with a certain homothety!) - the reader is encouraged to fill in the details. Now, it is clear that the midpoint of BC and the midpoint of the A-altitude belong to this line, since they are the centers of the two degenerate rectangles inscribed in ABC with one side on BC; hence the locus is precisely the line MX. Now, why does K lie on this line MX? Well, because K is the center of a rectangle inscribed in ABC which has one side on BC! Indeed, recall from the proof of Theorem 9 that $KZ_b = KY_c = KZ_a = KY_a$, so $Z_aY_aZ_bY_c$ is a rectangle inscribed in ABC with Z_aY_a on BC with center K. $\square$

Second Proof. Let D, E, F be the projections of the symmedian point K on the sides BC, CA, AB, respectively. Let D' be the reflection of D in K, so K is the midpoint of segment DD'. Since DD' is parallel to the A-altitude, showing that K lies on the line MX is the same thing as showing that D' lies on the A-median AM. But this is now rather straightforward to see. Notice that since K is the centroid of triangle DEF (Characterization 5), $FKED'$ is a parallelogram, as its diagonals bisect one another; thus $D'E \parallel KF$ and $D'F \parallel KE$. However, KF and KE are perpendicular to AB and AC, respectively; hence $D'E \parallel AF$ and $D'F \parallel AE$; hence D' is the orthocenter of triangle AEF; so AD' is perpendicular to EF. Hence, AD' needs to be the A-median. $\square$

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