Junior problems

J271. Find all positive integers n with the following property: if a, b, c are integers such that n divides $ab+bc+ca+1$, then n divides $abc(a+b+c+abc)$.

Proposed by Titu Andreescu, University of Texas at Dallas and Gabriel Dospinescu, Ecole Polytechnique, Lyon, France

Solution by the authors

We will prove that all positive divisors of 720 are solutions of the problem. If n is a solution, then by choosing $a=3, b=5, c=-2$ (for which $ab+bc+ca+1=0$) we obtain $n|720$. Conversely, let d be a divisor of 720 and suppose that d divides $ab+bc+ca+1$. We want to prove that d divides $abc(a+b+c+abc)$. We may assume that d is a power of a prime. The key ingredient is the following identity

$$(a^2-1)(b^2-1)(c^2-1) = (a+b+c+abc)^2 - (ab+bc+ca+1)^2.$$

This follows easily by multiplying the equalities

$$(a+1)(b+1)(c+1) = abc + a + b + c + ab + bc + ca + 1$$

and

$$(a-1)(b-1)(c-1) = abc + a + b + c - (ab+bc+ca+1).$$

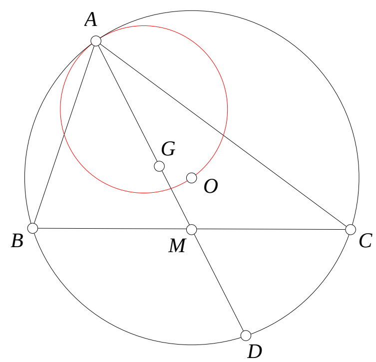
Define $f(a,b,c) = a^2b^2c^2(a^2-1)(b^2-1)(c^2-1)$. If at least two of the numbers a, b, c are odd, then $f(a,b,c)$ is a multiple of $4 \cdot 64 = 256$. Hence if 2^k divides $ab+bc+ca+1$ and $1 \le k \le 4$, then at least two of the numbers a, b, c are odd and the previous discussion yields $2^{2k}|a^2b^2c^2(a+b+c+abc)^2$, hence $2^k|abc(a+b+c+abc)$. A similar argument shows that if $3^k|ab+bc+ca+1$ and $1 \le k \le 2$, then 27 divides $f(a,b,c)$, hence 3^{2k-1} divides $(abc(a+b+c+abc))^2$ and then $3^k|abc(a+b+c+abc)$. Finally, suppose that 5 divides $ab+bc+ca+1$ and 5 does not divide $abc(a+b+c+abc)$. The previous identity shows that a, b, c are each congruent to 2 or 3 mod 5 and one can easily check that this contradicts the fact that 5 divides $ab+bc+ca+1$.

Also solved by Polyahedra, Polk State College, FL, USA.

J272. Let ABC be a triangle with centroid G and circumcenter O. Prove that if BC is its greatest side, then G lies in the interior of the circle of diameter AO.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Ercole Suppa, Teramo, Italy



Let $a = BC$, $b = CA$, $c = AB$, let M the midpoint of BC, let D be the second intersection point between AM and the circumcircle of $\triangle ABC$.

By the Power of Point theorem, we have $AM \cdot MD = BM \cdot MC$. Thus, we get

$$AD = AM + MD = m_a + \frac{\frac{a}{2} \cdot \frac{a}{2}}{m_a} = \frac{4m_a^2 + a^2}{4m_a} = \frac{2b^2 + 2c^2}{4m_a} = \frac{b^2 + c^2}{2m_a}$$

Now, taking into account that the circle of diameter AO is the locus of midpoints of chords of (O) that that pass through A, we have that G lies in the interior of the circle of diameter AO if and only if $AG < AD/2$ or equivalently

$$\frac{2}{3} m_a < \frac{1}{2} \cdot \frac{b^2 + c^2}{2m_a} \quad \Leftrightarrow \quad 8m_a^2 < 3\left(b^2 + c^2\right) \quad \Leftrightarrow \quad b^2 + c^2 < 2a^2$$

Since BC is the greatest side of $\triangle ABC$, the last inequality holds, so we are done.

Also solved by YoungSoo Kwon, St. Andrew's School, Delaware, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Polyahedra, Polk State College, FL, USA.

J273. Let a, b, c be real numbers greater than or equal to 1. Prove that

$$\frac{a^3+2}{b^2-b+1}+\frac{b^3+2}{c^2-c+1}+\frac{c^3+2}{a^2-a+1}\geq 9.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Note first that $a^3+2\geq 3(a^2-a+1)$ is equivalent to $a^3-3a^2+3a-1=(a-1)^3\geq 0$, clearly true by hypothesis and with equality iff $a=1$. Analogous inequalities hold for b, c. Therefore, application of the AM-GM produces

$$\frac{a^3+2}{b^2-b+1}+\frac{b^3+2}{c^2-c+1}+\frac{c^3+2}{a^2-a+1}\geq$$

$$\geq 3\sqrt[3]{\frac{a^3+2}{a^2-a+1}\cdot\frac{b^3+2}{b^2-b+1}\cdot\frac{c^3+2}{c^2-c+1}}\geq 3\sqrt[3]{3^3}=9.$$

The conclusion follows, and the necessary condition for equality $a=b=c=1$ is clearly also sufficient.

Also solved by Zarif Ibragimov, SamSU, Samarkand, Uzbekistan; Polyahedra, Polk State College, FL, USA; f; Arber Igrishta, Eqrem Qabej, Vushtrri, Kosovo; Arkady Alt , San Jose ,California, USA; Ercole Suppa, Teramo, Italy; Mathematical Group "Galaktika shqiptare", Albania; Harun Immanuel, ITS Surabaya; Jonathan Luke Lottes, The College at Brockport, State University of New York; Prithwijit De, HBCSE, Mumbai, India; Sayan Das, Indian Statistical Institute, Kolkata; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Shivang Jindal, Jaipur, India; Alessandro Ventullo, Milan, Italy; Ioan Viorel Codreanu, Satulung, Maramures, Romania.

J274. Let p be a prime and let k be a nonnegative integer. Find all positive integer solutions (x, y, z) to the equation

$$x^k(y-z)+y^k(z-x)+z^k(x-y)=p.$$

Proposed by Alessandro Ventullo, Milan, Italy

First solution by Polyahedra, Polk State College, USA

Let $A_k = x^k(y-z)+y^k(z-x)+z^k(x-y)$. Then for $k = 0, 1$, $A_k \equiv 0$, so no solution exists. Suppose that $k \geq 2$. Then

$$\begin{aligned} A_k &= (y-z)\left(x^k - y^k\right) + (x-y)\left(z^k - y^k\right) = (x-y)(y-z)\sum_{i=1}^{k-1}\left(x^i - z^i\right)y^{k-1-i} \\ &= (x-y)(y-z)(x-z)B_k, \end{aligned}$$

where

$$B_k = \sum_{i=1}^{k-1}\left(\sum_{j=0}^{i-1} x^j z^{i-1-j}\right) y^{k-1-i} = \sum_{i+j+l=k-2} x^i y^j z^l.$$

Now for $k \geq 3$, B_k is an integer greater than 2, so no solution to $A_k = p$ is possible. Finally, for $k = 2$, $A_k = (x-y)(y-z)(x-z) = p$ if and only if $p = 2$ and $(x, y, z) = (n+2, n+1, n)$, $(n, n+2, n+1)$, or $(n+1, n, n+2)$ for $n \geq 1$.

Second solution by Polyahedra, Polk State College, USA

Note that

$$A_k = x^k(y-z)+y^k(z-x)+z^k(x-y) = -\begin{vmatrix} x & x^k & 1 \\ y & y^k & 1 \\ z & z^k & 1 \end{vmatrix},$$

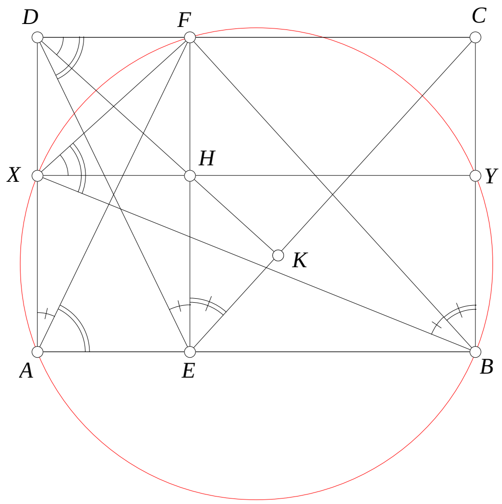
which is twice the area of $\triangle XYZ$, where $X = (x, x^k)$, $Y = (y, y^k)$, $Z = (z, z^k)$, and XYZ is in the clockwise orientation. Clearly, $A_k \equiv 0$ for $k = 0, 1$. Assume that $k \geq 2$ and $x > y > z$. By Pick's theorem, $A_k = 2(I_k - \frac{1}{2}B - 1)$, where I_k and B are the numbers of lattice points in the interior and on the boundary of $\triangle XYZ$, respectively. Since the slopes of XY, YZ, and ZX are integers, it is easy to see that $B = 2(x-z)$, thus $A_k = 2(I_k + x - z - 1)$. Therefore, for $A_k = p$, we must have $p = 2$, $x - z = 2$, and $I_k = 0$. Now $\frac{1}{2}\left[(z+2)^k + z^k\right] \geq (z+1)^k + 1$, with equality if and only if $k = 2$. So $W = (y, y^k + 1)$ is an interior point of $\triangle XYZ$ for $k \geq 3$. Finally, it is easy to see that when $k = 2$, the solutions are $(x, y, z) = (n+2, n+1, n)$, $(n, n+2, n+1)$, or $(n+1, n, n+2)$ for $n \geq 1$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain.

J275. Let $ABCD$ be a rectangle and let point E lie on side AB. The circle through A, B, and the orthogonal projection of E onto CD intersects AD and BC at X and Y. Prove that XY passes through the orthocenter of triangle CDE.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Ercole Suppa, Teramo, Italy



Let F be the orthogonal projection of E onto CD, let $H = EF \cap XY$, $K = DH \cap CE$.

Clearly $XY \parallel AB$ so $DXHF$ is cyclic. Thus a simple angle chasing gives

$$\begin{aligned} \angle EDH &= \angle EDC - \angle HDC = \angle FAB - \angle HDC = \\ &= \angle FXB - \angle FXY = \angle YXB \end{aligned} \tag{1}$$

From cyclic quadrilaterals $ABFX$ and $EBCF$ we get

$$\angle DEF = \angle DAF = \angle XAF = \angle XBF \tag{2}$$

$$\angle FEC = \angle FBC \tag{3}$$

By using (1),(2),(3) we obtain

$$\begin{aligned} \angle DKE &= 180^\circ - \angle EDK - \angle DEK = \\ &= 180^\circ - \angle EDH - \angle DEF - \angle FEC = \\ &= 180^\circ - \angle YXB - \angle XBF - \angle FBC = \\ &= 180^\circ - \angle YXB - \angle XBY = \\ &= \angle XYB = 90^\circ \end{aligned}$$

Therefore $DK \perp EC$. Since $EF \perp DC$ and $H = DK \cap EF$, it follows that H is the orthocenter of $\triangle CDE$, as we wanted to prove.

Second solution by Cosmin Pohoata, Princeton University, USA As in the previous solution, let F be the orthogonal projection of E onto CD and let H be the intersection of EF with XY. It is well-known that the reflections of the orthocenter of a triangle across is sidelines lie on the circumcircle of the triangle. Thus, H is the orthocenter of CDE if and onl only if $FC \cdot FD = FE \cdot FH$. But as before $XY \| AB$, so

$FC \cdot FD = XH \cdot HY$. On the other hand, if F' is the second intersection of FE with the circumcircle of FAB, the power of H with respect to (FAB) yields $F'H \cdot FH = XH \cdot XY$; hence, it follows that $FC \cdot FD = F'H \cdot FH$. However, $FE = F'H$, by symmetry, so, we get $FC \cdot FD = FE \cdot FH$, as desired.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; YoungSoo Kwon, St. Andrew's School, Delaware, USA; Polyahedra, Polk State College, USA.

J276. Find all positive integers m and n such that

$$10^n - 6^m = 4n^2.$$

Proposed by Tigran Akopyan, Vanadzor, Armenia

Solution by Alessandro Ventullo, Milan, Italy

It is easy to see that if $n = 1$, then $m = 1$ and $(1, 1)$ is a solution to the given equation. We prove that this is the only solution. Assume that $n > 1$ and n odd. Then, $10^n - 6^m \equiv -6^m \pmod 8$ and $4n^2 \equiv 4 \pmod 8$ and it is clear that $-6^m \equiv 4 \pmod 8$ if and only if $m = 2$. But $10^n > 36 + 4n^2$ for all positive integers $n > 1$, therefore there are no solutions when n is odd. Let n be an even number, i.e. $n = 2k$ for some $k \in \mathbb{Z}^+$. Hence,

$$(10^k - 4k)(10^k + 4k) = 6^m.$$

Since there are no solutions for $k = 1, 2$, let us assume that $k > 2$. Clearly $m \geq 4$ and simplifying by 2, we get

$$(2^{k-2} \cdot 5^k - k)(2^{k-2} \cdot 5^k + k) = 2^{m-4} \cdot 3^m. \quad (1)$$

If k is odd, then $m = 4$. But $2^{k-2} \cdot 5^k + k \geq 2 \cdot 5^3 + 3 > 3^4$, contradiction. Therefore, k must be even. Assume that $k = 2^\alpha h$, where $\alpha, h \in \mathbb{Z}^+$ and h is odd. If $\alpha \geq k-2$, then $2^{k-2} | k$, which implies that $2^{k-2} \leq k$. This gives $k = 3, 4$ and an easy check shows that there are no solutions for this values. If $\alpha < k - 2$, then equation (1) becomes

$$2^{2\alpha}(2^{k-2-\alpha} \cdot 5^k - h)(2^{k-2-\alpha} \cdot 5^k + h) = 2^{m-4} \cdot 3^m,$$

and by unique factorizaton we have $2\alpha = m - 4$. Therefore, from the inequality $k > \alpha + 2$, we obtain $n > 2\alpha + 4 = m$. But this implies that $10^n = 6^m + 4n^2 < 6^n + 4n^2$, which is false for all integers $n > 1$. Hence, there are no solutions for n even and the statement follows.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain;Polyahedra, Polk State College, USA; Arbër Avdullahu,Mehmet Akif College,Kosovo; David Xu; G. C. Greubel, Newport News, VA; Mathematical Group "Galaktika shqiptare", Albania; Harun Immanuel, ITS Surabaya; Toan Pham Quang, Dang Thai Mai Secondary School, Vinh, Vietnam; Tony Morse and Oiza Ochi, College at Brockport, State University of New York.

Senior problems

S271. Determine if there is an $n \times n$ square with all entries cubes of pairwise distinct positive integers such that the product of entries on each of the n rows, n columns, and two diagonals is 2013^{2013}.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Li Zhou, Polk State College, USA

Shown below is De Morgan's 11×11 additive magic square, where the sum of entries on each of the 11 rows, 11 columns, and two diagonals is $(1 + 2 + \cdots + 121)/11 = 671$.

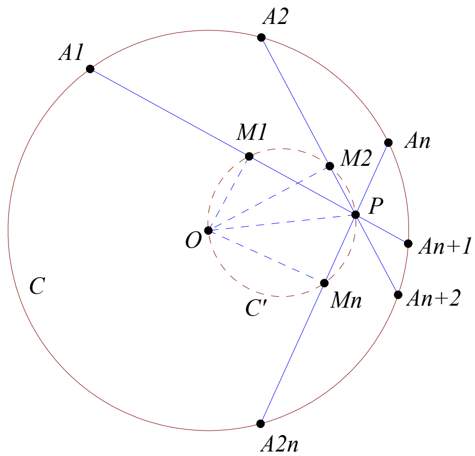
56	117	46	107	36	97	26	87	16	77	6
7	57	118	47	108	37	98	27	88	17	67
68	8	58	119	48	109	38	99	28	78	18
19	69	9	59	120	49	110	39	89	29	79
80	20	70	10	60	121	50	100	40	90	30
31	81	21	71	11	61	111	51	101	41	91
92	32	82	22	72	1	62	112	52	102	42
43	93	33	83	12	73	2	63	113	53	103
104	44	94	23	84	13	74	3	64	114	54
55	105	34	95	24	85	14	75	4	65	115
116	45	106	35	96	25	86	15	76	5	66

Raising 2013^3 to each entry of the the table, we obtain an 11×11 multiplicative magic square satisfying the requirements.

S272. Let $A_1, A_2, \ldots, A_{2n}$ be a polygon inscribed in a circle $C(O, R)$. Diagonals $A_1A_{n+1}, A_2A_{n+2}, \ldots, A_nA_{2n}$ intersect at point P. Let G be the centroid of the polygon. Prove that $\angle OPG$ is acute.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Li Zhou, Polk State College, USA



For $1 \le i \le n$, let M_i be the midpoint of A_iA_{n+i}. Then $OM_i \perp A_iA_{n+i}$, so the n-gon $M_1M_2\cdots M_n$ is circumscribed by the circle C' of diameter OP. Since G is also the centroid of $M_1M_2\cdots M_n$, it must be in the interior of C'. Hence, $\angle OPG$ is acute.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain.

S273. Let a, b, c be positive integers such that $a \geq b \geq c$ and $\frac{a-c}{2}$ is a prime. Prove that if

$$a^2 + b^2 + c^2 - 2(ab + bc + ca) = b,$$

then b is either a prime or a perfect square.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Stephanie Lash and Jessica Schuler, College at Brockport, State University of New York

We rewrite the equation as $(a-c)^2 = b(1 - b + 2a + 2c)$, from which we have immediately that $b|(a-c)^2$, i.e. $b|4\left(\frac{a-c}{2}\right)^2$.

We will consider two cases. First, suppose $\frac{a-c}{2} = 2$. This means that $b|16$ and so $b \in \{1, 2, 4, 8, 16\}$. To complete the problem in this case we need to show that $b \neq 8$. Suppose that $b = 8$. The rewritten equation becomes $16 = 8(1 - 8 + 2a + 2c)$ iff $2 = 2a + 2c - 7$, which is a contradiction because the left hand side is even while the right hand side is odd. Now, suppose that $\frac{a-c}{2}$ is an odd prime. Among other things this implies that $a - c \neq 0$ and it is possible to cancel it if needed. The divisors of $4\left(\frac{a-c}{2}\right)^2$ are

$$1, 2, 4, \frac{a-c}{2}, a-c, 2(a-c), \frac{(a-c)^2}{4}, \frac{(a-c)^2}{2}, (a-c)^2.$$

To complete the problem in this case we need to show that

$$b \notin \left\{a-c, 2(a-c), \frac{(a-c)^2}{2}\right\}.$$

Suppose that $b = a - c$. The rewritten equation becomes

$$(a-c)^2 = (a-c)(1 - a + c + 2a + 2c) \iff a - c = 1 + a + 3c \iff 0 = 1 + 3c,$$

a contradiction. Suppose that $b = 2(a-c)$. Then

$$(a-c)^2 = 2(a-c)(1 - 2a + 2c + 2a + 2c) \iff a - c = 2 + 8c \iff a = 2 + 9c.$$

In the same time, $b = 2(a-c) = 4 + 16c > 2 + 9c = a$, which is a contradiction. Finally, suppose that $b = \frac{(a-c)^2}{2}$. Then, since $a - c$ is even, b is also even and

$$(a-c)^2 = \frac{(a-c)^2}{2}(1 - b + 2a + 2c) \iff 1 = \frac{1}{2}(1 - b + 2a + 2b) \iff 1 = 2a + 2c - b$$

which is a contradiction because the left hand side is odd while the right hand side is even. This completes the proof.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Li Zhou, Polk State College, USA; Alessandro Ventullo, Milan, Italy.

S274. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{a}{ca+1} + \frac{b}{ab+1} + \frac{c}{bc+1} \le \frac{1}{2}\left(a^2+b^2+c^2\right).$$

Proposed by Sayan Das, Kolkata, India

Solution by Ercole Suppa, Teramo, Italy

We make the well-known substitution $a = \frac{x}{y}$, $b = \frac{y}{z}$ and $c = \frac{z}{x}$, where $x, y, z > 0$. The original inequality becomes:

$$\frac{2x}{y+z} + \frac{2y}{z+x} + \frac{2z}{x+y} \le \frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} \qquad (*)$$

According to AM-GM inequality we get

$$\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} = \frac{1}{2}\left(\frac{x^2}{y^2} + \frac{y^2}{z^2}\right) + \frac{1}{2}\left(\frac{y^2}{z^2} + \frac{z^2}{x^2}\right) + \frac{1}{2}\left(\frac{z^2}{x^2} + \frac{x^2}{y^2}\right) \ge \frac{x}{z} + \frac{y}{x} + \frac{z}{y} \qquad (1)$$

From AM-GM and Cauchy Schwarz inequalities we obtain

$$\begin{aligned}\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} &= \sqrt{\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2}} \cdot \sqrt{\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2}} \\ &\ge \sqrt{3} \cdot \sqrt{\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2}} \ge \frac{x}{y} + \frac{y}{z} + \frac{z}{x} \qquad (2)\end{aligned}$$

Summing up the inequalities (1),(2) we deduce that

$$\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} \ge \frac{1}{2}\left(\frac{x}{y} + \frac{x}{z}\right) + \frac{1}{2}\left(\frac{y}{x} + \frac{y}{z}\right) + \frac{1}{2}\left(\frac{z}{x} + \frac{z}{y}\right) \qquad (3)$$

Applying the AM-HM inequality for two numbers, we obtain

$$\begin{aligned}\frac{1}{2}\left(\frac{x}{y} + \frac{x}{z}\right) + \frac{1}{2}\left(\frac{y}{x} + \frac{y}{z}\right) + \frac{1}{2}\left(\frac{z}{x} + \frac{z}{y}\right) &\ge \frac{2}{\frac{y+z}{x}} + \frac{2}{\frac{x+z}{y}} + \frac{2}{\frac{x+y}{z}} = \\ &= \frac{2x}{y+z} + \frac{2y}{x+z} + \frac{2z}{x+y} \qquad (4)\end{aligned}$$

Finally, from (3) and (4), we get (*) which is exactly the desired result.

Also solved by Georgios Batzolis, Mandoulides High School, Thessaloniki, Greece; Daniel Lasaosa, Universidad Pública de Navarra, Spain; AN- anduud Problem Solving Group, Ulaanbaatar, Mongolia; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Li Zhou, Polk State College, USA; Arkady Alt , San Jose ,California, USA; Mathematical Group "Galaktika shqiptare", Albania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Zarif Ibragimov, SamSU, Samarkand, Uzbekistan.

S275. Let ABC be a triangle with incircle $\mathcal{I}$ and incenter I. Let A', B', C' be the intersections of $\mathcal{I}$ with the segments AI, BI, CI, respectively. Prove that

$$\frac{AB}{A'B'}+\frac{BC}{B'C'}+\frac{CA}{C'A'}\geq 12-4\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right).$$

Proposed by Marius Stanean, Zalau, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain
Lemma:
Clearly,

$$\angle A'IB'=\angle AIB=180^\circ-\frac{A+B}{2}=90^\circ+\frac{C}{2},$$

or $\angle A'C'B'=45^\circ+\frac{C}{4}$, and applying the Sine Law, and that $r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$, where r, R are respectively the inradius and circunradius of ABC, we find

$$\frac{A'B'}{AB}=\frac{r}{R}\frac{\sin\left(45^\circ+\frac{C}{4}\right)}{\sin C}=2\sin\frac{A}{2}\sin\frac{B}{2}\frac{\sin\left(45^\circ+\frac{C}{4}\right)}{\cos\frac{C}{2}}.$$

Now,

$$2\sin\frac{A}{2}\sin\frac{B}{2}=\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\leq 1-\sin\frac{C}{2},$$

with equality iff $A=B$, or denoting $\gamma=\frac{C}{4}+45^\circ$, we have $\sin\frac{C}{2}=-\cos\left(\frac{C}{2}+90^\circ\right)=-\cos(2\gamma)$, $\cos\frac{C}{2}=\sin\left(\frac{C}{2}+90^\circ\right)=\sin(2\gamma)$, and

$$\frac{A'B'}{AB}\leq(1+\cos(2\gamma))\frac{\sin\gamma}{\sin(2\gamma)}=\cos\gamma,$$

with equality iff $A=B$. Analogous relations may be found for the other terms in the LHS, defining $\alpha=\frac{A}{4}+45^\circ$ and $\beta=\frac{B}{4}+45^\circ$, yielding

$$\frac{AB}{A'B'}+\frac{BC}{B'C'}+\frac{CA}{C'A'}\geq\frac{1}{\cos\alpha}+\frac{1}{\cos\beta}+\frac{1}{\cos\gamma},$$

and using the definition of α, β, γ in the RHS of the proposed inequality, it suffices to show that

$$\frac{1}{\cos\alpha}+\frac{1}{\cos\beta}+\frac{1}{\cos\gamma}\geq 8\left(\cos^2\alpha+\cos^2\beta+\cos^2\gamma\right).$$

Note furthermore that, since $0<A<180^\circ$, we have $45^\circ<\alpha<90^\circ$, or α, β, γ are the angles of an acute triangle, or all cosines are positive reals. It is also well-known (or easily provable using the Cosine Law) that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma=1$, or it suffices to show that

$$\frac{1}{\cos\alpha}+\frac{1}{\cos\beta}+\frac{1}{\cos\gamma}+16\cos\alpha\cos\beta\cos\gamma\geq 8,$$

clearly true after applying the AM-GM inequality to the sum in the LHS, and with equality iff $\alpha=\beta=\gamma=60^\circ$, ie iff ABC is equilateral, which is also clearly necessary and sufficient for the proposed inequality to become an equality. The conclusion follows.

Also solved by Stephanie Lash and Jessica Schuler, College at Brockport, State University of New York; Arkady Alt , San Jose ,California, USA; Li Zhou, Polk State College, USA.

S276. Let a, b, c be real numbers such that

$$\frac{2}{a^2+1}+\frac{2}{b^2+1}+\frac{2}{c^2+1}\geq 3.$$

Prove that $(a-2)^2+(b-2)^2+(c-2)^2\geq 3.$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Arkady Alt , San Jose ,California, USA

Notice that

$$\sum_{cyc}\frac{2}{a^2+1}\geq 3 \iff \sum_{cyc}\left(\frac{2}{a^2+1}-1\right)\geq 0 \iff \sum_{cyc}\frac{1-a^2}{1+a^2}\geq 0.$$

Now, the key part is to see that

$$\begin{aligned}(a-2)^2-\frac{2}{a^2+1} &= \frac{a^4-4a^3+5a^2-4a+2}{a^2+1}=\frac{a^4-4a^3+6a^2-4a+1+\left(1-a^2\right)}{a^2+1}\\ &= \frac{(a-2)^4}{a^2+1}+\frac{1-a^2}{1+a^2}.\end{aligned}$$

It follows that

$$\sum_{cyc}(a-2)^2=\sum_{cyc}\frac{2}{a^2+1}+\sum_{cyc}\frac{(a-2)^4}{a^2+1}+\sum_{cyc}\frac{1-a^2}{1+a^2}\geq\sum_{cyc}\frac{2}{a^2+1}\geq 3.$$

Also solved by Zarif Ibragimov, SamSU, Samarkand, Uzbekistan; Mathematical Group "Galaktika shqiptare", Albania; Daniel Lasaosa, Universidad Pública de Navarra, Spain; AN-anduud Problem Solving Group, Ulaanbaatar,Mongolia; Li Zhou, Polk State College, USA.

U271. Let $a > b$ be positive real numbers and let n be a positive integer. Prove that

$$\frac{\left(a^{n+1}-b^{n+1}\right)^{n-1}}{\left(a^{n}-b^{n}\right)^{n}} > \frac{n}{(n+1)^2}\cdot\frac{e}{a-b},$$

where e is the Euler number.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by AN-anduud Problem Solving Group, Ulaanbaatar,Mongolia

Let us consider following function

$$f(x) = (n-1)\log(x^n + x^{n-1} + \cdots + 1) - n\log(x^{n-1} + x^{n-2} + \cdots + 1)$$

Taking the derivative respect with x, we have

$$f'(x) = \frac{(n-1)(nx^{n-1} + (n-1)x^{n-2} + \cdots + 1)}{x^n + x^{n-1} + \cdots + 1} - \frac{n\left((n-1)x^{n-2} + (n-2)x^{n-3} + \cdots + 1\right)}{x^{n-1} + x^{n-2} + \cdots + 1}.$$

If we prove following inequality

$$(n-1)\left(\sum_{k=1}^{n} kx^{k-1}\right)\cdot\left(\sum_{k=1}^{n} x^{k-1}\right) \geq n\left(\sum_{k=1}^{n-1} kx^{k-1}\right)\cdot\left(\sum_{k=1}^{n+1} x^{k-1}\right) \qquad (*)$$

then for any x with $x > 1$ we have $f'(x) > 0$. Hence $f(x)$ is increasing on the open interval $(0, \infty)$.

$$\forall x > 1 : f(x) > f(1) \Leftrightarrow \log\frac{(x^n + x^{n-1} + \cdots + 1)^{n-1}}{(x^{n-1} + x^{n-2} + \cdots + 1)^n} \geq \log\frac{(n+1)^{n-1}}{n^n}$$

$$\frac{(x^n + x^{n-1} + \cdots + 1)^{n-1}}{(x^{n-1} + x^{n-2} + \cdots + 1)^n} \geq \frac{(n+1)^{n-1}}{n^n}$$

Thus (1) inequality is proved. Now we will prove $(*)$.

$$(*) \Leftrightarrow (n-1)\left(\sum_{k=1}^{n} kx^{k-1}\right)(x^n - 1) \geq n\left(\sum_{k=1}^{n-1} kx^{k-1}\right)(x^{n+1} - 1)$$

$$\Leftrightarrow (n-1)\sum_{k=1}^{n} kx^{n+k-1} + n\sum_{k=1}^{n-1} kx^{k-1} \geq n\sum_{k=1}^{n-1} kx^{n+k} + (n-1)\sum_{k=1}^{n} kx^{k-1}$$

$$\Leftrightarrow \sum_{k=1}^{n-1}(n-k)x^{n-1+k} + \sum_{k=1}^{n-1} \geq (n-1)nx^{n-1} \qquad (**)$$

Now we will use AM-GM inequality for $(n-1)n$ numbers.Then $(**)$ is proved, so our lemma is proved.$\square$
Let us solve the posed problem using our lemma.

$$\frac{(a^{n+1}-b^{n+1})^{n-1}}{(a^n-b^n)^n} = \frac{(a-b)^{n-1}(a^n+a^{n-1}b+\cdots+b^n)^{n-1}}{(a-b)^n(a^{n-1}+a^{n-2}b+\cdots+b^{n-1})^n}$$

$$= \frac{1}{a-b}\cdot\frac{\left(\left(\frac{a}{b}\right)^n+\left(\frac{a}{b}\right)^{n-1}+\cdots+1\right)^{n-1}}{\left(\left(\frac{a}{b}\right)^{n-1}+\left(\frac{a}{b}\right)^{n-2}+\cdots+1\right)^n}$$

$$\overset{\frac{a}{b}=x>1}{=} \frac{1}{a-b}\cdot\frac{(x^n+x^{n-1}+\cdots+1)^{n-1}}{(x^{n-1}+x^{n-2}+\cdots+1)^n}$$

by the lemma

$$\geq \frac{1}{a-b}\cdot\frac{(n+1)^{n-1}}{n^n} = \frac{n}{(n+1)^2}\cdot\frac{1}{a-b}\cdot\left(1+\frac{1}{n}\right)^{n+1}$$

using well known inequality $\left(1+\frac{1}{n}\right)^{n+1} > e$

$$> \frac{n}{(n+1)^2}\cdot\frac{e}{a-b}.$$

Hence our desired inequality is proved.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Arkady Alt , San Jose ,California, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Jedrzej Garnek, Adam Mickiewicz University, Poznan, Poland.

U272. Let a be a positive real number and let $(a_n)_{n\geq 0}$ be the sequence defined by $a_0 = \sqrt{a}$, $a_{n+1} = \sqrt{a_n + a}$, for all positive integers n. Prove that there are infinitely many irrational numbers among the terms of the sequence.

Proposed by Marius Cavachi, Constanţa, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Assume that the result is false. Then, either all terms in the sequence are rational, or there is a last irrational term a_N, and $a_{N+1}, a_{N+2}, \ldots$ are all rational. If two consecutive terms a_{N+1}, a_{N+2} are rational, then $a = {a_{N+2}}^2 - a_{N+1}$ must clearly be rational. Therefore, if a_N is irrational, ${a_{N+1}}^2 = a_N + a$ is irrational, and so is a_{N+1}, or if the proposed result is false, then a is rational, and so is every term in the sequence.

From the previous argument, assuming that the proposed result is false, a positive rational a exists such each a_n is rational for all $n \geq 0$. Clearly $a_1 = \sqrt{a + \sqrt{a}} > \sqrt{a} = a_0$, and if $a_n > a_{n-1}$, then $a_{n+1} = \sqrt{a + a_n} > \sqrt{a + a_{n-1}} = a_n$, or by trivial induction the sequence is strictly increasing. Moreover, if $\sqrt{a} = \frac{u}{v}$ for u, v coprime, an increasing sequence of integers u_n exists such that $u_0 = u$, and $u_{n+1} = \sqrt{u^2 + u_n v}$. Indeed, with such a definition it follows that if $a_n = \frac{u_n}{v}$, then $a_{n+1} = \frac{u_{n+1}}{v}$. Moreover, since a_{n+1} is rational, if u_n is an integer, then u_{n+1} is rational and at the same time the square root of an integer, hence an integer too, or since u_0 is an integer, then by trivial induction so is every $u_n = va_n$, or the u_n's are a strictly increasing sequence of positive integers, hence unbounded. Or for sufficiently large n, u_n will be as large as we desire, and at some point we will have $a_n = vu_n > \frac{1+\sqrt{1+4a}}{2}$. Then,

$$a < \frac{(2a_n - 1)^2 - 1}{4} = {a_n}^2 - a_n, \qquad a_{n+1} = \sqrt{a + a_n} < \sqrt{{a_n}^2} = a_n,$$

and the sequence would decrease, contradiction, hence the proposed result cannot be false. The conclusion follows.

U273. Let Φ_n be the nth cyclotomic polynomial, defined by

$$\Phi_n(X) = \prod_{1\le m\le n,\gcd(m,n)=1} \left(X - e^{\frac{2i\pi m}{n}}\right).$$

a) Let k and n be positive integers with k even and $n > 1$. Prove that

$$\pi^{k\varphi(n)} \cdot \prod_p \Phi_n\left(\frac{1}{p^k}\right) \in \mathbb{Q},$$

where the product is taken over all primes and φ is the Euler totient function.

b) Prove that

$$\prod_p \left(1 - \frac{1}{p^2} + \frac{1}{p^6} - \frac{1}{p^8} + \frac{1}{p^{10}} - \frac{1}{p^{14}} + \frac{1}{p^{16}}\right) = \frac{192090682746473135625}{3446336510402\pi^{16}}.$$

Proposed by Albert Stadler, Herrliberg, Switzerland

Solution by Konstantinos Tsouvalas, University of Athens, Athens, Greece
a)We will use induction. First of all, we will use the following formula:

$$\zeta(2n) = (-1)^{n+1} \frac{B_{2n}(2\pi)^{2n}}{2(2n)!}$$

where ζ denotes Riemann's function and B_{2n} the 2n-nth Bernoulli number.
For the cuclotomic polynomial it is known that:

$$\Phi_n(X) = (-1)^{\sum_{d|n}\mu(n/d)} \prod_{d|n} \left(1 - X^d\right)^{\mu(n/d)} = \prod_{d|n} \left(1 - X^d\right)^{\mu(n/d)}$$

$$\prod_{d|n} \left(1 - X^d\right)^{\mu(n/d)}$$

We also have:

$$n = \sum_{d|n} \phi(d)$$

hence from Mobius inversion formula:

$$\phi(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) d$$

Then:

$$\pi^{k\phi(n)} \prod_p \Phi_n\left(p^{-k}\right) = \pi^{k\phi(n)} \prod_{d|n} \left(\prod_p \left(1 - \frac{1}{p^{kd}}\right)\right)^{\mu(n/d)}$$

$$= \prod_{d|n} \left(\pi^{kd} \prod_p \left(1 - \frac{1}{p^{kd}}\right)\right)^{\mu(n/d)}$$

$$= \prod_{d|n} \left(\frac{(kd)!}{(-1)^{kd/2+1} B_{kd} 2^{kd-1}}\right)^{\mu(n/d)} \in \mathbb{Q}$$

b)We oserve that:

$$\Phi_{15}\left(p^{-2}\right)\frac{\left(1-p^{-30}\right)\left(1-p^{-2}\right)}{\left(1-p^{-6}\right)\left(1-p^{-10}\right)}$$

.

Finally we have:

$$\pi^{16}\prod_{p}\Phi_{15}\left(p^{-2}\right)=\frac{\zeta(10)\zeta(6)}{\zeta(2)\zeta(30)}$$

which is the desired number.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Jedrzej Garnek, Adam Mickiewicz University, Poznan, Poland.

U274. Let $A_1, \ldots, A_m \in M_n(\mathbb{C})$ satisfying $A_1 + \cdots + A_m = mI_n$ and ${A_1}^2 = \cdots = {A_m}^2 = I_n$. Prove that $A_1 = \cdots = A_m$.

Proposed by Marius Cavachi, Constanţa, Romania

Solution by Jedrzej Garnek, Adam Mickiewicz University, Poznan, Poland Note that since $A_i^2 = I_n$, the eigenvalues of A_i satisfy equation $\lambda^2 = 1$, i.e. all eigenvalues of A_i are equal to ± 1. Thus, since trace of a matrix is sum of its eigenvalues, $tr\, A_i \leq n$, with equality iff all eigenvalues are equal to 1.
On the other hand: $mn = tr\, mI_n = tr\,(A_1 + \ldots + A_m) = tr\, A_1 + \ldots + tr\, A_m \leq n + \ldots + n = mn$. Thus for all i all eigenvalues of A_i are equal to 1.

Finally, since $A_i^2 - I_n = 0$, the minimal polynomial of A_i divides $x^2 - 1$, and has no multiple roots, so that A_i is diagonalizable. But the only diagonalizable $n \times n$ matrix with all eigenvalues equal to 1 is the identity matrix – thus $A_i = I_n$ for all i.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Nikolaos Zarifis, National Technical University of Athens, Athens, Greece; Moubinool Omarjee, Lycée Henri IV, Paris, France; Konstantinos Tsouvalas, University of Athens, Athens, Greece.

U275. Let m and n be positive integers and let $(a_k)_{k\geq 1}$ be real numbers. Prove that

$$\sum_{d|m,e|n,g|\gcd(d,e)} \frac{\mu(g)}{g} de \cdot a_{de/g} = \sum_{k|mn} k a_k.$$

Here, μ is the usual Möbius function.

Proposed by Darij Grinberg, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Since the result is proposed for any sequence of real numbers, and a_k only appears in the sum in the LHS when $\frac{de}{g} = k$, then the proposed result is equivalently formulated as follows: given any positive integer k, for any pair of positive integers m, n such that mn is a multiple of k, we have

$$\sum_{d|m,e|n,\frac{de}{k}|\gcd(d,e)} \mu\left(\frac{de}{k}\right) = 1.$$

We prove the proposed result, expressed in this latter form, by induction on the number of distinct primes that divide simultaneously m and n.

If m, n are coprime, then so are d, e, and $\gcd(d,e) = 1$, or $\frac{de}{k} = 1$, ie, there is exactly one term in the sum, occurring when $d = \gcd(k,m)$ and $e = \gcd(k,n)$, for which $\mu\left(\frac{de}{k}\right) = \mu(1) = 1$, and the result clearly follows in this case.

Let p be a prime that divides m, n, with respective multiplicities $U, V \geq 1$. Let t, u, v the multiplicities with which p divides k, d, e, where clearly $u \in \{0, 1, \ldots, U\}$ and $v \in \{0, 1, \ldots, V\}$. The multiplicities of p in $\frac{de}{k}$ and $\gcd(d,e)$ are respectively $u + v - t$ and $\min(u,v)$. Since k must divide de, but at the same time a nonzero contribution to the sum will happen iff $\frac{de}{k}$ is square-free, we must have $u + v - t \in \{0, 1\}$ for all nonzero contributions to the sum, while at the same time, since $\frac{de}{k}$ divides $\gcd(d,e)$, we must have $u + v - t \leq \min(u,v)$, or equivalently $t \geq \max(u,v)$. The set of values that the pair (u,v) can take therefore satisfies $\max(0, t-V) \leq u \leq \min(t,U)\}$ and $v = t-u$, or $\max(1, t+1-V) \leq u \leq \min(t,U)\}$ and $u = t+1-V$. Note therefore that there is always exactly one more pair (u,v) such that $u + v - t = 0$, than pairs such that $u + v - t = 1$.

Denote now by m', n', d', e', k' the respective results of removing all factors p from m, n, d, e, k. For each set (m', n', d', e', k'), consider all possible pairs (u,v) as described above. For the pairs of the first kind, we have $\mu\left(\frac{de}{k}\right) = \mu\left(\frac{d'e'}{k'}\right)$, since p does not appear in $\frac{de}{k}$, while for the pairs of the second kind, we have $\mu\left(\frac{de}{k}\right) = \mu\left(p\frac{d'e'}{k'}\right) = -\mu\left(\frac{d'e'}{k'}\right)$, since p appears with multiplicity 1 in $\frac{de}{k}$, but does not appear in $\frac{d'e'}{k'}$. Therefore, the net contribution to the sum of all possible pairs (u,v) given m', n', d', e', k' is $\mu\left(\frac{d'e'}{k'}\right)$. In other words, adding one more distinct prime factor to m, n, k does not change the value of the sum, or after trivial induction, the proposed result follows.

U276. Let K be a finite field. Find all polynomials $f \in K[X]$ such that $f(X) = f(aX)$ for all $a \in K^*$.

Proposed by Mihai Piticari, Campulung Moldovenesc, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

It is well known that, if K is a finite field, then there exists a prime p such that the characteristic of K is p, the order of K is p^n for some positive integer n, and this finite field is unique up to isomorphism. Moreover, the multiplicative group of this finite field (ie, the group with respect to multiplication of K^*), has order $p^n - 1$ and $\varphi(p^n - 1)$ primitive generators g, such that $K^* = \left\{g, g^2, \dots, g^{p^n-1} = 1\right\}$, where 1 denotes the unit of the multiplication operation.

Moreover, for all $a \in K^*$, we can take $X = 1$, or $f(1) = f(a)$ for all $a \in K^*$. Therefore, any such polynomial takes a certain value b for all $a \in K^*$, and a certain value (not necessarily distinct) c for 0 (where 0 denotes the unit of the sum operation). Thus, any such polynomial f can be written as $f(X) = dX^{p^n-1} + c$, where $c = f(0)$ and $d = b - c = f(1) - f(0) = f(a) - f(0)$ for all $a \in K^*$. Any other polynomial that satisfies the conditions given in the problem statement can be written equivalently in the proposed form.

O271. Let $(a_n)_{n\geq 0}$ be the sequence given by $a_0 = 0$, $a_1 = 2$ and $a_{n+2} = 6a_{n+1} - a_n$ for $n \geq 0$. Let $f(n)$ be the highest power of 2 that divides n. Prove that $f(a_n) = f(2n)$ for all $n \geq 0$.

Proposed by Albert Stadler, Herrliberg, Switzerland

Solution by G. C. Greubel, Newport News, VA
First consider the difference equation

$$a_{n+2} = 6a_{n+1} - a_n \tag{2}$$

where $a_0 = 0$ and $a_1 = 2$. The solution of the difference equation can be obtained by making the approximation $a_m = r^m$ for which r must satisfy the quadratic equation $r^2 - 6r + 1 = 0$. It is seen that $r = 3 \pm 2\sqrt{2}$ and leads to the general form

$$a_n = A(3 + 2\sqrt{2})^n + B(3 - 2\sqrt{2})^n. \tag{3}$$

The values of A and B can be obtained from the initial values and leads to

$$a_n = \frac{1}{2\sqrt{2}} \left[(3 + 2\sqrt{2})^n - (3 - 2\sqrt{2})^n \right]. \tag{4}$$

Since $3 + 2\sqrt{2} = (1 + \sqrt{2})^2$ and $3 - 2\sqrt{2} = (1 - \sqrt{2})^2$ then a_n becomes

$$a_n = \frac{(1 + \sqrt{2})^{2n} - (1 - \sqrt{2})^{2n}}{(1 + \sqrt{2}) - (1 - \sqrt{2})}. \tag{5}$$

This last form can readily be seen as the Pell numbers of even values, namely, $a_n = P_{2n}$.

The function $f(n)$ is the highest power of 2 that divides n. This works as follows: $f(0) = 0$, $f(1) = 0$, $f(2) = 1$, $f(3) = 0$, $f(4) = 2$, $f(5) = 0$, and so on. The function $f(n)$ is the set

$$f(n) \in \{0, 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0, 4, 0, 1, 0, 2, \cdots\}$$

and is the sequence A007814 of the On-line Encyclopedia of Integers Sequences. The following table defines, for each n value the corresponding P_{2n}, $f(2n)$ and $f(P_{2n})$ values.

n	P_{2n}	$f(2n)$	$f(P_{2n})$
0	0	0	0
1	2	1	1
2	12	2	2
3	70	1	1
4	408	3	3
5	2378	1	1
6	13860	2	2
$\cdots$	$\cdots$	$\cdots$	$\cdots$

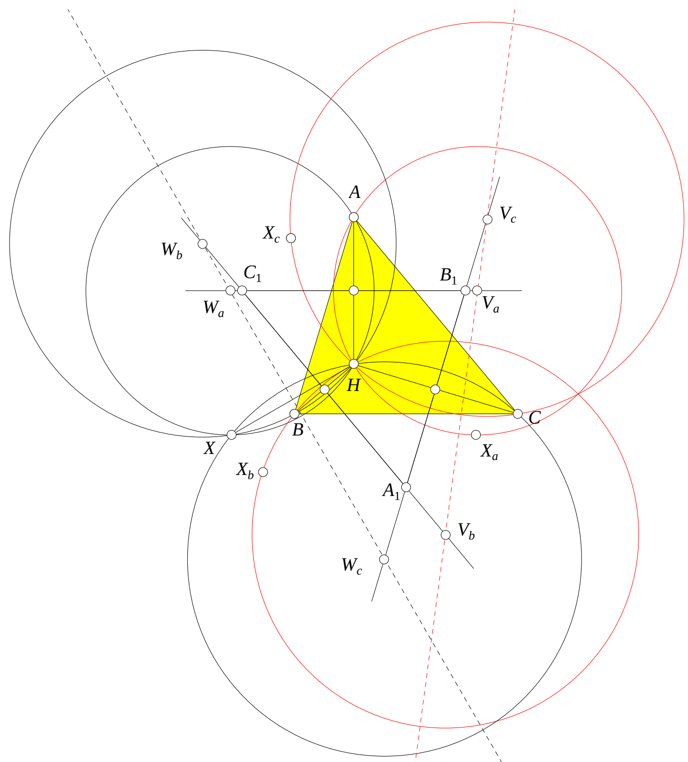
Indeed it is seen that $f(P_{2n}) = f(2n)$ and thus the requirement of the problem is shown.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; AN- anduud Problem Solving Group, Ulaanbaatar, Mongolia; Li Zhou, Polk State College, USA.

O272. Let ABC be an acute triangle with orthocenter H and let X be a point in its plane. Let X_a, X_b, X_c be the reflections of X across AH, BH, CH, respectively. Prove that the circumcenters of triangles AHX_a, BHX_b, CHX_c are collinear.

Proposed by Michal Rolinek, Institute of Science and Technology, Vienna and Josef Tkadlec, Charles University, Prague

Solution by Sebastiano Mosca, Pescara, Italy and Ercole Suppa, Teramo, Italy



Let V_a, V_b, V_c be the circumcenters of triangles AHX_a, BXH_b, CXH_c and let W_a, W_b, W_c be the reflections of V_a, V_b, V_c across AH, BH, CH, respectively; let ℓ_a, ℓ_b, ℓ_c be the perpendicular bisectors of AH, BH, CH and denote $A_1 = \ell_b \cap \ell_c$, $B_1 = \ell_a \cap \ell_c$, $C_1 = \ell_a \cap \ell_b$, as shown in figure.

Observe that the circles $(V_a) \equiv \odot(AHX_a)$, $(V_b) \equiv \odot(BHX_b)$, $(V_c) \equiv \odot(CHX_c)$ are respectively congruent to $(W_a) \equiv \odot(AHX)$, $(W_b) \equiv \odot(BHX)$, $(W_c) \equiv \odot(CHX)$.

The points W_a, W_b, W_c belong to the perpendicular bisector of HX. Thus by applying the Menelaus' theorem to the triangle A_1, B_1, C_1 and to the transversal W_aW_bWc we get

$$\frac{A_1W_c}{W_cB_1} \cdot \frac{B_1W_a}{W_aC_1} \cdot \frac{C_1W_b}{W_bA_1} = -1 \tag{1}$$

Because of the symmetry clearly we have

$$A_1W_c = B_1V_c \quad , \quad B_1W_a = C_1V_A \quad , \quad C_1W_b = A_1V_b \tag{2}$$

$$W_cB_1 = V_cA_1 \quad , \quad W_aC_1 = V_aB_1 \quad , \quad W_bA_1 = V_bC_1 \tag{3}$$

From (1),(2),(3) it follows that

$$\frac{A_1V_b}{V_bC_1} \cdot \frac{C_1V_a}{V_aB_1} \cdot \frac{B_1V_c}{V_cA_1} = \frac{A_1W_c}{W_cB_1} \cdot \frac{B_1W_a}{W_aC_1} \cdot \frac{C_1W_b}{W_bA_1} = -1$$

so, by converse of Menelaus' theorem, the points V_a, V_b, V_c are collinears, as required.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Georgios Batzolis, Mandoulides High School, Thessaloniki, Greece; Li Zhou, Polk State College, USA.

O273. Let P be a polygon with perimeter L. For a point X, denote by $f(X)$ the sum of the distances to the vertices of P. Prove that for any point X in the interior of P, $f(X) < \frac{n-1}{2}L$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Claim: It suffices to show the proposed result for convex polygons.

Proof: If the polygon is concave, a sequence of consecutive vertices $V_1, V_2, \ldots, V_k$ with $k \geq 3$ exists such that line V_1V_k leaves all the polygon on the same half-plane, and the region determined by $V_1, V_2, \ldots, V_k$ is outside the given polygon. Let $W_2, W_3, \ldots, W_{k-1}$ be the symmetric of $V_2, V_3, \ldots, V_{k-1}$ with respect to line V_1V_k. Note that for any point X inside the original polygon, $XW_i > XV_i$ for $i = 2, 3, \ldots, k-1$, while the perimeter of the polygon resulting from deleting vertices $V_2, V_3, \ldots, V_{k-1}$ and substituting them by vertices $W_2, W_3, \ldots, W_{k-1}$ is the same as the perimeter of the original polygon. After a finite number of these transformations, we will obtain a convex polygon, and if the result is true for it, it will also be true for the original concave polygon. The Claim follows.

The proposed result is true for any triangle ABC and any point X inside it. Indeed, consider the ellipse with foci B, C through A. This ellipse contains X inside it, or $BX + CX < AB + CA$, and similarly for the result of cyclically permuting A, B, C. Adding all such inequalities and dividing by 2 we obtain

$$f(X) = AX + BX + CX < AB + BC + CA = \frac{3-1}{2}L.$$

Assume now that the result is true for any convex polygon with n sides. Note that, when polygon P has $n+1$ sides, there are always three consecutive vertices that we can denote V_n, V_{n+1}, V_1 such that X is outside or on the perimeter of triangle $V_nV_{n+1}V_1$. Denote by P' the polygon $V_1V_2\ldots V_n$ (which clearly holds X inside it or on its perimeter), by $f'(X)$ the sum of distances from X to the vertices of P', and by L' the perimeter of $V_1V_2\ldots V_n$. Then, $f(X) = f'(X) + XV_{n+1}$, while by the triangle inequality, $L = L' + V_nV_{n+1} + V_{n+1}V_1 - V_nV_1 > L'$. Now, by hypothesis of induction, $f'(X) < \frac{n-1}{2}L' < \frac{(n+1)-2}{2}L$, or it suffices to show that $XV_{n+1} \leq \frac{L}{2}$. Now, line XV_{n+1} intersects the perimeter of P at a second point Y, such that $XV_{n+1} < YV_{n+1}$. When we move around the perimeter of P from Y to V_{n+1}, there is one direction in which the distance traveled is at most $\frac{L}{2}$, and by the triangle inequality $YV_{n+1} < \frac{L}{2}$. The conclusion follows.

Also solved by Li Zhou, Polk State College, USA.

O274. Let a, b, c be positive integers such that a and b are relatively prime. Find the number of lattice points in

$$D = \{(x, y) | x, y \geq 0, bx + ay \leq abc\}.$$

Proposed by Arkady Alt, San Jose, California, USA

Solution by Li Zhou, Polk State College, USA

Let $X = (ac, 0)$ and $Y = (0, bc)$. Since the equation of XY is $y = b\left(c - \frac{x}{a}\right)$ and a, b are relatively prime, y is an integer if and only if $a|x$. Hence, the number of lattice points in the interior of segment XY is $c - 1$. Then it is easy to see that the number of lattice points on the boundary of D is $B = ac + bc + c$. Let I be the the number of lattice points in the interior of D. By Pick's theorem, $I + \frac{1}{2}B - 1$ equals the area of D, which is $\frac{1}{2}abc^2$. Therefore, the number of lattice points in D is

$$I + B = \frac{1}{2}(abc^2 + B) + 1 = \frac{c(abc + a + b + 1)}{2} + 1.$$

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain.

O275. Let ABC be a triangle with circumcircle $\Gamma(O)$ and let ℓ be a line in the plane which intersects the lines BC, CA, AB at X, Y, Z, respectively. Let ℓ_A, ℓ_B, ℓ_C be the reflections of ℓ across BC, CA, AB, respectively. Furthermore, let M be the Miquel point of triangle ABC with respect to line ℓ.

a) Prove that lines ℓ_A, ℓ_B, ℓ_C determine a triangle whose incenter lies on the circumcircle of triangle ABC.

b) If S is the incenter from (a) and O_a, O_b, O_c denote the circumcenters of triangles AYZ, BZX, CXY, respectively, prove that the circumcircles of triangles SOO_a, SOO_b, SOO_c are concurrent at a second point, which lies on Γ.

Proposed by Cosmin Pohoata, Princeton University, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

a) Let $A' = \ell_B \cap \ell_C$, $B' = \ell_C \cap \ell_A$ and $C' = \ell_A \cap \ell_B$. Note first that Y is the point where ℓ, ℓ_B, AC concur, or since ℓ, ℓ_B are symmetric around AC, they are also symmetric around the perpendicular to AC at Y, and we have $\angle ZYA' = 180° - 2\angle AYZ$, and similarly $\angle YZA' = 180° - 2\angle AZY$, hence

$$\angle YA'Z = 2(\angle AYZ + \angle AZY) - 180° = 180° - 2\angle YAZ = 180° - \angle YO_aZ,$$

hence YO_aZA' is cyclic. Moreover,

$$\angle YO_aA' = \angle YZA' = 180° - 2\angle AZY = 180° - \angle AO_aY,$$

hence A, O_a, A' are collinear. Finally, $\angle YA'A = \angle YA'O_a = \angle YZO_a$, and similarly $\angle ZA'A = \angle ZYO_a$, hence AA' is the angle bisector of $\angle YA'Z$, hence of $\angle B'A'C'$. Or the incenter of $A'B'C'$ is the point where AO_a, BO_b, CO_c concur.

Let S_a, S_b be the points where AO_a, BO_b meet the circumcircle of ABC for the second time. Clearly,

$$\angle CAS_a = \angle YAO_a = 90° - \frac{1}{2}\angle AO_aY = 90° - \angle AZY,$$

and similarly $\angle CBS_b = 90° - \angle BZX$. But $\angle AZY = \angle BZX$ since it is the angle between ℓ and AB, or $S_a = S_b$. Similarly, $S_b = S_c$, or AO_a, BO_b, CO_c meet at a point $S = S_a = S_b = S_c$ on the circumcircle of ABC, which is also the incenter of $A'B'C'$. The conclusion to part a) follows.

Note: The previous argument breaks down when ABC is obtuse because some of the angles cannot be added or subtracted as needed. Indeed, in this case, we can draw triangles ABC for which the statement proposed in part a) is not true, and the incenter of $A'B'C'$ may be inside triangle ABC and not on its circumcircle.

b) It is well known that the Miquel point for $X \in BC$, $Y \in CA$, $Z \in AB$ such that X, Y, Z are collinear, is on the circumcircle of ABC, hence M is on the circumcircle of ABC. Assume wlog (since we may cyclically permute the vertices of ABC without altering the problem) that M is on the arc AB that does not contain C. Since M is on the circumcircle of AYZ, the perpendicular bisector of AM passes through O, O_a. At the same time, denoting by T the point diametrally opposite S in the circumcircle of ABC, we have, if for example $\angle SAM$ is acute and $\angle SBM$ is obtuse,

$$\angle OSM = 90° - \angle STM = 90° - \angle SAM = 90° - \angle O_aAM = \frac{1}{2}\angle AO_aM = 180° - \angle MO_aO,$$

and M, O, O_a, S are concyclic, while at the same time

$$\angle OSM = 90° - \angle STM = \angle SBM - 90° = 90° - \angle O_bBM = \frac{1}{2}\angle BO_bM = \angle MO_bO,$$

and again M, O, O_b, S are concyclic. Similarly, since $\angle SCM$ is either acute or obtuse, we have one of these two cases, and M, O, O_c, S are also concyclic. Hence the circumcircles of SOO_a, SOO_b, SOO_c all meet again at M, which is a point on the circumcircle of ABC. The conclusion to part b) follows.

O276. For a prime p, let $S_1(p) = \{(a,b,c) \in \mathbf{Z}^3,\quad p|a^2b^2+b^2c^2+c^2a^2+1\}$ and $S_2(p) = \{(a,b,c) \in \mathbf{Z}^3,\quad p|a^2b^2c^2(a^2+b^2+c^2+a^2b^2c^2)\}$. Find all p for which $S_1(p) \subset S_2(p)$.

Proposed by Titu Andreescu, University of Texas at Dallas and Gabriel Dospinescu, Ecole Polytechnique, Lyon, France

Solution by the authors

The answer is $2, 3, 5, 13$ and 17. From now on we will work in $\mathbf{Z}/p\mathbf{Z}$ and consider $X_1(p) = \{(a,b,c) \in (\mathbf{Z}/p\mathbf{Z})^3, a^2b^2+b^2c^2+c^2a^2+1 = 0\}$ and $X_2(p) = \{(a,b,c) \in (\mathbf{Z}/p\mathbf{Z})^3, a^2b^2c^2(a^2+b^2+c^2+a^2b^2c^2) = 0\}$. The condition $S_1(p) \subset S_2(p)$ is equivalent to $X_1(p) \subset X_2(p)$.

First, we prove that $2, 3, 5, 13$ and 17 are solutions of the problem. Suppose that p is one of these primes, that $(a,b,c) \in X_1(p)$ and that $(a,b,c) \notin X_2(p)$. In particular, $a^2b^2c^2 \neq 0$. If one of a^2, b^2, c^2 equals 1 or -1, then $a^2b^2+b^2c^2+c^2a^2+1$ and $a^2+b^2+c^2+a^2b^2c^2$ are equal or opposite, a contradiction. Hence $a^2, b^2, c^2 \notin \{0, \pm 1\}$, which already settles the cases $p = 2$, $p = 3$ and $p = 5$. If $p = 13$, the squares mod p are $0, \pm 1, \pm 3, \pm 4$, hence $a^2, b^2, c^2 \in \{\pm 3, \pm 4\}$ which means that two of them add up to 0, say $a^2 + b^2 = 0$. This readily yields $(a,b,c) \in X_2(p)$, a contradiction. Finally, suppose that $p = 17$, so that the squares mod p are $0, \pm 1, \pm 2, \pm 4, \pm 8$ and $a^2, b^2, c^2 \in \{\pm 2, \pm 4, \pm 8\}$. Moreover, no two of a^2, b^2, c^2 add up to 0 (same argument as before), and no two of them have product -1. Hence up to permutation we have $(a^2, b^2, c^2) \in \{(2, \pm 4, -8), (-2, \pm 4, 8)\}$, contradicting the fact that $(a,b,c) \in X_1(p)$.

Next, we prove that if $p > 3$ is of the form $4k+3$, then $X_1(p)$ is nonempty and disjoint from $X_2(p)$, hence p is not a solution of the problem. Pick $c \in \mathbf{Z}/p\mathbf{Z}$ such that $c^2 \notin \{0, 1\}$ (such c exists, since $p > 3$). We will constantly use the fact that if $x, y \in \mathbf{Z}/p\mathbf{Z}$ satisfy $x^2 + y^2 = 0$, then $x = y = 0$. This implies that the map

$$f : \{0, 1, ..., \frac{p-1}{2}\} \to \mathbf{Z}/p\mathbf{Z}, \quad f(a) = -\frac{a^2c^2+1}{a^2+c^2}$$

is well-defined and we claim that f is injective. Indeed, if $f(a) = f(a_1)$, then an easy computation gives $(a^2 - a_1^2)(c^4 - 1) = 0$, hence $a = a_1$ (because $c^2 \neq \pm 1$). Since f is injective and since there are $\frac{p+1}{2}$ quadratic residues mod p, it follows that there are $a, b \in \mathbf{Z}/p\mathbf{Z}$ such that $f(a) = b^2$, which is equivalent to $(a,b,c) \in X_1(p)$. Hence $X_1(p) \neq \emptyset$. Suppose that $(a,b,c) \in X_1(p) \cap X_2(p)$. Since $p \equiv 3 \pmod 4$, we have $abc \neq 0$, hence $a^2(b^2c^2+1) + b^2 + c^2 = 0$ and $a^2(b^2+c^2) + b^2c^2 + 1 = 0$. This yields $(a^4-1)(b^2+c^2) = 0$, then $a^2 = 1$ and finally $(1+b^2)(1+c^2) = 0$, a contradiction.

Suppose now that $p \equiv 1 \pmod 4$ and $p > 17$. We will construct an element of $X_1(p)$ which is not in $X_2(p)$, finishing the solution. Since $p \equiv 1 \pmod 4$, there exists $x \in \mathbf{Z}/p\mathbf{Z}$ such that $x^2 + 1 = 0$. We will need the following

Lemma. The equation $a^2 + ab + b^2 = x$ has at least $p - 1$ solutions $(a,b) \in \mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}$.

Proof. Write the equation as $(2a+b)^2 + 3b^2 = 4x$. So it is enough to prove that the equation $u^2 + 3v^2 = t$ has at least $p-1$ solutions in $\mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}$, when $t \neq 0$. For each $v \in \mathbf{Z}/p\mathbf{Z}$, the equation $u^2 = t - 3v^2$ has $1 + \left(\frac{t-3v^2}{p}\right)$ solutions, where $\left(\frac{\cdot}{p}\right)$ is Legendre's symbol. Hence the number of solutions (u,v) is $p + \sum_{v \in \mathbf{Z}/p\mathbf{Z}} \left(\frac{t-3v^2}{p}\right)$. It is not difficult to check that $\sum_{v \in \mathbf{Z}/p\mathbf{Z}} \left(\frac{t-3v^2}{p}\right)$ equals ± 1, according to whether -3 is a quadratic residue mod p or not. This finishes the prof of the lemma. □

Now let S be the set of solutions of the previous equation. For each $(a,b) \in S$ we have an element (a,b,c) of $X_1(p)$, where $c = -a - b$. Indeed,

$$a^2b^2 + b^2c^2 + c^2a^2 + 1 = (ab + bc + ca)^2 + 1 = (a^2 + ab + b^2)^2 + 1 = x^2 + 1 = 0,$$

hence $(a,b,c) \in X_1(p)$. Suppose that $(a,b,c) \in X_2(p)$. If $a = 0$, then $b^2 = x$ and (a,b) takes at most two values. Similarly the cases $b = 0$ and $c = 0$ yield each at most 2 values for (a,b), hence we have at most 6 solutions missed until now. Suppose that $a^2+b^2+c^2+a^2b^2c^2=0$. Since $a^2+b^2+c^2=2(a^2+b^2+ab)=2x$ and $a^2c^2=(a(a+b))^2=(x-b^2)^2$, we obtain $2x+b^2(x-b^2)^2=0$. This equation has at most 6 solutions in $\mathbf{Z}/p\mathbf{Z}$ and for each solution b we loose at most two solutions (a,b). Hence we loose at most 12 solutions of the equation $a^2+ab+b^2=x$ if $a^2+b^2+c^2+a^2b^2c^2=0$, and in total we loose at most 18 solutions. Since $p>17$ and $p\equiv 1 \pmod 4$, we still have one solution (a,b) and this yields an element of $X_1(p)$ which is not in $X_2(p)$.