ON THE MAXIMUM AREA OF A TRIANGLE WITH THE FIXED DISTANCES FROM ITS VERTICES TO A GIVEN POINT

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Introduction. In problem solving, the search for ideas that lead to a solution or to a better understanding of mathematical concepts is more important than the ability to conclude that a problem is solved. The problem that we discuss in this article is a good example of a question whose answer is not particularly of interest, but rather the different approaches that lead to it.

Problem 1. Let $P$ be a point in the plane. Points $A, B, C$ are located at distances $R_1 = 1$, $R_2 = 2$, $R_3 = 3$ from $P$, respectively. Find the maximum area of triangle $ABC$.

Such a problem could appear in a mathematical contest, where participants might be asked to find, with proof, a particular lower or upper bound. There is also the general problem, for which we would like to find the exact value of the maximum area.

Problem 2. Let $P$ be a point in the plane. Points $A, B, C$ are located at distances $R_1, R_2, R_3$ from $P$, respectively. Find the maximum area of triangle $ABC$.

In the first part of this article we explore some sharp bounds that estimate the maximum of $K_{ABC}$ for the general problem. We use the conditions from the first problem to verify the effectiveness of our approaches. In the second part, we solve the general problem and show how to find the exact value of the maximum area.

Lower and Upper Bounds. To find a lower bound for $K_{ABC}$ it is sufficient to provide a construction of a triangle $ABC$ and compute its area. The simplest method is to construct segments $AP, BP, CP$ at equal angles of $120^\circ$ from each other. Summing up the areas of triangles $APB, BPC, CPA$ we obtain

$$K_{ABC} \geq \frac{\sqrt{3}}{4} (R_1 R_2 + R_2 R_3 + R_3 R_1).$$

We could also construct triangle $ABC$ by placing $P$ on the segment joining two of the points from the triplet $(A, B, C)$ and take the line joining $P$ with the third point perpendicular to this segment. There are three possible constructions from which we conclude that

$$K_{ABC} \geq \max \left( \frac{1}{2} R_1 (R_2 + R_3), \frac{1}{2} R_2 (R_1 + R_3), \frac{1}{2} R_3 (R_1 + R_2) \right).$$

Using the values $R_1 = 1$, $R_2 = 2$, $R_3 = 3$ from Problem 1, we find $K_{ABC} \geq \frac{\sqrt{3}}{4} \cdot 11 \approx 4.7631$ and $K_{ABC} \geq \max(\frac{5}{2}, \frac{8}{2}, \frac{9}{2}) = 4.5$. We observe that our construction with equal angles separating the segments yields a better lower bound.

The upper bounds require totally different approaches. The first one is unique in its nature. We reformulate the problem and map it in the coordinate plane. Let $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ be
the coordinates of points $A, B, C$, respectively with $P$ the origin. We have
\[ x_1^2 + y_1^2 = R_1^2, \]
\[ x_2^2 + y_2^2 = R_2^2, \]
\[ x_3^2 + y_3^2 = R_3^3, \]
and
\[ K_{ABC} = \pm \frac{1}{2} \left| \begin{array}{ccc} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{array} \right|. \]

We will use Hadamard’s Inequality.

**Hadamard’s Inequality.** If $A$ is a $n \times n$ matrix with entries $a_{ij}$ and determinant $|A|$, then
\[ |A|^2 \leq \prod_{i=1}^{n} \left( \sum_{j=1}^{n} |a_{ij}|^2 \right), \]
where equality holds if and only if the rows are orthogonal.

From Hadamard’s inequality,
\[ \left| \begin{array}{ccc} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{array} \right|^2 \leq (R_1^2 + 1)(R_2^2 + 1)(R_3^2 + 1), \]
yielding
\[ K_{ABC} \leq \frac{1}{2} \sqrt{(R_1^2 + 1)(R_2^2 + 1)(R_3^2 + 1)}. \]
Using this result in Problem 1, we obtain $K_{ABC} \leq \frac{1}{2} \sqrt{2 \cdot 5 \cdot 10} = 5$.

It is difficult to interpret this result geometrically, because when we add $R_1^2$ and 1 we lose the metric context. The upper bound we found using Hadamard’s inequality is surprising. It may not be that useful in the general case, but it seems effective if we restrict $P, A, B, C$ to be lattice points.

Another approach to find an upper bound would be to use properties of the Fermat-Torricelli point.

**Lemma 1.** Let $T$ be the point in the plane of triangle $ABC$ that minimizes the sum of the distances to the vertices of triangle: $f(M) = AM + BM + CM$. Then
\[ 4\sqrt{3}K_{ABC} \leq f^2(T). \]

**Proof of Lemma 1.** It is known that, if all angles of triangle $ABC$ are less than $120^\circ$, then $T$ is the Fermat-Torricelli point. Otherwise, if there is an angle of triangle $ABC$ that is greater than or equal to $120^\circ$, say $\angle A \geq 120^\circ$, then $T \equiv A$. In the first case, the area of triangle $ABC$ can be expressed as
\[ K_{ABC} = \frac{\sqrt{3}}{4} (AT \cdot BT + BT \cdot CT + CT \cdot AT). \]
Using the basic inequality $AT \cdot BT + BT \cdot CT + CT \cdot AT \leq \frac{1}{4} (AT + BT + CT)^2$, we obtain the desired result. In the second case, $\angle A \geq 120^\circ$, hence
\[ K_{ABC} \leq \frac{1}{2} AB \cdot AC \cdot \sin(\angle A) \leq \frac{\sqrt{3}}{4} AB \cdot AC. \]
But $f(T) = f(A) = AB + AC$, yielding
\[
K_{ABC} \leq \frac{\sqrt{3}}{4} AB \cdot AC \leq \frac{\sqrt{3}}{4} \cdot \frac{(AB + AC)^2}{4} = \frac{\sqrt{3}}{16} f^2(T),
\]
an even tighter bound than we desired. Thus, the lemma is proved.

Returning back to the problem, assume that we found triangle $ABC$ with maximum area. Let $T$ be the point that minimizes the sum of the distances to the vertices of the triangle. Then $AP + BP + CP \geq f(T)$, and using Lemma 1, we obtain
\[
K_{ABC} \leq \frac{1}{4\sqrt{3}}(R_1 + R_2 + R_3)^2.
\]
Substituting $R_1 = 1$, $R_2 = 2$, $R_3 = 3$, we find $K_{ABC} \leq 3\sqrt{3} \approx 5.1961$. This upper bound is close, but nonetheless weaker than the one obtained by using Hadamard’s inequality.

**Finding the exact value.** Denote $u = \angle BPC$, $v = \angle CPA$, $w = \angle APB$. Observe that our problem is equivalent to the following maximization problem:

Given the distances $R_1, R_2, R_3$, and angles $u, v, w$, find the maximum value of the function
\[
f(u, v, w) = \frac{1}{2} \left(R_1 R_2 \sin w + R_2 R_3 \sin u + R_3 R_1 \sin v\right),
\]
on the compact set $K = \{(u, v, w) \in [0, \pi]^3 : u + v + w = 2\pi\}$.

Define the function
\[
L(u, v, w, \lambda) = \frac{1}{2} \left(R_1 R_2 \sin w + R_2 R_3 \sin u + R_3 R_1 \sin v\right) + \lambda(u + v + w - 2\pi).
\]

By Weierstrass’s Theorem, the maximum of $K_{ABC}$ exists; it is attained either on the boundary of the set $K$ or when all partial derivatives are equal to 0. The boundary points of $K$ describe the situation as when one of the angles $(u, v, w)$ is equal to 0, then $K_{ABC} = 0$ or when one of the angles $(u, v, w) = \pi$. In the latter case, the maximum of $K_{ABC}$ occurs when the other two angles are both equal to $\frac{\pi}{2}$. This brings us to the case already discussed, where we looked for lower bounds of $K_{ABC}$, and found that
\[
K_{ABC} \geq \frac{1}{2} \max \left(R_1(R_2 + R_3), R_2(R_1 + R_3), R_3(R_1 + R_2)\right).
\]

After inspecting the maximum of $K_{ABC}$ on the boundary of $K$, we look for the extreme points inside this set. We have
\[
\frac{\partial L}{\partial u} = \frac{1}{2} R_2 R_3 \cos u - \lambda = 0, \\
\frac{\partial L}{\partial v} = \frac{1}{2} R_3 R_1 \cos v - \lambda = 0, \\
\frac{\partial L}{\partial w} = \frac{1}{2} R_1 R_2 \cos w - \lambda = 0, \\
\frac{\partial L}{\partial \lambda} = u + v + w - 2\pi = 0.
\]
Solving the system of equations, we find that there is a number $\nu$ such that $R_1 = 2\nu \cos u$, $R_2 = 2\nu \cos v$, $R_3 = 2\nu \cos w$, and $\lambda = 2\nu^2 \cos u \cos v \cos w$. Note that $u + v + w = 2\pi$ if and only if
\[
1 + 2 \cos u \cos v \cos w = \cos^2 u + \cos^2 v + \cos^2 w.
\]
Because \( \cos w = -\cos(u + v) \), we have

\[
1 + 2 \cos u \cos v \cos w = 1 - \cos w(\cos(u + v) + \cos(u - v))
\]
\[
= 1 - \cos w(- \cos w + \cos(u - v))
\]
\[
= 1 + \cos^2 w - \cos w \cos(u - v)
\]
\[
= 1 + \cos^2 w + \cos(u + v) \cos(u - v)
\]
\[
= 1 + \cos^2 w + \frac{1}{2} (\cos 2u + \cos 2v)
\]
\[
= \cos^2 u + \cos^2 v + \cos^2 w.
\]

Substituting \( \cos u = \frac{R_1}{2\nu}, \cos v = \frac{R_2}{2\nu}, \cos w = \frac{R_3}{2\nu} \), we find that \( \nu \) should satisfy the following equation

\[
1 + 2 \frac{R_1 R_2 R_3}{8\nu^3} - \left( \frac{R_1^2}{4\nu^2} + \frac{R_2^2}{4\nu^2} + \frac{R_3^2}{4\nu^2} \right) = 0.
\]

It follows that \( \nu \) is solution of a cubic equation

\[
x^3 - \frac{1}{4}(R_1^2 + R_2^2 + R_3^2)x + \frac{1}{4}R_1 R_2 R_3 = 0.
\]

Let \( p = -\frac{1}{4}(R_1^2 + R_2^2 + R_3^2) \) and \( q = \frac{1}{4}R_1 R_2 R_3 \). Then our equation becomes \( x^3 + px + q = 0 \). The discriminant is equal to

\[
\Delta = \frac{q^2}{4} + \frac{p^3}{27} = \frac{1}{64} \left(R_1^2 R_2^2 R_3^2 - \frac{1}{27}(R_1^2 + R_2^2 + R_3^2)^3\right).
\]

By the AM-GM inequality, \( \Delta \leq 0 \), therefore the cubic has three real roots. If \( x_1, x_2, x_3 \) are the roots of the cubic, then by Vi`{e}ta’s Formulas \( x_1 x_2 x_3 = -q < 0 \) and \( x_1 + x_2 + x_3 = 0 \). From these equations it follows that two of the roots are positive and, the other one is negative.

Assume \( \nu > 0 \). Then equations \( \cos u = \frac{R_1}{2\nu}, \cos v = \frac{R_2}{2\nu}, \cos w = \frac{R_3}{2\nu} \), imply \( u, v, w < \frac{\pi}{2} \) so \( u + v + w = 2\pi \) does not hold. Therefore the only root that we should consider is the negative one.

Suppose \( \nu < 0 \). Then we can find angles \( u, v, w \), where \( \frac{\pi}{2} < u, v, w < \pi \). Construct a circle with radius \( |\nu| \) and a triangle \( ABC \) with sides \( 2|\nu| \sin(180^\circ - u), 2|\nu| \sin(180^\circ - v), \) and \( 2|\nu| \sin(180^\circ - w) \). The angles of this triangle are equal to \( 180^\circ - u, 180^\circ - v, \) and \( 180^\circ - w \). Let \( H \) be the orthocenter of triangle \( ABC \). Then \( \angle BHC = u, \angle CHA = v, \angle AHB = w, \) and \( AH = 2\nu \cos u, BH = 2\nu \cos v, CH = 2\nu \cos w \). Thus in triangle \( ABC \), the orthocenter satisfies all the necessary conditions. It follows that \( ABC \) is a candidate that produces the maximum area, with \( P \) being its orthocenter and \( \nu \) being its circumradius.

Using this method we determine the fourth value that is a candidate for the maximum of \( K_{ABC} \). It remains to find a negative root \( \nu \) of the cubic \( x^3 - \frac{1}{4}(R_1^2 + R_2^2 + R_3^2)x + \frac{1}{4}R_1 R_2 R_3 = 0 \). Since

\[
\sin u = \frac{\sqrt{4\nu^2 - R_1^2}}{2|\nu|}, \quad \sin v = \frac{\sqrt{4\nu^2 - R_2^2}}{2|\nu|}, \quad \sin w = \frac{\sqrt{4\nu^2 - R_3^2}}{2|\nu|},
\]

we get that the fourth value, which we denote by \( K_{\nu} \), is equal to

\[
K_{\nu} = \frac{1}{4|\nu|} \left(R_1 R_2 \sqrt{4\nu^2 - R_2^2} + R_2 R_3 \sqrt{4\nu^2 - R_2^2} + R_3 R_1 \sqrt{4\nu^2 - R_1^2}\right).
\]
Finally, 
\[ \max(K_{ABC}) = \max \left( \frac{1}{2} R_1(R_2 + R_3), \frac{1}{2} R_2(R_1 + R_3), \frac{1}{2} R_3(R_1 + R_2), K_\nu \right). \]

If we want the maximum area of the triangle in Problem 1, we need the negative root of the cubic equation \( x^3 - \frac{7}{2} x + \frac{3}{2} = 0 \). Using Maple, we obtain \( \nu = -2.0565 \), yielding \( \max(K_{ABC}) \approx 4.9044 \).

We also determine angles \( u, v, w \) as
\[
\begin{align*}
  u &= \arccos \left( \frac{R_1}{2\nu} \right) \approx 104^\circ, \\
  v &= \arccos \left( \frac{R_2}{2\nu} \right) \approx 119^\circ, \\
  w &= \arccos \left( \frac{R_3}{2\nu} \right) \approx 137^\circ.
\end{align*}
\]

Setting \( A(-1,0) \), the position of points \( B \) and \( C \) is shown in the figure:

We observe that point \( P \) is the orthocenter of triangle \( ABC \).

The main result that we proved is that, except for three extreme boundary cases, triangle \( ABC \) that has the maximum area must have \( P \) as its orthocenter. The conditions of the problem imply that we know the distances \( AH, BH, CH \). In order to find the maximum area, we need the circumradius \( R \), found by solving a cubic equation. In conclusion, by knowing \( R \), we can construct triangle \( ABC \) and find its maximum area.

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