An Analytic Proof of an Interesting Combinatorial Identity

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Abstract

In this paper, we will prove a rather remarkable combinatorial identity with some analytic methods. The identity we wish to prove is

$$\sum_{k=0}^{n} \frac{(-4)^k k!}{(2k+1)!(n-k)!} = \frac{1}{(2n+1) \cdot n!}.$$ 

To do so, consider the function

$$I(\alpha) = \int_{0}^{\infty} e^{-x^2 - 2\alpha x} \, dx.$$ 

Its $n$th derivative is

$$I^{(n)}(\alpha) = \int_{0}^{\infty} (-2)^n x^n e^{-x^2 - 2\alpha x} \, dx.$$ 

Thus, if the Taylor series of $I(\alpha)$ centered at 0 converges to the function, we must have

$$I(\alpha) = \sum_{n=0}^{\infty} a_n \alpha^n,$$

where

$$a_n = \frac{(-2)^n}{n!} \int_{0}^{\infty} x^n e^{-x^2} \, dx = \frac{(-1)^n \cdot 2^{n-1}}{n!} \Gamma \left( \frac{n+1}{2} \right).$$

Now note that

$$e^{-\alpha^2} I(\alpha) = \int_{0}^{\infty} e^{-(x+\alpha)^2} \, dx = \int_{0}^{\infty} e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2} - \int_{0}^{\alpha} e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1) \cdot n!} \alpha^{2n+1}. $$
Expressing everything in terms of series, we get the identity
\[
\left( \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \alpha^{2n} \right) \left( \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^{n-1}}{n!} \Gamma \left( \frac{n+1}{2} \right) \alpha^n \right) = \frac{\sqrt{\pi}}{2} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1) \cdot n!} \alpha^{2n+1}.
\]

By separating even and odd powers, we can rewrite the second series as
\[
\sum_{n=0}^{\infty} \left[ \frac{2^{2n-1}}{(2n)!} \Gamma \left( \frac{n+1}{2} \right) \alpha^{2n} - \frac{2^{2n}}{(2n+1)!} \Gamma(n+1) \alpha^{2n+1} \right].
\]

We can simplify the first coefficient by observing that
\[
\frac{2^{2n-1}}{(2n)!} \Gamma \left( \frac{n+1}{2} \right) = \frac{2^{2n-1}}{2^n n! \cdot (2n-1)!} \cdot \left( \frac{n-1}{2} \right) \left( \frac{n-3}{2} \right) \cdots \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \right)
\]
\[
= \frac{\sqrt{\pi}}{2} \cdot \frac{2^{2n-1}}{(2n-1)!} \frac{(2n-1)!}{2^n}
\]
\[
= \frac{\sqrt{\pi}}{2} \cdot \frac{1}{n!}.
\]

Note that this holds for all nonnegative integers, even though $(2n-1)!$ is not defined for $n = 0$. The second coefficient can also be simplified somewhat:
\[
\frac{2^{2n}}{(2n+1)!} \Gamma(n+1) = \frac{2^{2n} n!}{(2n+1)!}.
\]

Thus, the prior identity can be expressed as
\[
\left( \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \alpha^{2n} \right) \left( \sum_{n=0}^{\infty} \left[ \frac{\sqrt{\pi}}{2 \cdot n!} \alpha^{2n} - \frac{2^{2n} n!}{(2n+1)!} \alpha^{2n+1} \right] \right) = \frac{\sqrt{\pi}}{2} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1) \cdot n!} \alpha^{2n+1}.
\]

Equating the even coefficients gives us the identity
\[
\sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!} = \begin{cases} 1, & n = 0 \\ 0, & n > 0, \end{cases}
\]
which is a clear consequence of the binomial theorem. Equating the odd coefficients yields
\[
\sum_{k=0}^{n} \frac{(-1)^{n-k} \cdot 2^{2k} k!}{(2k+1)! (n-k)!} = \frac{(-1)^n}{(2n+1) \cdot n!},
\]
or equivalently,
\[
\sum_{k=0}^{n} \frac{(-4)^k k!}{(2k+1)! (n-k)!} = \frac{1}{(2n+1) \cdot n!},
\]
as desired.

Note that we could have also proved this identity by using the fact that
\[
I'(\alpha) - 2\alpha I(\alpha) = \int_{0}^{\infty} \frac{(-2x - 2\alpha) e^{-x^2 - 2\alpha x}}{x} \, dx
\]
\[
= \int_{0}^{\infty} -e^{-u} \, du
\]
\[
= -1,
\]

which is the result we desired.
where in the second equation we make the substitution \( u = x^2 + 2\alpha x \). We could then use this differential equation to find a Taylor series for \( I(\alpha) \). This method, while perhaps slicker than the first, is much less useful as a general strategy for finding series representations of functions like these. One can rarely expect to find such a nice differential equation as this one.

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