Junior problems

J307. Prove that for each positive integer n there is a perfect square whose sum of digits is equal to 4^n.

Proposed by Mihaly Bencze, Brasov, Romania

Solution by Albert Stadler, Herrliberg, Switzerland

Let $a_j = 10^j$ and consider

$$m = \left(\sum_{j=1}^{k} 10^{a_j}\right)^2 = \sum_{j=1}^{k} 10^{2a_j} + 2 \sum_{1\le i\le j\le k} 10^{a_i+a_j}$$

m is a perfect square whose only digits in the base 10 representation are only zeros, ones and twos, since $a_i + a_j = a_r + a_s$, $1 \le i \le j \le k$, $1 \le r \le s \le k$, implies $i = r$, $j = s$. The sum of the digits of m equals

$$k + \frac{2k(k-1)}{2} = k^2.$$

This result goes beyond what is asserted, since we have produced for a given natural number k a perfect square m whose digit sum equals k^2. In particular, if $k = 2^n$, then $k^2 = 4^n$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Bedri Hajrizi, Gjimnazi "Frang Bardhi", Mitrovicë, Kosovë; Corneliu Mănescu- Avram, Transportation High School, Ploieşti, Romania; Henry Ricardo, New York Math Circle; Alessandro Ventullo, Milan, Italy; Zachary Chase, University School of NSU, FL, USA; Jaesung Son, Ridgewood, NJ, USA; Jongyeob Lee, Stuyvesant High School, NY, USA; Yeonjune Kang, Peddie School, Hightstown, NJ, USA; William Kang, Bergen County Academies, Hackensack, NJ, USA; Chaeyeon Oh, Episcopal High School, Alexandra, VA, USA; Kyoung A Lee, The Hotchkiss School, Lakeville, CT, USA; Seung Hwan An, Taft School, Watertown, CT, USA; Polyahedra, Polk State College, USA; Seonmin Chung, Stuyvesant High School, New York, NY, USA; Alyssa Hwang, Kent Place School Summit, NJ, USA; Woosung Jung, Korea International School, South Korea; Daniel Jhiseung Hahn, Phillips Exeter Academy, Exeter, NH, USA; Seong Kweon Hong,The Hotchkiss School, Lakeville, CT; Radouan Boukharfane, Sidislimane, Morocco; Farrukh Mukhammadiev, Academic lyceum under the SamIES Nr.1, Samarkand, Uzbekistan.

J308. Are there triples (p, q, r) of primes for which $(p^2 - 7)(q^2 - 7)(r^2 - 7)$ is a perfect square?

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Michael Tang, Edina High School, MN, USA

We claim that there is no triple (p, q, r) of primes so that $S = (p^2 - 7)(q^2 - 7)(r^2 - 7)$ is a perfect square. Suppose for the sake of contradiction that there is such a triple. If, say, $p = 2$, then $p^2 - 7 < 0$, so we must have $(q^2 - 7)(r^2 - 7) < 0$, since S must be nonnegative. However, if $q, r \geq 3$, then $q^2 - 7, r^2 - 7 > 0$, a contradiction, so one of q, r must equal 2. Without loss of generality, take $q = 2$. Then the expression becomes

$$S = (2^2 - 7)(2^2 - 7)(r^2 - 7) = 9(r^2 - 7),$$

so $r^2 - 7$ must be a perfect square. Let $r^2 - 7 = k^2$ for some nonnegative integer k. Rearranging and factoring, we get $(r - k)(r + k) = 7$. Since $r + k$ is positive and greater than $r - k$, we must have $r + k = 7$ and $r - k = 1$, which gives $(r, k) = (4, 3)$. But r is prime, a contradiction.

Therefore, none of p, q, r can equal 2, so they must all be odd. Then, by a well-known result, $p^2, q^2, r^2 \equiv 1 \pmod 8$. (To see this, write $p = 2n - 1$, giving $p^2 = 4n^2 - 4n + 1 = 4n(n - 1) + 1$, where $n(n - 1)$ is clearly even, so $4n(n - 1) + 1 \equiv 0 + 1 = 1 \pmod 8$.) Thus, $p^2 - 7, q^2 - 7, r^2 - 7 \equiv 2 \pmod 8$. This means that the power of 2 in each of $p^2 - 7, q^2 - 7, r^2 - 7$ is exactly one (since they are of the form $8m + 2 = 2(4m + 1)$, twice an odd integer), so the power of 2 in their product, S, is exactly 3. But the power of 2 must be even for S to be a perfect square, a contradiction.

Therefore, we conclude that there is no triple (p, q, r) of primes so that $(p^2 - 7)(q^2 - 7)(r^2 - 7)$ is a perfect square.

Also solved by Adnan Ali, A.E.C.S-4, Mumbai, India; Daniel Lasaosa, Pamplona, Spain; Ali Baouan, Rabat, Morocco; Jaesung Son, Ridgewood, NJ, USA; Ioan-Andrei Nicolae, Romania; Jongyeob Lee, Stuyvesant High School, NY, USA; Yeonjune Kang, Peddie School, Hightstown, NJ, USA; William Kang, Bergen County Academies, Hackensack, NJ, USA; Chaeyeon Oh, Episcopal High School, Alexandra, VA, USA; Kyoung A Lee, The Hotchkiss School, Lakeville, CT, USA; Seung Hwan An, Taft School, Watertown, CT, USA; Polyahedra, Polk State College, USA; Seonmin Chung, Stuyvesant High School, New York, NY, USA; Alyssa Hwang, Kent Place School Summit, NJ, USA; Woosung Jung, Korea International School, South Korea; Arber Avdullahu, Mehmet Akif College, Kosovo; Arkady Alt, San Jose, CA, USA; Ilyes Hamdi, Lycée Voltaire, Doha, Qatar; Evgenidis Nikolaos, M. N. Raptou High School, Larissa, Greece; Prithwijit De, HBCSE, Mumbai, India; Alessandro Ventullo, Milan, Italy; Zachary Chase, University School of NSU, FL, USA; Radouan Boukharfane, Sidislimane, Morocco.

J309. Let n be an integer greater than 3 and let S be a set of n points in the plane that are not the vertices of a convex polygon and such that no three are collinear. Prove that there is a triangle with the vertices among these points having exactly one other point from S in its interior.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

First solution by Daniel Lasaosa, Pamplona, Spain
We solve the problem by induction on n. The base case is clearly $n = 4$, and if 4 points in the plane are not the vertices of a convex polygon, and no 3 are collinear, then three of them form a triangle which has the fourth in its interior, or the proposed result is clearly true. Assume that the result is true for $4, 5, \ldots, n-1$, and consider n points in the plane which are not the vertices of a convex polygon, and such that no three are collinear. There is a convex polygon, with vertices among the n points, such that the rest of the n points are in its interior. We consider three cases:

- If the convex polygon is a triangle ABC, consider any point P of the remaining $n-3$, which is clearly in its interior, and consider triangles PAB, PBC, PCA. Since $n \geq 5$, at least one of these triangles, wlog PAB, has at least one point in its interior. Apply now the hypothesis of induction to the set formed by the vertices of PAB and all points in its interior, which contains at least 4 and at most $n-1$, and the proposed result is true in this case.

- If the convex polygon is not a triangle and three consecutive of its vertices A, B, C are such that none of the n points are in the interior of triangle ABC, consider the set of $n-1$ points once B is removed, and apply the hypothesis of induction to this set, and the proposed result is also true in this case.

- If the convex polygon is not a triangle and three consecutive of its vertices A, B, C are such that at least one of the n points is in the interior of triangle ABC, consider the set of points formed by A, B, C, and all points inside triangle ABC, which clearly contains at least 4 points and at most $n-1$, and apply the hypothesis of induction to this set, and the proposed result is true again in this case.

The conclusion follows.

Second solution by Alessandro Ventullo, Milan, Italy
We use the following theorem.
Carathéodory's Theorem. If $\mathbf{x} \in \mathbb{R}^d$ lies in the convex hull of a set S, there is a subset S' of S consisting of $d+1$ or fewer points such that $\mathbf{x}$ lies in the convex hull of S'.

Let $\mathbf{x} \in S$ such that $\mathbf{x}$ lies in the interior of the convex hull of S. Hence, by Caratheodory's Theorem ($d = 2$), there is a subset S' of S having at most 3 points such that $\mathbf{x}$ lies in the convex hull of S'. Since the points are not collinear, then $|S'| \neq 2$ and since $\mathbf{x}$ lies in the interior of the convex hull of S, then $|S'| \neq 1$. So, $|S'| = 3$ and there exists a triangle T whose vertices are in S which contains $\mathbf{x}$. If $\mathbf{x}$ is the unique point of S in T, we are done. If there is another point $\mathbf{y} \neq \mathbf{x}$ of S in T, joining $\mathbf{x}$ with the three points in S' we have that $\mathbf{y}$ lies in the interior of one among these triangles. Proceeding in this way and using the fact that S has finitely many points, then there exist a triangle whose vertices are in S which has exactly one point of S in its interior.

Also solved by Jaesung Son, Ridgewood, NJ, USA; Jongyeob Lee, Stuyvesant High School, NY, USA; Yeonjune Kang, Peddie School, Hightstown, NJ, USA; William Kang, Bergen County Academies, Hackensack, NJ, USA; Chaeyeon Oh, Episcopal High School, Alexandra, VA, USA; Kyoung A Lee, The Hotchkiss School, Lakeville, CT, USA; Seung Hwan An, Taft School, Watertown, CT, USA; Polyahedra, Polk State College, USA; Adnan Ali, A.E.C.S-4, Mumbai, India; Arber Igrishta, Eqrem Qabej, Vushtrri, Kosovo; Ioan-Andrei Nicolae, Romania; Zachary Chase, University School of NSU, FL, USA.

J310. Alice puts checkers in some cells of an 8×8 board such that:

a) there is at least one checker in any 2×1 or 1×2 rectangle.
b) there are at least two adjacent checkers in any 7×1 or 1×7 rectangle.
Find the least amount of checkers that Alice needs to satisfy both conditions.

Proposed by Roberto Bosch Cabrera, Havana, Cuba

Solution by Polyahedra, Polk State College, USA
First, the two placements of 37 checkers below satisfy both conditions.

	1	1		1		1	
1		1	1		1		1
	1		1	1		1	
1		1		1	1		1
1	1		1		1	1	
	1	1		1		1	1
1		1	1		1		1
	1		1	1		1	

	1		1	1		1	
1		1	1		1		1
	1		1	1		1	
1		1		1	1		1
1	1		1		1	1	1
	1	1		1		1	
1		1	1		1		1
	1		1	1		1	

Now each row requires at least 4 checkers. Also, if a row has only 4 checkers, then the two squares at the ends of the row must be both empty. Thus two such 4-checker rows cannot be adjacent, to avoid two empty 1×2 rectangles in the first and last columns. So there cannot be 5 or more 4-checker rows. If there are 3 or fewer 4-checker rows (such as in the left figure above), then Alice needs at least $s \times 2 + 5 \times 5 = 37$ checkers. Next, suppose that there are 4 nonadjacent 4-checker rows. Then the two adjacent checkers in the first and last columns must appear in the same rows (such as the fourth and fifth rows in the right figure above). Now between the 4 checkers in the first and last columns, these two adjacent rows require at least 7 more checkers in the middle columns: 6 for the 6 1×2 rectangles and 1 more to ensure condition b). Again, Alice needs at least $4 \times 4 + 2 \times 5 + (4 + 7) = 37$ checkers.

Also solved by Ioan-Andrei Nicolae, Romania; Jaesung Son, Ridgewood, NJ, USA; Jongyeob Lee, Stuyvesant High School, NY, USA; Yeonjune Kang, Peddie School, Hightstown, NJ, USA; William Kang, Bergen County Academies, Hackensack, NJ, USA; Chaeyeon Oh, Episcopal High School, Alexandra, VA, USA; Kyoung A Lee, The Hotchkiss School, Lakeville, CT, USA; Seung Hwan An, Taft School, Watertown, CT, USA; Seonmin Chung, Stuyvesant High School, New York, NY, USA; Alyssa Hwang, Kent Place School Summit, NJ, USA; Woosung Jung, Korea International School, South Korea; Ilyes Hamdi, Lycée Voltaire, Doha, Qatar; Zachary Chase, University School of NSU, FL, USA; Adnan Ali, A.E.C.S-4, Mumbai, India.

J311. Let a, b, c be real numbers greater than or equal to 1. Prove that

$$\frac{a\left(b^2+3\right)}{3c^2+1}+\frac{b\left(c^2+3\right)}{3a^2+1}+\frac{c\left(a^2+3\right)}{3b^2+1}\geq 3.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Arkady Alt, San Jose, CA, USA

By AM-GM Inequality

$$\sum_{cyc}\frac{a(b^2+3)}{3c^2+1}\geq 3\sqrt[3]{\prod_{cyc}\frac{a(b^2+3)}{3c^2+1}}=3\sqrt[3]{\prod_{cyc}\frac{a(a^2+3)}{3a^2+1}}\geq 3$$

since

$$\frac{a(a^2+3)}{3a^2+1}\geq 1 \iff (a-1)^3\geq 0.$$

Also solved by Bedri Hajrizi, Gjimnazi "Frang Bardhi", Mitrovicë, Kosovë; Daniel Lasaosa, Pamplona, Spain; Jaesung Son, Ridgewood, NJ, USA; Jongyeob Lee, Stuyvesant High School, NY, USA; Yeonjune Kang, Peddie School, Hightstown, NJ, USA; William Kang, Bergen County Academies, Hackensack, NJ, USA; Chaeyeon Oh, Episcopal High School, Alexandra, VA, USA; Kyoung A Lee, The Hotchkiss School, Lakeville, CT, USA; Seung Hwan An, Taft School, Watertown, CT, USA; Polyahedra, Polk State College, USA; Alyssa Hwang, Kent Place School Summit, NJ, USA; Woosung Jung, Korea International School, South Korea; Daniel Jhiseung Hahn, Phillips Exeter Academy, Exeter, NH, USA; Seong Kweon Hong,The Hotchkiss School, Lakeville, CT; Michael Tang, Edina High School, MN, USA; Adnan Ali, A.E.C.S-4, Mumbai, India; Bodhisattwa Bhowmik, RKMV, Agartala, Tripura, India; Ilyes Hamdi, Lycée Voltaire, Doha, Qatar; Prithwijit De, HBCSE, Mumbai, India; Alessandro Ventullo, Milan, Italy; Zachary Chase, University School of NSU, FL, USA; George Gavrilopoulos, Nea Makri High School, Athens, Greece; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Florin Stanescu, Cioculescu Serban High School, Gaesti, Romania; Radouan Boukharfane, Sidislimane, Morocco; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Moubinool Omarjee, Lycée Henri IV, Paris, France.

J312. Let ABC be a triangle with circumcircle Γ and let P be a point in its interior. Let M be the midpoint of side BC and let lines AP, BP, CP intersect BC, CA, AB at X, Y, Z, respectively. Furthermore, let line YZ intersect Γ at points U and V. Prove that M, X, U, V are concyclic.

Proposed by Cosmin Pohoata, Princeton University, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Denote $\rho = \frac{AZ}{ZB}$ and $\kappa = \frac{CY}{YA}$. Clearly, by Ceva's theorem $\rho\kappa = 1$ iff $M = X$. In this case, by Thales' theorem $YZ \parallel BC$, and parallel chords BC and UV in the circumcircle of BC have a common perpendicular bisector, which is an axis of symmetry. A circle with center in this axis through U, V, and tangent to BC at $M = X$ clearly exists, which is the limiting case of the general case with $\rho\kappa \neq 1$, which we study next.

Assume wlog (since we may invert the roles of B, C without altering the problem) that $\rho\kappa > 1$, ie line YZ intersects line BC at a point T such that B is inside segment TC, and by Menelaus' theorem, $TC = \rho\kappa TB$. Since $TC = TB + BC$, it follows that

$$TB = \frac{BC}{\rho\kappa - 1}, \qquad TC = \frac{\rho\kappa}{\rho\kappa - 1}, \qquad TM = \frac{BC}{2} + TB = \frac{BC(\rho\kappa + 1)}{2(\rho\kappa - 1)},$$

and since by Ceva's theorem we have $CX = \rho\kappa BX$, or $BX = \frac{BC}{\rho\kappa+1}$, it follows that

$$TX = BX + TB = \frac{2\rho\kappa BC}{\rho^2\kappa^2 - 1}, \qquad TM \cdot TX = \frac{\rho\kappa BC^2}{(\rho\kappa - 1)^2} = TB \cdot TC.$$

Now, since UV is a common chord of the circumcircle of ABC and the circumcircle of XUV, it is also their radical axis, or the power of T with respect to the circumcircle of ABC (which clearly equals $TB \cdot TC$) also equals the power of T with respect to the circumcircle of XUV, and since this power equals $TM \cdot TX$, the circumcircle of XUV also passes through M. The conclusion follows.

Also solved by Jaesung Son, Ridgewood, NJ, USA; Jongyeob Lee, Stuyvesant High School, NY, USA; Yeonjune Kang, Peddie School, Hightstown, NJ, USA; William Kang, Bergen County Academies, Hackensack, NJ, USA; Chaeyeon Oh, Episcopal High School, Alexandra, VA, USA; Kyoung A Lee, The Hotchkiss School, Lakeville, CT, USA; Seung Hwan An, Taft School, Watertown, CT, USA; Polyahedra, Polk State College, USA; George Gavrilopoulos, Nea Makri High School, Athens, Greece; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Prithwijit De, HBCSE, Mumbai, India; Zachary Chase, University School of NSU, FL, USA; Titu Zvonaru, Comănești, Romania and Neculai Stanciu, Buzău, Romania.

Senior problems

S307. Let ABC be a triangle such that $\angle ABC - \angle ACB = 60^\circ$. Suppose that the length of the altitude from A is $\frac{1}{4}BC$. Find $\angle ABC$.

Proposed by Omer Cerrahoglu and Mircea Lascu, Romania

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain

Let a and h be lengths of the side BC and the altitude from A respectively. Then the area of $\triangle ABC$ may be expressed as $\frac{1}{2}ah = \frac{1}{8}a^2$, and also as $\frac{1}{2}a^2\frac{\sin B \sin C}{\sin(B+C)} = \frac{1}{2}a^2\frac{\sin B \sin(B-60^\circ)}{\sin(2B-60^\circ)}$.
Therefore

$$\frac{\sin B \sin(B-60^\circ)}{\sin(2B-60^\circ)} = \frac{1}{4}$$

and

$$\begin{aligned}\sin(2B-60^\circ) &= 4\sin B \sin(B-60^\circ)\\ &= -2\left[\cos(2B-60^\circ) - \cos 60^\circ\right]\\ &= -2\cos(2B-60^\circ) + 1\end{aligned}$$

i.e.

$$\sin(2B-60^\circ) + 2\cos(2B-60^\circ) = 1 \tag{1}$$

Let $t = \tan(B-30^\circ)$. Then

$$\sin(2B-60^\circ) = \frac{2t}{1+t^2} \text{ and } \cos(2B-60^\circ) = \frac{1-t^2}{1+t^2} \tag{2}$$

Substitute (2) into (1) and get

$$3t^2 - 2t - 1 = 0$$

and the only valid solution is $t = 1$. We conclude that $B - 30^\circ = 45^\circ$, and therefore $\angle ABC = 75^\circ$

Also solved by George Tsapakidis, "Panagia Prousiotissa" High School, Agrinio, Greece; Daniel Lasaosa, Pamplona, Spain; Li Zhou, Polk State College, Winter Haven, FL, USA; Titu Zvonaru, Comănești, Romania and Neculai Stanciu, Buzău, Romania; Yassine Hamdi, Lycée du Parc, Lyon, France; Alessandro Ventullo, Milan, Italy; Prithwijit De, HBCSE, Mumbai, India; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Jongyeob Lee, Stuyvesant High School, NY, USA; Corneliu Mănescu- Avram, Transportation High School, Ploieşti, Romania; Bodhisattwa Bhowmik, RKMV, Agartala, Tripura, India; Arkady Alt, San Jose, CA, USA; Alok Kumar, Delhi, India; Adnan Ali, A.E.C.S-4, Mumbai, India; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Ali Baouan, Rabat, Morocco; Alyssa Hwang, Kent Place School Summit, NJ, USA; Woosung Jung, Korea International School, South Korea; Seung Hwan An, Taft School, Watertown, CT, USA; Kyoung A Lee, The Hotchkiss School, Lakeville, CT, USA; Chaeyeon Oh, Episcopal High School, Alexandra, VA, USA; William Kang, Bergen County Academies, Hackensack, NJ, USA; Radouan Boukharfane, Sidislimane, Morocco.

S308. Let n be a positive integer and let $\mathcal{G}_n$ be an $n \times n$ grid with the number 1 written in each of its unit squares. An operation consists of multiplying all entries of a column or all entries of a row by -1. Determine the number of distinct grids that can be obtained after applying a finite number of operations on $\mathcal{G}_n$.

Proposed by Marius Cavachi, Constanta, Romania

Solution by Daniel Lasaosa, Pamplona, Spain

Operations as defined are clearly commutative and associative. Note further that applying twice (or any even number of times) an operation over the same row, or over the same column, leaves the grid unchanged. Therefore, all possible sequences of a finite number of operations on $\mathcal{G}_n$ are equivalent to choosing any number of the n rows and n columns, and multiplying exactly the chosen rows and columns by -1, but not the rest. This results in *a priori* 2^{2n} distinct grids. However, note that two different choices of rows and columns may provide the same final grid. Indeed, we say that two sequences of operations are complementary if the rows and columns which are multiplied by -1 in the first sequence, are exactly those that are not multiplied by -1 in the second sequence. Note that applying two complementary sequences leaves the grid unchanged, since each number is multiplied by -1 exactly twice, once when its row is chosen, and once when its column is chosen. It follows that two complementary sequences yield the same grid.

Assume now that two sequences yield the same grid. For every position in the grid where a 1 appears, either both its row and column, or niether its row and column, were chose, whereas for every position where a -1 appears, either its row was chosen but not its column, or its column was chosen but not its row. It follows that, if a row was chosen in both sequences, then every column was either chosen in both sequences, or not chosen in both sequences, and similarly all rows, or both sequences are the same. On the other hand, if a row was chosen in one sequence but not in the other, then all columns chosen in the first sequence were not chosen in the second, and *vice versa*, and similarly with all rows, resulting in both sequences being complementary.

We conclude that for every grid, there are exactly two sequences that produce it, which are complementary, and the total number of distinct grids is therefore 2^{2n-1}.

Also solved by Erlang Wiratama Surya, Ipeka International, Indonesia; Li Zhou, Polk State College, Winter Haven, FL, USA; Kyoung A Lee, The Hotchkiss School, Lakeville, CT, USA; Seung Hwan An, Taft School, Watertown, CT, USA.

S309. Let $ABCD$ be a circumscriptible quadrilateral, which lies strictly inside a circle ω. Let ω_A be the circle outside of $ABCD$ that is tangent to AB, AD, and to ω at A'. Similarly, define B', C', D'. Prove that lines AA', BB', CC', DD' are concurrent.

Proposed by Khakimboy Egamberganov, Tashkent, Uzbekistan

Solution by the author

Let the circle k is incircle of $ABCD$ and point P is insimilicenter of the circles ω and k. Now, we will prove that the lines AA', BB', CC', DD' are pass through the point P.

For the line AA', we have :

$A^{'}$ is exsimilicenter of the circles ω and ω_A ;

A is insimilicenter of the circles k and ω_A ;

Since P is the insimilicenter of the circles ω and k and by Monge -D'Alemberts' circles theorem we can that A', A and P are collinear. The line AA' passes through P. Anologously, the lines BB', CC' and DD' are pass through P. Hence AA', BB', CC' and DD' are concurrent at the point P, which the insimilicenter of the circles k and ω and we are done.

Also solved by Li Zhou, Polk State College, Winter Haven, FL, USA; Yassine Hamdi, Lycée du Parc, Lyon, France.

S310. Let a, b, c be nonzero complex numbers such that $|a| = |b| = |c| = k$. Prove that

$$\sqrt{|-a+b+c|} + \sqrt{|a-b+c|} + \sqrt{|a+b-c|} \le 3\sqrt{k}.$$

Proposed by Marcel Chirita, Bucharest, Romania

Solution by Daniel Lasaosa, Pamplona, Spain

Since we may divide both sides by $\sqrt{k}$, it follows that we may assume wlog $k = 1$. Moreover, since we may rotate a, b, c by the same angle without altering the problem, we may assume wlog that $c = 1$, with $a = \cos\alpha + i \sin\alpha$, $b = \cos\beta + i\sin\beta$ for appropriately chosen angles $0 \le \alpha \le \beta < 2\pi$. Then,

$$|-a+b+c|^2 = (1+\cos\beta - \cos\alpha)^2 + (\sin\beta - \sin\alpha)^2 = 3 - 2\cos\alpha + 2\cos\beta - 2\cos(\alpha-\beta) =$$

$$= 3 - 4\sin\mu\sin\delta - 2\cos^2(\delta) + 2\sin^2\delta = 1 - 4\sin\mu\sin\delta + 4\sin^2\delta,$$

where we have defined $\mu = \frac{\alpha+\beta}{2}$ and $\delta = \frac{\beta-\alpha}{2}$. Similarly,

$$|a-b+c|^2 = 3 + 2\cos\alpha - 2\cos\beta - 2\cos(\alpha-\beta) = 1 + 4\sin\mu\sin\delta + 4\sin^2\delta,$$

$$|a+b-c|^2 = 3 - 2\cos\alpha - 2\cos\beta + 2\cos(\alpha-\beta) = 1 - 4\cos\mu\cos\delta + 4\cos^2\delta.$$

By the power mean inequality, we have

$$\sqrt{|-a+b+c|} + \sqrt{|a-b+c|} + \sqrt{|a+b-c|} = \sqrt[4]{1 - 4\sin\mu\sin\delta + 4\sin^2\delta} +$$

$$+ \sqrt[4]{1 + 4\sin\mu\sin\delta + 4\sin^2\delta} + \sqrt[4]{1 - 4\cos\mu\cos\delta + 4\cos^2\delta} \le$$

$$3\sqrt[4]{\frac{7 - 4\cos\mu\cos\delta + 4\sin^2\delta}{3}},$$

with equality iff

$$1 - 4\sin\mu\sin\delta + 4\sin^2\delta = 1 + 4\sin\mu\sin\delta + 4\sin^2\delta = 1 - 4\cos\mu\cos\delta + 4\cos^2\delta,$$

ie iff $\sin\mu\sin\delta = 0$ and $\cos\mu\cos\delta = \cos(2\delta)$. If $\sin\mu = 0$, then either $\mu = \alpha = \beta = 0$ and $\delta = 0$ yielding the bound given in the problem statement, or $\mu = \pi$ and $(2\cos\delta - 1)(\cos\delta + 1) = 0$. If $\cos\delta = -1$, then $\alpha = 0$ and $\beta = 2\pi$, yielding the same result as in the previous case, while if $\cos\delta = \frac{1}{2}$, then $\alpha = \frac{2\pi}{3}$ and $\beta = \frac{4\pi}{3}$.

Moreover, note that the maximum of the expression $7 - 4\cos\mu\cos\delta + 4\sin^2\delta$ is reached for values of μ, δ such that their cosines have opposite signs (or one of them is zero), otherwise we could add or subtract π to one of them, making the second term positive instead of negative, thus producing a larger value than the maximum. It follows that it suffices to find the maximum of $7 + 4\cos\delta + 4\sin^2\delta$. This expression has first and second derivatives with respect to δ which are respectively equal to

$$-4\sin\delta(1 - 2\cos\delta), \qquad -4\left(2 + \cos\delta - 4\cos^2\delta\right).$$

The first expression is zero for $\delta = 0$, in which case the second expression is positive, or a minimum is reached, and for $\delta = \frac{\pi}{3}$, in which case the second expression is negative, and the maximum of the expression is reached. Since $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, we conclude that

$$7 - 4\cos\mu\cos\delta + 4\sin^2\delta \le 7 + 4\cdot\frac{1}{2} + 4\cdot\frac{3}{4} = 12,$$

with equality iff $\alpha = \frac{2\pi}{3}$ and $\beta = \frac{4\pi}{3}$, which is also clearly a case of equality in the previous inequality. Restoring generality, we have

$$\sqrt{|-a+b+c|} + \sqrt{|a-b+c|} + \sqrt{|a+b-c|} \le 3\sqrt[4]{4}\sqrt{k} = 3\sqrt{2k},$$

with equality iff a, b, c are the vertices of an equilateral triangle centered at the origin of the complex plane. Note that this maximum is larger than the one proposed in the problem statement.

Also solved by Florin Stanescu, Cioculescu Serban High School, Gaesti, Romania; George Gavrilopoulos, Nea Makri High School, Athens, Greece; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Li Zhou Polk State College, FL; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy.

S311. Let n be a positive integer. Prove that

$$\prod_{j=0}^{\lfloor \frac{n}{2} \rfloor} (x+2j+1)^{\binom{n}{2j+1}} - \prod_{j=0}^{\lfloor \frac{n}{2} \rfloor} (x+2j)^{\binom{n}{2j}}$$

is a polynomial of degree $2^{n-1}-n$, whose highest term's coefficient is $(n-1)!$.

Proposed by Albert Stadler, Herrliberg, Switzerland

Solution by Khakimboy Egamberganov, Tashkent, Uzbekistan

Since $(x-1)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1} + ... + \binom{n}{n}x^0$ we have that at $x=1$

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j+1} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} = 2^{n-1}.$$

Let $P(x) = \prod_{j=0}^{\lfloor \frac{n}{2} \rfloor} (x+2j+1)^{\binom{n}{2j+1}}$ and $G(x) = \prod_{j=0}^{\lfloor \frac{n}{2} \rfloor} (x+2j)^{\binom{n}{2j}}$, $K_i(A(x))$ is i- coefficient of the polynomial $A(x)$, whose the coefficient of x^i. Since

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j+1} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j}$$

we have that

$$x^{\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j+1}} = x^{\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j}}$$

$$K_{\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j+1}}(P(x)) = K_{\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j}}(G(x)).$$

Let $f(x) = (x-1)^n$ is a function(polynomial) and $x=1$ is the root(n times) of the polynomial. Then $x=1$ be root of the polynomials $f'(x)$, $f''(x)$, ..., $f^{''..''}(x)$ - $(n-1)$-derivative of the $f(x)$ and n-derivative of the $f(x)$ is equal to $n!$. So $f(1) = f'(1) = ... = f^{''..''}(1) = 0$. Now, we can

$$\sum_{\alpha_1+\alpha_2+...+\alpha_{\lfloor \frac{n}{2} \rfloor}=k<n,(\alpha_i \geq 0)} \prod_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{\binom{n}{2j+1}}{\alpha_j} = \sum_{\alpha_1+\alpha_2+...+\alpha_{\lfloor \frac{n}{2} \rfloor}=k<n,(\alpha_i \geq 0)} \prod_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{\binom{n}{2j}}{\alpha_j}$$

and

$$\sum_{\alpha_1+\alpha_2+...+\alpha_{\lfloor \frac{n}{2} \rfloor}=n,(\alpha_i \geq 0)} \prod_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{\binom{n}{2j+1}}{\alpha_j} - \sum_{\alpha_1+\alpha_2+...+\alpha_{\lfloor \frac{n}{2} \rfloor}=n,(\alpha_i \geq 0)} \prod_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{\binom{n}{2j}}{\alpha_j} = (n-1)!.$$

Since

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j+1} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} = 2^{n-1},$$

we get that

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j+1} - n = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} - n = 2^{n-1} - n.$$

and we are done.

Also solved by Li Zhou, Polk State College, Winter Haven, FL, USA.

S312. Let a, b, c, d be positive real numbers such that $a^3+b^3+c^3+d^3=1$. Prove that

$$\frac{1}{1-bcd}+\frac{1}{1-bcd}+\frac{1}{1-bcd}+\frac{1}{1-bcd}\leq\frac{16}{3}.$$

Proposed by An Zhen-ping, Xianyang Normal University, China

Solution by Adnan Ali, A.E.C.S-4, Mumbai, India

From the AM-GM Inequality, we obtain that

$$abcd\leq\frac{1}{4\sqrt[3]{4}}\iff abc\leq\frac{1}{4\sqrt[3]{4}d}\iff\frac{1}{1-abc}\leq\frac{1}{1-\dfrac{1}{4\sqrt[3]{4}d}}$$

So, we obtain

$$\sum_{cyc}\frac{1}{1-abc}\leq\sum_{cyc}\frac{4\sqrt[3]{4}d}{4\sqrt[3]{4}d-1}$$

Hence it suffices to show that

$$\sum_{cyc}\frac{4\sqrt[3]{4}d}{4\sqrt[3]{4}d-1}\leq\frac{16}{3}\iff 4-\sum_{cyc}\frac{-1}{4\sqrt[3]{4}d-1}\leq\frac{16}{3}$$

$$\iff\sum_{cyc}\frac{-1}{4\sqrt[3]{4}d-1}\geq\frac{-4}{3}\iff\sum_{cyc}\frac{1}{1-4\sqrt[3]{4}d}\geq\frac{-4}{3}$$

So, from Cauchy-Schwartz Inequality(Titu's Lemma), we obtain

$$\sum_{cyc}\frac{1}{1-4\sqrt[3]{4}d}\geq\frac{16}{4-4\sqrt[3]{4}\,(a+b+c+d)}$$

Hence it suffices to show that

$$\frac{16}{4-4\sqrt[3]{4}\,(a+b+c+d)}\geq\frac{-4}{3}\iff\frac{1}{1-\sqrt[3]{4}\,(a+b+c+d)}\geq\frac{-1}{3}$$

$$\iff 3\geq-1+\sqrt[3]{4}\,(a+b+c+d)\iff 4\geq\sqrt[3]{4}\,(a+b+c+d)$$

$$\iff\sqrt[3]{16}\geq a+b+c+d.$$

The last inequality follows from Hölder Inequality as

$$(a^3+b^3+c^3+d^3)(1+1+1+1)(1+1+1+1)\geq(a+b+c+d)^3$$

$$\iff\sqrt[3]{16}\geq a+b+c+d.$$

Hence the conclusion follows and equality holds for $a=b=c=d=\dfrac{1}{\sqrt[3]{4}}$.

Also solved by Marius Stanean, Zalau, Romania; Navid Safei, Sharif University of Technology, Tehran, Iran; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy.

Undergraduate problems

U307. Prove that any polynomial $f \in \mathbb{R}[X]$ can be written as a difference of increasing polynomials.

Proposed by Jishnu Bose, Calcutta, India

Solution by G.R.A.20 Problem Solving Group, Roma, Italy

Let $h(x) = (2d+1)x^{2d} + f'(x)$ where d is the degree of f. If $d > 0$ then $\lim_{x\to\pm\infty} h(x) = +\infty$, while if $d = 0$ then $h \equiv 1$. In either case the polynomial h attains a minimum value a.

Let $Q(x) = x^{2d+1} + (|a|+1)x$ and $P(x) = Q(x) + f(x)$ then P and Q are polynomials such that $P - Q = f$ and

$$Q'(x) = (2d+1)x^{2d} + (|a|+1) \geq 1 > 0,$$
$$P'(x) = Q'(x) + f'(x) = h(x) + (|a|+1) \geq 1 > 0,$$

so they are increasing.

Also solved by Daniel Lasaosa, Pamplona, Spain; Li Zhou, Polk State College, Winter Haven, FL, USA; Khakimboy Egamberganov, Tashkent, Uzbekistan; Moubinool Omarjee, Lycée Henri IV, Paris, France; Alessandro Ventullo, Milan, Italy; Radouan Boukharfane, Sidislimane, Morocco; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy.

U308. Let $a_1, b_1, c_1, a_2, b_2, c_2$ be positive real numbers. Consider the functions $X(x,y)$ and $Y(x,y)$ which satisfy the functional equations

$$\frac{x}{X} = 1 + a_1x + b_1y + c_1Y$$
$$\frac{y}{Y} = 1 + a_2x + b_2y + c_2X.$$

Prove that if $0 < x_1 \leq x_2$ and $0 < y_2 \leq y_1$, then $X(x_1, y_1) \leq X(x_2, y_2)$ and $Y(x_1, y_1) \geq Y(x_2, y_2)$.

Proposed by Razvan Gelca, Texas Tech University, USA

Solution by the author

We will show first that if $0 < x_1 \leq x_2$ then $X(x_1, y) \leq X(x_2, y)$ and $Y(x_1, y) \geq Y(x_2, y)$. Set $1 + b_1y = d_1$, $1 + b_2y = d_2$. Rewrite the system as

$$a_1xX + c_1XY + d_1X = x$$
$$a_2xY + c_2XY + d_2Y = y.$$

Differentiating with respect to x and rearranging the terms we obtain

$$(a_1x + c_1Y + d_1)X' + (c_1X)Y' = 1 - a_1X$$
$$(c_2Y)X' + (a_2x + c_2X + d_2)Y' = -a_2Y.$$

Solving for the unknowns X and Y we obtain

$$X' = \frac{\begin{vmatrix} 1 - a_1X & c_1Y \\ -a_2Y & a_2x + c_2X + d_2 \end{vmatrix}}{\begin{vmatrix} a_1x + c_1Y + d_1 & c_1X \\ c_2Y & a_2x + c_2X + d_2 \end{vmatrix}}$$
$$\frac{(1 - a_1X)(a_2x + c_2X + d_2) + c_1b_2XY}{a_1a_2x^2 + a_1c_2xX + a_1xd_2 + a_2c_1xY + c_1d_2Y + d_1a_2x + d_1c_2X + d_1d_2}.$$

The denominator is positive. The $(1 - a_1x)$ in the numerator equals $\frac{1}{x}(c_1XY + d_1X)$, which is also positive. This shows that $X' > 0$. Because

$$Y = \frac{y}{d_2 + a_2x + c_2X}$$

it follows that Y is decreasing.

Exchanging the roles of x, y respectively X, Y we deduce that if $0 < x_1 = x_2 = x$ and $0 < y_2 \leq y_1$ then $X(x, y_1) \leq X(x, y_2)$ and $Y(x, y_1) \geq Y(x, y_2)$.

So for x_1, x_2, y_1, y_2 as specified in the statement we have

$$X(x_1, y_1) \leq X(x_2, y_1) \leq X(x_2, y_2) \quad \text{and} \quad Y(x_1, y_1) \geq Y(x_2, y_1) \geq Y(x_2, y_2),$$

as desired.

U309. Let $a_1, \ldots, a_n$ be positive real numbers such that $a_1 + \cdots + a_n = 1$, $n \geq 2$. Prove that for every positive integer m,

$$\sum_{k=1}^{n} \frac{{a_k}^{m+1}}{1 - {a_k}^m} \geq \frac{1}{n^m - 1}.$$

Proposed by Titu Zvonaru, Comanesti and Neculai Stanciu, Romania

Solution by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

Function $f(x) = \frac{x^{m+1}}{1-x^m}$ is convex for $x \in (0,1)$ because

$$f''(x) = \frac{2m^2 x^(-1+3m)}{(1-x^m)^3} + \frac{2m^2 x^{-1+2m})}{(1-x^m)^2} + \frac{m(1+m)x^{-1+2m}}{(1-x^m)^2} + \frac{m(1+m)x^{-1+m}}{1-x^m} > 0$$

for $x \in (0,1)$

Therefore, by Jensen's inequality

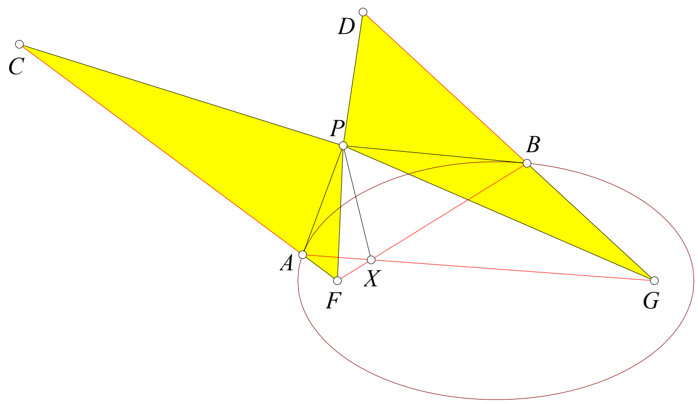
$$\sum_{k=1}^{n} \frac{a_k^{m+1}}{1-a_k^m} \geq nf\left(\frac{\sum \frac{a_k}{n}}{\sum \frac{1}{n}}\right) = nf\left(\frac{1}{n}\right) = n\ \frac{\left(\frac{1}{n}\right)^{m+1}}{1-\left(\frac{1}{n}\right)^m} = \frac{1}{n^m - 1}.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Prithwijit De, HBCSE, Mumbai, India; Titouan Morvan, Lyée Millet, France; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Khakimboy Egamberganov, Tashkent, Uzbekistan; Bodhisattwa Bhowmik, RKMV, Agartala, Tripura, India; Arkady Alt, San Jose, CA, USA; Adnan Ali, A.E.C.S-4, Mumbai, India; Li Zhou, Polk State College, Winter Haven, FL, USA; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Ali Baouan, Rabat, Morocco; Moubinool Omarjee, Lycée Henri IV, Paris, France; Radouan Boukharfane, Sidislimane, Morocco; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Prasanna Ramakrishnan, International School of Port of Spain, Trinidad and Tobago.

U310. Let $\mathcal{E}$ be an ellipse with foci F and G, and let P be a point in its exterior. Let A and B be the points where the tangents from P to $\mathcal{E}$ intersect $\mathcal{E}$, such that A is closer to F. Furthermore, let X be the intersection of AG with BF. Prove that XP bisects $\angle AXB$.

Proposed by Jishnu Bose, Calcutta, India

Solution by Li Zhou, Polk State College, Winter Haven, FL, USA



Locate C on the ray FA and D on the ray GB such that $AC = AG$ and $BD = BF$. Since PA is tangent to $\mathcal{E}$, $\angle PAC = \angle PAG$, so $\triangle PAC \cong \triangle PAG$. Hence $PC = PG$. Likewise $PD = PF$. Also, $FC = FA + AC = FA + AG = GB + BF = GB + BD = GD$, thus $\triangle PCF \cong \triangle PGD$. Therefore $\angle CFP = \angle GDP = \angle BFP$, which implies that P is the excenter, opposite F, of $\triangle AFX$. Hence PX bisects $\angle AXB$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Radouan Boukharfane, Sidislimane, Morocco; Prasanna Ramakrishnan, International School of Port of Spain, Trinidad and Tobago.

U311. Let $f : [0,1] \to [0,1]$ be a nondecreasing concave function such that $f(0) = 0$ and $f(1) = 1$. Prove that

$$\int_0^1 \left(f(x)f^{-1}(x)\right)^2 dx \geq \frac{1}{12}.$$

Proposed by Marcel Chirita, Bucharest, Romania

Solution by Daniel Lasaosa, Pamplona, Spain

Let $f(x) = x^{\frac{1}{k}}$ for some positive real $k > 1$, which clearly satisfies the conditions given in the problem statement. Note that $f^{-1}(x) = x^k$, or

$$\int_0^1 \left(f(x)f^{-1}(x)\right)^2 dx = \int_0^1 x^{\frac{2k^2+2}{k}} dx = \frac{k}{2k^2+k+2}\, x^{\frac{2k^2+k+2}{k}}\Big|_0^1 = \frac{k}{2k^2+k+2},$$

which can be made as small as desired by letting k be as large as needed. It follows that the proposed result is not necessarily true.

Also solved by Li Zhou, Polk State College, Winter Haven, FL, USA; Moubinool Omarjee, Lycée Henri IV, Paris, France; Radouan Boukharfane, Sidislimane, Morocco; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy.

U312. Let p be a prime and let R be a commutative ring with characteristic p. Prove that the sets $S_k = \{x \in R | x^p = k\}$, where $k \in \{1, \ldots, p\}$, have the same number of elements.

Proposed by Corneliu Manescu-Avram, Ploiesti, Romania

Solution by Alessandro Ventullo, Milan, Italy

If for any $x \in R$ and $k \in \{1, \ldots, p\}$ it holds $x^p \neq k$, then there is nothing to prove. Assume that there is some $x \in R$ such that $x^p = k$ for some $k \in \{1, \ldots, p\}$. Hence, $k \in R$. Consider the set $S = \{x + k \mid x \in R\}$. Observe that the mapping $\varphi : R \longrightarrow S$ defined by $\varphi(x) = x + k$ is bijective. Indeed, if $\varphi(x) = \varphi(y)$, then $x + k = y + k$, i.e. $x = y$ and this proves that φ is injective. If $y \in S$, then $y = x + k$ for some $x \in R$, so it's enough to take $x = y - k \in R$ in order to have $y = \varphi(x)$, and this proves that φ is surjective. Let $n \in \{1, \ldots, p\}$ and consider the set $S_n = \{x \in R \mid x^p = n\}$. Since $(x + k)^p = x^p + k^p = x^p + k$, then

$$\varphi(S_n) = \begin{cases} S_{n-k} & \text{if } n > k \\ S_{p-k+n} & \text{if } k \leq n. \end{cases}$$

It follows that $|S_n| = |S_{n-k}|$ or $|S_n| = |S_{p-k+n}|$. By arbitrarily of $n \in \{1, \ldots, p\}$, we conclude that $|S_1| = |S_2| = \ldots = |S_p|$.

Also solved by Daniel Lasaosa, Pamplona, Spain.

O307. Let a, b, c, d be positive real numbers such that $a+b+c+d=4$. Prove that

$$\frac{1}{a+3}+\frac{1}{b+3}+\frac{1}{c+3}+\frac{1}{d+3} \leq \frac{1}{abcd}.$$

Proposed by An Zhen-ping, Xianyang Normal University, China

Solution by Daniel Lasaosa, Pamplona, Spain

Denote $s_1=a+b+c+d=4$, $s_2=ab+bc+cd+da+ac+bd$, $s_3=abc+bcd+cda+dab$, $s_4=abcd$. After multiplying both sides by $s_4(a+3)(b+3)(c+3)(d+3)$, and rearranging terms, the inequality is equivalent to

$$3\left(s_2-6\sqrt{s_4}\right)+2\left(s_3-4\sqrt[4]{s_4{}^3}\right)+18\sqrt{s_4}\left(1-\sqrt{s_4}\right)+8\sqrt[4]{s_4{}^3}\left(1-\sqrt[4]{s_4}\right)+$$

$$+\left(81+27s_1+6s_2+s_3\right)\left(1-s_4\right) \geq 0.$$

Now, by the AM-GM inequality, we have

$$1=\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}=\sqrt[4]{s_4},$$

$$\frac{s_2}{6}=\frac{ab+bc+cd+da+ac+bd}{6} \geq \sqrt[6]{a^3b^3c^3d^3}=\sqrt{s_4},$$

$$\frac{s_3}{4}=\frac{abc+bcd+cda+dab}{4} \geq \sqrt[4]{a^3b^3c^3d^3}=\sqrt[4]{s_4{}^3},$$

or clearly all terms in the LHS are non-negative, being all simultaneously zero iff equality holds in all the AM-GM inequalities, ie iff $a=b=c=d=1$. The conclusion follows.

Also solved by Ioan Viorel Codreanu, Satulung, Maramures, Romania; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Adnan Ali, A.E.C.S-4, Mumbai, India; Arkady Alt, San Jose, CA, USA; Khakimboy Egamberganov, Tashkent, Uzbekistan; Marius Stanean, Zalau, Romania; Navid Safei, Sharif University of Technology, Tehran, Iran; Radouan Boukharfane, Sidislimane, Morocco; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy.

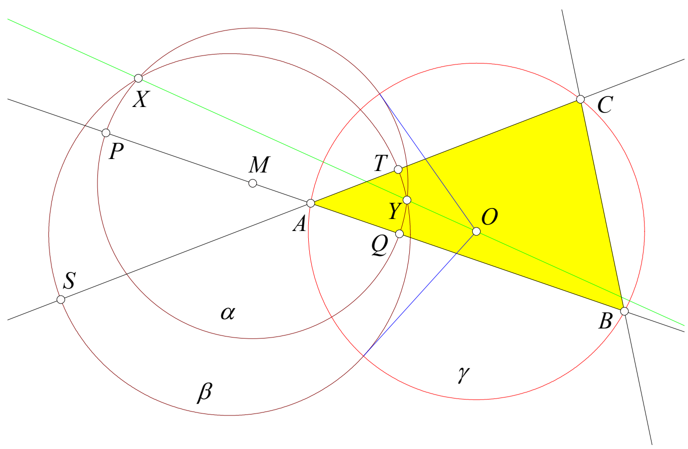
O308. Let ABC be a triangle and let X, Y be points in its plane such that

$$AX : BX : CX = AY : BY : CY.$$

Prove that the circumcenter of triangle ABC lies on the line XY.

Proposed by Cosmin Pohoata, USA and Josef Tkadlec, Czech Republic

Solution by Li Zhou, Polk State College, Winter Haven, FL, USA



Let $u = AX : BX = AY : BY$ and $v = AX : CX = AY : CY$. Then the locus of all points U such that $AU : BU = u$ is the circle α of diameter PQ, where P, Q are on AB with $PA : PB = AQ : QB = u$. Likewise, the locus of all points V such that $AV : CV = v$ is the circle β of diameter ST, where S, T are on AC with $SA : SC = AT : TC = v$. Let M be the midpoint of PQ. Then

$$(PM + MA)(MB - PM) = PA \cdot QB = PB \cdot AQ = (PM + MB)(PM - MA),$$

that is, $PM^2 = MA \cdot MB$. Thus α is orthogonal to the circumcircle γ of $\triangle ABC$. Likewise, β is orthogonal to γ. Therefore, the center O of γ is on the radical axis XY of α and β.

Also solved by Daniel Lasaosa, Pamplona, Spain; Khakimboy Egamberganov, Tashkent, Uzbekistan; Prasanna Ramakrishnan, International School of Port of Spain, Trinidad and Tobago.

O309. Determine the least real number μ such that

$$\mu\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)+\sqrt{a^2+b^2+c^2}\geq a+b+c$$

for all nonnegative real numbers a,b,c with $ab+bc+ca>0$. Find when equality holds.

Proposed by Albert Stadler, Herrliberg, Switzerland

Solution by Daniel Lasaosa, Pamplona, Spain

Let $\mathcal{R}$ be the region of three-dimensional space defined by the curve $g(a,b,c)=\sqrt{ab}+\sqrt{bc}+\sqrt{ca}=k$ for some positive real k, where moreover at most one of a,b,c is zero, and a,b,c are all non-negative. The problem is equivalent to findind the maximum of $f(a,b,c)=a+b+c-\sqrt{a^2+b^2+c^2}$, and dividing this maximum by k, thus yielding the minimum value of μ which satisfies the proposed inequality. Given the nature of the region $\mathcal{R}$, we need to analyze wlog (by symmetry in the variables) the following cases: (1) $a\geq b>0$ and $c=0$; (2) the limit of $f(a,b,c)$ when $b,c\to 0$; (3) the interior of region $\mathcal{R}$, ie a,b,c all positive.

(1) We have

$$\mu\geq\frac{a+b-\sqrt{a^2+b^2}}{\sqrt{ab}}.$$

Taking $a=b$, note that $\mu\geq 2-\sqrt{2}$, or it suffices to show that for all positive reals a,b, the following inequality holds:

$$\left(\sqrt{a}-\sqrt{b}\right)^2\leq\frac{\left(\sqrt{a}+\sqrt{b}\right)^2\left(\sqrt{a}-\sqrt{b}\right)^2}{\sqrt{a^2+b^2}+\sqrt{2ab}}.$$

In turn, and unless $a=b$ (in which case the inequality clearly holds with equality), it suffices to prove that

$$\sqrt{a^2+b^2}+\sqrt{2ab}\leq a+b+2\sqrt{ab},$$

clearly true and strict since a,b are positive reals, hence $(a+b)^2>a^2+b^2$. It follows that $\mu\geq 2-\sqrt{2}$, and the inequality holds for $\mu=2-\sqrt{2}$.

(2) If $b,c\to 0$ for fixed k, then $a\to\infty$, while

$$\sqrt{a^2+b^2+c^2}=a+\frac{b^2+c^2}{2a}+O\left(b^4\right)+O\left(c^4\right),$$

or

$$\lim_{b,c\to 0}\mu=\lim_{b,c\to 0}\frac{a+b+c-a-\frac{b^2+c^2}{2a}}{\sqrt{a}\left(\sqrt{b}+\sqrt{c}\right)+\sqrt{bc}}=\lim_{b,c\to 0}\frac{b+c}{\sqrt{a}\left(\sqrt{b}+\sqrt{c}\right)}=0,$$

or the proposed inequality holds for any positive μ under these conditions.

(3) For the interior of $\mathcal{R}$, we may use Lagrange's multiplier method, or a constant λ to be determined exists, such that

$$1-\frac{a}{\sqrt{a^2+b^2+c^2}}=\lambda\frac{\sqrt{b}+\sqrt{c}}{2\sqrt{a}},$$

and similarly for the cyclic permutations of a,b,c. If $a\neq b$ when $f(a,b,c)$ reaches its maximum for fixed k, then subtracting their corresponding equalities and dividing by $(\sqrt{a}-\sqrt{b})^2$ (clearly nonzero) yields

$$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a^2+b^2+c^2}}=\lambda\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{2\sqrt{ab}}.$$

Therefore, if a,b,c are all distinct and positive, it follows that

$$2\sqrt{ab}\left(\sqrt{a}+\sqrt{b}\right)=\sqrt{a^2+b^2+c^2}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)=2\sqrt{bc}\left(\sqrt{b}+\sqrt{c}\right),$$

or equivalently

$$\left(\sqrt{a}-\sqrt{c}\right)\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)=0,$$

contradiction. Or at least two of a,b,c are equal when $f(a,b,c)$ reaches its maximum in the interior of $\mathcal{R}$. If a,b,c are all equal, then $g(a,b,c)=k=3a$, and $f(a,b,c)=a(3-\sqrt{3})$, yielding $\mu\geq 1-\frac{1}{\sqrt{3}}$. Note that $1-\frac{1}{\sqrt{3}}<2-\sqrt{2}$ is equivalent to $\sqrt{24}<2+\sqrt{12}$, clearly true because $\sqrt{24}<5$ and $\sqrt{12}>3$, hence if the maximum inside $\mathcal{R}$ occurs when $a=b=c$, then μ is determined by case (1). Otherwise, if a maximum occurs in the interior of $\mathcal{R}$ which supersedes case (1), it happens when wlog $a\neq b=c$. Denote therefore $\Delta=2\sqrt{ab}$, or $k=b+\Delta$, $a+b+c=\frac{\Delta^2+8b^2}{4b}$, and $a^2+b^2+c^2=\frac{\Delta^4+32b^4}{16b^2}$, or

$$\mu\geq\frac{\Delta^2+8b^2-\sqrt{\Delta^4+32b^4}}{4b(b+\Delta)}.$$

Assume that the RHS is larger than $2-\sqrt{2}$ for some combination of b,Δ. Then,

$$\Delta^2-4(2-\sqrt{2})b\Delta+4\sqrt{2}b^2>\sqrt{\Delta^4+32b^4},$$

$$8\sqrt{2}(\sqrt{2}-1)b\Delta\left(\Delta^2-(5-\sqrt{2})b\Delta+4\sqrt{2}b^2\right)<0.$$

The quadratic expression in the LHS needs therefore to be negative, or its discriminant must be positive, ie

$$16\sqrt{2}<(5-\sqrt{2})^2=29-10\sqrt{2},\qquad\qquad 26\sqrt{2}<29,$$

clearly false. Therefore, no point inside $\mathcal{R}$ produces $\mu>2-\sqrt{2}$

We conclude that the minimum allowed value of μ is $2-\sqrt{2}$, in which case equality is reached in the proposed inequality iff one of a,b,c is zero, and the other two are equal, ie iff (a,b,c) is a permutation of $(k,k,0)$ for some positive real k.

Also solved by Bodhisattwa Bhowmik, RKMV, Agartala, Tripura, India; Arkady Alt, San Jose, CA, USA; Khakimboy Egamberganov, Tashkent, Uzbekistan; Radouan Boukharfane, Sidislimane, Morocco; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy.

O310. Let ABC be a triangle and let P be a point in its interior. Let X, Y, Z be the intersections of AP, BP, CP with sides BC, CA, AB, respectively. Prove that

$$\frac{XB}{XY} \cdot \frac{YC}{YZ} \cdot \frac{ZA}{ZX} \le \frac{R}{2r}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Note first that, because of Ceva's theorem, we have

$$XB \cdot YC \cdot ZA = XC \cdot YA \cdot ZB,$$

while as it is well known, $r = 4R \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$, or squaring the proposed inequality and substituting these results, the problem is equivalent to

$$\frac{YA \cdot ZA}{YZ^2} \cdot \frac{ZB \cdot XB}{ZX^2} \cdot \frac{XC \cdot YC}{XY^2} = \frac{XB^2}{XY^2} \cdot \frac{YC^2}{YZ^2} \cdot \frac{ZA^2}{ZX^2} \le \frac{R^2}{4r^2} = \frac{1}{64 \sin^2\frac{A}{2} \sin^2\frac{B}{2} \sin^2\frac{C}{2}},$$

and by cyclic symmetry, it suffices to show that

$$YZ^2 \ge 4YA \cdot ZA \sin^2\frac{A}{2} = 2YA \cdot ZA(1 - \cos A).$$

Now, by the Cosine Law, $YZ^2 = YA^2 + ZA^2 - 2YA \cdot ZA \cos A$, or the inequality is clearly true by the AM-GM, with equality iff $YA = ZA$. The conclusion follows, equality holds iff simultaneously $YA = ZA$, $ZB = XB$ and $XC = YC$, or iff X, Y, Z are the points where the incircle of ABC is tangent to its sides, ie iff P is the Gergonne point of ABC.

Also solved by Ioan Viorel Codreanu, Satulung, Maramures, Romania; Arkady Alt, San Jose, CA, USA; Khakimboy Egamberganov, Tashkent, Uzbekistan; Neculai Stanciu, Buzău, Romania; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Li Zhou, Polk State College, Winter Haven, FL, USA.

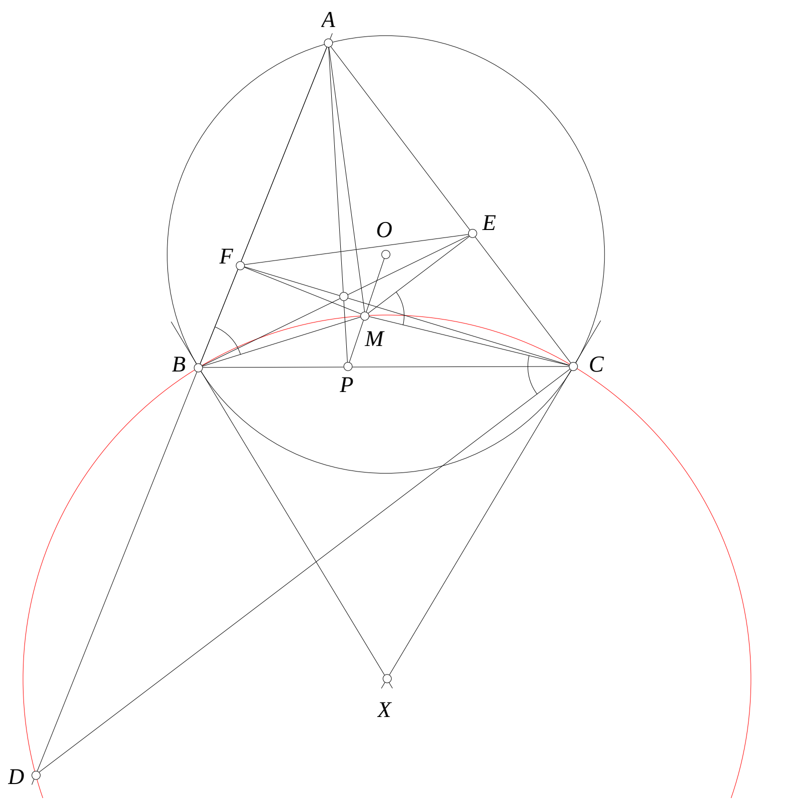
O311. Let ABC be a triangle with circumcircle Γ centered at O. Let the tangents to Γ at vertices B and C intersect each other at X. Consider the circle χ centered at X with radius XB, and let M be the point of intersection of the internal angle bisector of angle A with χ such that M lies in the interior of triangle ABC. Denote by P the intersection of OM with the side BC and by E and F be the orthogonal projections of M on CA and AB, respectively. Prove that PE and FP are perpendicular.

Proposed by Cosmin Pohoata, Princeton University, USA

Solution by Ercole Suppa, Teramo, Italy

We begin with a preliminary result.

Lemma. The lines BE, CF and AP are concurrent.
Proof.



Let D be the second intersection point between AB and χ. We have

$$\angle BDC = \frac{1}{2} \cdot \angle BXC = \frac{1}{2}\left(180^\circ - 2A\right) = 90^\circ - A \quad \Rightarrow \quad CA \perp CD$$

Therefore $CD \parallel ME$ so, from the cyclic quadrilateral $BMCD$ we get

$$\angle MBF = \angle MCD = \angle CME \quad \Rightarrow \quad \triangle MBF \sim \triangle CME \quad \Rightarrow$$
$$\frac{MB}{MC} = \frac{MF}{CE} = \frac{BF}{ME} \quad \Rightarrow \quad \left(\frac{MB}{MC}\right)^2 = \frac{MF}{CE} \cdot \frac{BF}{ME} = \frac{BF}{CE} \qquad (1)$$

On the other hand, since OM is the M-symmedian of $\triangle MBC$ we have

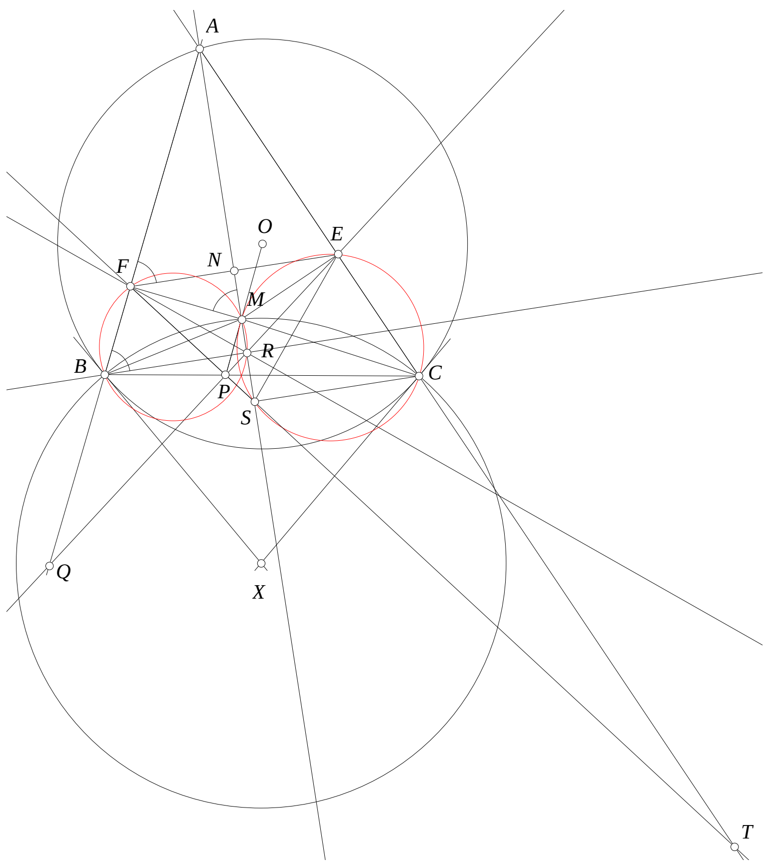
$$\left(\frac{MB}{MC}\right)^2 = \frac{PB}{PC} \tag{2}$$

From (1) and (2) it follows that

$$\frac{PB}{PC}\cdot\frac{EC}{FB} = 1 \qquad \Rightarrow \qquad \frac{PB}{PC}\cdot\frac{EC}{EA}\cdot\frac{FA}{FB} = 1$$

and this implies that AP, BE, CF are concurrent by Ceva's theorem. ■

Returning to initial problem, let $Q = PE \cap AB$, $R = PE \cap AM$, $S = PF \cap AM$, $T = PF \cap AC$, $N = EF \cap AM$, as shown in figure.



By the previous lemma BE, CF, AP are concurrent so $(A, B; F, Q) = -1$ and the pencil $R(A, B, F, Q)$ is harmonic. Since $FN = NE$, using a well known property, we have $BR \parallel FE$ so

$$\angle FBR = \angle AFE = \angle FMA$$

Therefore the quadrilateral $BRMF$ is cyclic, so

$$\angle MBF = \angle MRF = \angle NRF = \angle NRE \tag{3}$$

Similarly we have $S(A, C, E, T) = -1$, $CS \parallel FE$, $CEMS$ is cyclic and

$$\angle MCE = \angle MSE = \angle MSF = \angle NSP \tag{4}$$

From (3) and (4), since $\angle NRE + \angle NER = 90^\circ$ and $\angle MBF + \angle MCE = 90^\circ$ (by Lemma), we have

$$\begin{aligned} \angle NRE + \angle NER = \angle MBF + \angle MCE \qquad \Rightarrow \\ \angle NEP = \angle NER = \angle MCE = \angle MSE = \angle NSP \end{aligned} \tag{5}$$

hence the quadrilateral $ENPS$ is cyclic.

Therefore $\angle SPE = \angle SNE = 90^\circ$, i.e. $PE \perp PF$ and we are done.

Also solved by Khakimboy Egamberganov, Tashkent, Uzbekistan; Sebastiano Mosca, Pescara, Italy; Radouan Boukharfane, Sidislimane, Morocco; Prasanna Ramakrishnan, International School of Port of Spain, Trinidad and Tobago.

O312. Find all increasing bijections $f : (0, \infty) \to (0, \infty)$ satisfying

$$f(f(x)) - 3f(x) + 2x = 0$$

and for which there exists $x_0 > 0$ such that $f(x_0) = 2x_0$.

Proposed by Razvan Gelca, Texas Tech University, USA

Solution by Khakimboy Egamberganov, Tashkent, Uzbekistan

Suppose that there is $\xi > 0$ such that $f(\xi) \neq 2\xi$. We have $f(f(\xi)) - 3f(\xi) + 2\xi = 0$ and

$$f^{n+1}(\xi) - 3f^n(\xi) + 2f^{n-1}(\xi) = 0,$$

where $f^n(x) = f(f(..f(f(x))..))$, n times f.

Let $f^n(\xi) = a_n$, and $(a_n)_{n\in\mathbb{Z}}$ is positive sequence. By Characteristic equation for a_n, we get that $t^2 - 3t + 2 = 0$ and $t_{1,2} = \{1, 2\}$. So

$$f^n(\xi) = a_n = A + B \cdot 2^n \qquad (1)$$

for all integer $n \in \mathbb{Z}$ and some constant real numbers A and B. Why we said $n \in \mathbb{Z}$? Because, us given that f is bijective function. So $f^{-1}(x)$ exists and satisfies the condition (1). Similarly, the $f^{-n}(x)$ also exists.

Clearly, $B > 0$ and we will prove that $A = 0$. If $A < 0$ then there is a $n_0 \in \mathbb{Z}$ such that $f^{n_0}(\xi) = A + B \cdot 2^{n_0} \leq 0$, for example $n_0 \to -\infty$. Therefore, $A \geq 0$.

Assume that $A > 0$. By the condition, there exists $x_0 > 0$ such that $f(x_0) = 2x_0$. We can see that there exist infinitely many $x > 0$ such that $f(x) = 2x$, it's not hard look at the (1). For x_0 also there is one linear equation similar to (1) and there will be $A = 0$,

$$f^n(x_0) = 2^n x_0$$

for all $n \in \mathbb{Z}$.

So there exists $x_0 > 0$ such that satisfies the condition $f(x_0) = 2x_0$ and $x_0 < 2A$. If $x_0 \leq A$ then there exists $n_0 \in \mathbb{Z}$ such that $x_0 < A + B \cdot 2^{n_0} < x_0 + A$. If $A < x_0 < 2A$ then

$$(\log_2 x_0 - \log_2 B) - (\log_2(x_0 - A) - \log_2 B) = \log_2 \frac{x_0}{x_0 - A} > 1$$

and there exists $n_0 \in \mathbb{Z}$ such that $\log_2(x_0 - A) - \log_2 B < n_0 < \log_2 x_0 - \log_2 B$ and $x_0 < A + B \cdot 2^{n_0} < x_0 + A$. Hence there exists $n_0 \in \mathbb{Z}$ such that

$$x_0 < A + B \cdot 2^{n_0} < x_0 + A.$$

Let $\alpha = A + B \cdot 2^{n_0}$. Since the f is increasing bijection, we get that $2x_0 < f(x_0) < f(\alpha) = A + B \cdot 2^{n_0+1} = 2\alpha - A$ and $x_0 + \frac{A}{2} < \alpha$. So $2^m x_0 = f^m(x_0) < f^m(\alpha) = 2^m \alpha - (2^m - 1) \cdot A$ for all $m \in \mathbb{Z}$ and as $m \to +\infty$

$$\alpha > \lim_{m\to\infty} (x_0 + \frac{2^m - 1}{2^m} \cdot A),$$

$$\alpha \geq x_0 + A$$

and a contradiction.

Hence $A = 0$ and $f(\xi) = 2\xi$ again a contradiction. There isn't $\xi > 0$ such that $f(\xi) \neq 2\xi$ and for all $\xi > 0$ we get that $f(\xi) = 2\xi$. The solution of the equation

$$f(f(x)) - 3f(x) + 2x = 0$$

is $f(x) = 2x$ for all $x > 0$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Henry Ricardo, New York Math Circle; Navid Safei, Sharif University of Technology, Tehran, Iran; Radouan Boukharfane, Sidislimane, Morocco; Moubinool Omarjee, Lycée Henri IV, Paris, France.