Triangle Identities via Elimination Theory

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Abstract

The aim of this paper is to present some new identities via resultants and elimination theory, by presenting the sidelengths, the lengths of the altitudes and the exradii of a triangle as roots of cubic polynomials.

1 Introduction

Finding identities between elements of a triangle represents a classical topic in elementary geometry. In *Recent Advances in Geometric Inequalities* [3], Mitrinovic et al presents the sidelengths, the lengths of the altitudes and the various radii as roots of cubic polynomials, and collects many interesting equalities and inequalities between these quantities. As far as we know, almost of them are built using only elementary algebra and geometry. In this paper, we use transformations of rational functions and resultants to build similar nice equalities and inequalities might be difficult to prove with different methods.

2 Main results

To fix notations, suppose we are given a triangle ΔABC with sidelengh a, b, c. Denote the radius of the circumcircle by R, the radius of incircle by r, the area of ΔABC by S, the semiperimeter as p, the radii of the excircles as r_1, r_2, r_3, and the altitudes from sides a, b and c, respectively, as h_a, h_b and h_c. We have the results from [3, Chapter 1].

Theorem 1. [3] *Using the above notations,*

(i) a, b, c *are the roots of the cubic polynomial*

$$x^3 - 2px^2 + (p^2 + r^2 + 4Rr)x - 4Rrp. \tag{1}$$

(ii) h_a, h_b, h_c *are the roots of the cubic polynomial*

$$x^3 - \frac{S^2 + 4Rr^3 + r^4}{2Rr^2}x^2 + \frac{2S^2}{Rr}x - \frac{2S^2}{R}. \tag{2}$$

(iii) r_1, r_2, r_3 *are the roots of the cubic polynomial*

$$x^3 - (4R + r)x^2 + p^2x - p^2r. \tag{3}$$

Next, we present how to use transformations of rational functions to build new geometric equalities and inequalities. The transformation that we use here is given by $y = \dfrac{ax + b}{cx + d}, a, b, c, d \in \mathbb{R}$.

Proposition 2. *Let*

$$\begin{aligned}
T_1 &= \frac{3r_1 - 2r}{2r_1 + r} + \frac{3r_2 - 2r}{2r_2 + r} + \frac{3r_3 - 2r}{2r_3 + r} \\
T_2 &= \frac{3r_1 - 2r}{2r_1 + r}\frac{3r_2 - 2r}{2r_2 + r} + \frac{3r_2 - 2r}{2r_2 + r}\frac{3r_3 - 2r}{2r_3 + r} + \frac{3r_3 - 2r}{2r_3 + r}\frac{3r_1 - 2r}{2r_1 + r} \\
T_3 &= \frac{3r_1 - 2r}{2r_1 + r}\frac{3r_2 - 2r}{2r_2 + r}\frac{3r_3 - 2r}{2r_3 + r}.
\end{aligned}$$

We have
$$T_3 = \frac{3r_1-2r}{2r_1+r}\frac{3r_2-2r}{2r_2+r}\frac{3r_3-2r}{2r_3+r} \geqslant \frac{100r^2+9p^2}{19r^2+12p^2}.$$

Proof. Let $y_i = \dfrac{3r_i-2r}{2x+r}, i=1,2,3$ we have $y_i \neq \dfrac{3}{2}$ and $r_i = \dfrac{r(y_i+2)}{3-2y_i}$. Since r_1, r_2, r_3 are the roots of the polynomial (3), if we substitute r_i into the polynomial (3), we obtain

$$(3r^2+8Rr+12p^2)y_i^3 + (11r^2+20Rr-40p^2)y_i^2 + (8r^2-16Rr+39p^2)y_i - (4r^2+48Rr+9p^2) = 0;$$

therefore y_1, y_2, y_3 are the roots of

$$(3r^2+8Rr+12p^2)y^3 + (11r^2+20Rr-40p^2)y^2 + (8r^2-16Rr+39p^2)y - (4r^2+48Rr+9p^2).$$

By Viete's formula, we have

$$\begin{aligned} T_1 &= y_1+y_2+y_3 = \frac{40p^2-20Rr-11r^2}{3r^2+8Rr+12p^2} \\ T_2 &= y_1y_2+y_2y_3+y_3y_1 = \frac{8r^2-16Rr+39p^2}{3r^2+8Rr+12p^2} \\ T_3 &= y_1y_2y_3 = \frac{4r^2+48Rr+9p^2}{3r^2+8Rr+12p^2}. \end{aligned}$$

Let $f(t) = \dfrac{4r^2+48rt+9p^2}{3r^2+8rt+12p^2}$ then f is an increasing function over $[2r, +\infty]$. Therefore, from $R \geq 2r$ we have $T_3 \geq f(2r)$ i.e
$$\frac{3r_1-2r}{2r_1+r}\frac{3r_2-2r}{2r_2+r}\frac{3r_3-2r}{2r_3+r} \geqslant \frac{100r^2+9p^2}{19r^2+12p^2}.$$

□

Proposition 3. *Using the above notations, let*

$$\begin{aligned} T_1 &= \frac{2r_1-r}{r_1+r} + \frac{2r_2-r}{r_2+r} + \frac{2r_3-r}{r_3+r} \\ T_2 &= \frac{2r_1-r}{r_1+r}\frac{2r_2-r}{r_2+r} + \frac{2r_2-r}{r_2+r}\frac{2r_3-r}{r_3+r} + \frac{2r_3-r}{r_3+r}\frac{2r_1-r}{r_1+r} \\ T_3 &= \frac{2r_1-r}{r_1+r}\frac{2r_2-r}{r_2+r}\frac{2r_3-r}{r_3+r}. \end{aligned}$$

then we have an equality $4T_3 + T_2 - 2T_1 = 5$.

Proof. Let $y_i = \dfrac{2r_i-r}{r_i+r}, i=1,2,3$ we have $y_i \neq 2$ and $x = \dfrac{r(y_i+1)}{2-y_i}$. Since r_1, r_2, r_3 are the roots of the polynomial (3), if we substitute r_i into the polynomial (3), we obtain

$$2(r^2+2Rr+p^2)y_i^3 - 3(-r^2+3p^2)y_i^2 - 12(Rr-p^2)y_i - (r^2+8Rr+4p^2) = 0$$

Therefore, y_1, y_2, y_3 are the roots of the polynomial

$$2(r^2+2Rr+p^2)y^3 - 3(-r^2+3p^2)y^2 - 12(Rr-p^2)y - (r^2+8Rr+4p^2).$$

Applying the Viete's formulas for this polynomial, we obtain $4T_3 + T_2 - 2T_1 = 5$. □

Elimination theory [1] is one of the most effective method to solve polynomial equations, so if we combine this method and Theorem 1, we can obtain some nice equalities. Consequently, we give a brief overview of *the resultant* of two polynomials. Given a field $\mathbb{K} \subseteq \mathbb{C}$ and two polynomials $f, g \in \mathbb{K}[x]$ of positive degree

$$\begin{aligned} f &= a_0x^m + a_1x^{m-1} + \cdots + a_m, a_0 \neq 0, m > 0 \\ g &= b_0x^n + b_1x^{n-1} + \cdots + b_n, b_0 \neq 0, n > 0. \end{aligned}$$

We build the square matrix of size $(m+n)\times(m+n)$ associated to f,g (called Sylvester matrix) as follows:

$$S(f,g) := \left\{\begin{matrix} a_0 & a_1 & \cdots & a_m & & & \\ & a_0 & a_1 & \cdots & a_m & & \\ & & \ddots & \ddots & & \ddots & \\ & & & a_0 & a_1 & \cdots & a_m \\ b_0 & b_1 & \cdots & b_n & & & \\ & b_0 & b_1 & \cdots & b_n & & \\ & & \ddots & \ddots & & \ddots & \\ & & & b_0 & b_1 & \cdots & b_n \end{matrix}\right\} \tag{4}$$

where n first rows only depend on the coefficients of f, m last rows depend on the coefficients of g and the blank spaces are filled with 0. Then the *resultant* of f and g, denoted $\operatorname{Res}(f,g)$, is the determinant of Sylvester matrix $S(f,g)$.

If we want to emphasize the dependence on x, we write $\operatorname{Res}(f,g,x)$ instead of $\operatorname{Res}(f,g)$. Two important properties of resultants [2, Chapter 3] are

- (Common factor) $\operatorname{Res}(f,g)=0$ if and only if f and g have a non trivial common factor in $\mathbb{K}[x]$.
- (Elimination) There are polynomials $U,V\in\mathbb{K}[x]$ such that $Uf+Vg=\operatorname{Res}(f,g)$. The coefficients of U,V are integer polynomial in the coefficients of f and g.

Proposition 4. *Using the above notations,*

(i) $(a^2+2Rr)(b^2+2Rr)(c^2+2Rr)=2Rr(ab+bc+ca-2Rr)^2$.

(ii) $(a^2-2Rr)(b^2-2Rr)(c^2-2Rr)=4a^2b^2c^2-2Rr(ab+bc+ca+2Rr)^2$.

(iii) $\dfrac{(r_1+r_2+r_3+r)^2}{S}=\dfrac{a^2+p^2+4Rr+r^2}{S}\dfrac{b^2+p^2+4Rr+r^2}{S}\dfrac{c^2+p^2+4Rr+r^2}{S}$.

(iv) $S=\dfrac{\sqrt{(a^2+p^2+4Rr+r^2)(b^2+p^2+4Rr+r^2)(c^2+p^2+4Rr+r^2)}}{r_1+r_2+r_3+r}$.

(v) $(r_1^2+2Rr)(r_2^2+2Rr)(r_3^2+2Rr)=\big[r_1r_2r_3-2Rr(r_1+r_2+r_3)\big]^2+2Rr\big[2Rr-\frac{r_1r_2r_3}{r}\big]^2$.

(vi) $\Big(\frac{r_1^2}{p^2}-1\Big)\Big(\frac{r_2^2}{p^2}-1\Big)\Big(\frac{r_3^2}{p^2}-1\Big)=4\Big[\frac{(2R+r)^2}{p^2}-1\Big]$

Proof. (i) Let $f:=x^3-2px^2+(p^2+4Rr+r^2)x-4Rrp=0$ and $g:=x^2+2Rr-y$. By elimination property, two polynomials $f(x),g(x)$ have a common root if and only if resultant $\operatorname{Res}(f,g;x)=0$. So,

$$y^3-(4p^2+2Rr-2T)y^2+(T^2-2Rr)y-2RrT^2=0 \quad \text{where } T=p^2+2Rr+r^2. \tag{5}$$

Let $y_1=a^2+2Rr, y_2=b^2+2Rr$ and $y_3=c^2+2Rr$. Because a,b and c are the roots of $f(x)$, y_1,y_2,y_3 are the roots of the equation (5). Using Viete's formulas, we have

$$y_1y_2y_3=2RrT^2=2Rr(ab+bc+ca-2Rr)^2.$$

(ii) Let $f:=x^3-2px^2+(p^2+4Rr+r^2)x-4Rrp=0$ and $g:=x^2-2Rr-y$ then

$$y^3+(2Rr+2T-4p^2)y^2+(T^2+4RrT-32Rrp^2)y+2RrT^2-64R^2r^2p^2=0 \quad \text{where } T=p^2+6Rr+r^2. \tag{6}$$

Let $y_1=a^2-2Rr, y_2=b^2-2Rr$ and $y_3=c^2-2Rr$, then y_1,y_2,y_3 are the roots of the equation (6). Using Viete's formulas, we have

$$y_1y_2y_3=4a^2b^2c^2-2Rr(ab+bc+ca+2Rr)^2.$$

(iii) Let $f(x) = x^3 - 2px^2 + (p^2 + 4Rr + r^2)x - 4Rrp$ and $g(x) = x^2 + p^2 + 4Rr + r^2 - y$, we have

$$y^3 - (p^2 + 4Rr + r^2)y^2 - (2py - 4Rrp - 2r^2p)^2 = 0. \tag{7}$$

Let $y_1 = a^2 + p^2 + 4Rr + r^2, y_2 = b^2 + p^2 + 4Rr + r^2$ and $y_3 = c^2 + p^2 + 4Rr + r^2$, then y_1, y_2, y_3 are the roots of the equation (7). Using Viete's formulas, we have $y_1y_2y_3 = (4R + 2r)^2r^2p^2$. Hence,

$$(a^2 + p^2 + 4Rr + r^2)(b^2 + p^2 + 4Rr + r^2)(c^2 + p^2 + 4Rr + r^2) = (4Rr + 2r^2)^2p^2.$$

Otherwise, $r_1 + r_2 + r_3 = 4R + r$, we have

$$\frac{(r_1 + r_2 + r_3 + r)^2}{S} = \frac{a^2 + p^2 + 4Rr + r^2}{S}\frac{b^2 + p^2 + 4Rr + r^2}{S}\frac{c^2 + p^2 + 4Rr + r^2}{S}.$$

(iv) This equality is deduced from (iii).
(v) Let $f := x^3 - (4R + r)x^2 + p^2x - p^2r$ and $g := x^2 + 2Rr - y$. Similar to (i), we have

$$(r_1^2 + 2Rr)(r_2^2 + 2Rr)(r_3^2 + 2Rr) = \big[r_1r_2r_3 - 2Rr(r_1 + r_2 + r_3)\big]^2$$
$$+2Rr\big[2Rr - \frac{r_1r_2r_3}{r}\big]^2.$$

(vi) Similarly to (i), let $f := x^3 - (4R + r)x^2 + p^2x - p^2r$ and $g := x^2 - p^2 - y$; the conclusion follows.

□

If we combine them with the well-known inequality $R \geq 2r$ [3], we can yield some interesting inequalities. In particular,

Corollary 5. *Using the above notations,*

(i) $(a^2 + 2Rr)(b^2 + 2Rr)(c^2 + 2Rr) \leqslant R^2(ab + bc + ca - 2Rr)^2$.

(ii) $(a^2 + 2Rr)(b^2 + 2Rr)(c^2 + 2Rr) \leq 2R^6$.

(iii) $(a^2 - 2Rr)(b^2 - 2Rr)(c^2 - 2Rr) \geqslant 4a^2b^2c^2 - R^2(ab + bc + ca + R^2)^2$.

(iv) $(a^2 - 2Rr)(b^2 - 2Rr)(c^2 - 2Rr) \leqslant 4a^2b^2c^2 - 4r^2(ab + bc + ca + 4r^2)^2$.

Proof. (i) By Proposition 4(i) and $R \geq 2r$, we are done.
(ii) Let $T = (a^2+2Rr)(b^2+2Rr)(c^2+2Rr)$, by Proposition 4(i) we have $(a^2+2Rr)(b^2+2Rr)(c^2+2Rr) = 2Rr(ab + bc + ca - 2Rr)^2$. Combining with $ab + bc + ca \leq \frac{4p^2}{3}$ and $4p^2 \leq 27R^2$, we obtain

$$T \leq 2Rr(9R^2 - 2Rr)^2 \Rightarrow T \leqslant R^3.2r(9R - 2r)^2.$$

Consider the function $f(x) = x(9R - x)^2$ where $0 < x \leqslant R$. We have $f'(x) > 0$ with each $x \in (0, R]$, deduce that the function f is the increasing function in $(0, R]$. Therefore $f(x) \leq f(R) = 64R^3$ with each $x \in (0, R]$. Notice that $R \geq 2r$, so $T \leq 64R^6$ and the equality holds if and only if $\triangle ABC$ is equilateral.
By Proposition 4(ii) and $R \geq 2r$, we have the proof of (iii) and (iv). □

Corollary 6. *Suppose given a convex quadrilateral $ABCD$ of $AB = a, BC = b, CD = c, DA = d$ inscribed the circle of center O, radius R. Denoted r_1, r_2, r_3, r_4 as the radius of incircle the triangles $\triangle ABC$, $\triangle BCD$, $\triangle CDA$ and $\triangle DAB$, respectively. Let $T = \frac{2^6R^6}{ac + bd + 4\sqrt{r_1r_2r_3r_4}}$ then*

$$T \geq \sqrt{(ab + 4r_1^2)(bc + 4r_2^2)(cd + 4r_3^2)(da + 4r_4^2)}.$$

Proof. Let $AC = x, BD = y$, by Proposition 4 we have

$$(a^2 + 2Rr_1)(b^2 + 2Rr_1)(x^2 + 2Rr_1) \leqslant 64R^6.$$

Otherwise, we have $(a^2 + 2Rr_1)(b^2 + 2Rr_1) \geq (ab + 2Rr_1)^2$, so we deduce that

$$(ab + 2Rr_1)^2(x^2 + 2Rr_1) \leq 64R^6.$$

Similar, we obtain

$$\begin{aligned}(bc + 2Rr_2)^2(y^2 + 2Rr_2) &\leq 64R^6\\(cd + 2Rr_3)^2(x^2 + 2Rr_3) &\leq 64R^6\\(da + 2Rr_4)^2(y^2 + 2Rr_4) &\leq 64R^6\end{aligned}$$

Multiplying, we obtain

$$\big[(ab+2Rr_1)(bc+2Rr_2)(cd+2Rr_3)(da+2Rr_4)\big]^2(x^2+2Rr_1)(x^2+2Rr_3)(y^2+2Rr_2)(y^2+2Rr_4) \leq 2^{24}R^{24}. \tag{8}$$

Consequently,

$$\begin{aligned}(x^2 + 2Rr_1)(x^2 + 2Rr_3) &\geqslant (x^2 + 2R\sqrt{r_1r_3})^2\\(y^2 + 2Rr_2)(y^2 + 2Rr_4) &\geqslant (y^2 + 2R\sqrt{r_2r_4})^2\\&\text{and}\\(x^2 + 2R\sqrt{r_1r_3})(y^2 + 2R\sqrt{r_2r_4}) &\geqslant (xy + 2R\sqrt[4]{r_1r_2r_3r_4})^2.\end{aligned}$$

$$\Rightarrow (xy + 2R\sqrt[4]{r_1r_2r_3r_4})^2 \leq (x^2 + 2Rr_1)(x^2 + 2Rr_3)(y^2 + 2Rr_2)(y^2 + 2Rr_4) \tag{9}$$

From (8) and (9), it follows that

$$(ab + 2Rr_1)(bc + 2Rr_2)(cd + 2Rr_3)(da + 2Rr_4)(xy + 2R\sqrt[4]{r_1r_2r_3r_4}) \leq 2^{12}R^{12}. \tag{10}$$

From $2r_i \leq R, i = 1, 2, 3, 4$ and by Ptolemy's theorem, we furthermore have that

$$xy + 2R\sqrt[4]{r_1r_2r_3r_4} = ac + bd + 2R\sqrt[4]{r_1r_2r_3r_4} \geq ac + bd + 4\sqrt{r_1r_2r_3r_4}.$$

Therefore, we deduce that

$$\begin{aligned}T^2 &= \frac{2^{12}R^{12}}{(ac + bd + 4\sqrt{r_1r_2r_3r_4})^2} \geqslant \frac{2^{12}R^{12}}{(ac + bd + 2R\sqrt[4]{r_1r_2r_3r_4})^2}\\&\geqslant (ab + 2Rr_1)(bc + 2Rr_2)(cd + 2Rr_3)(da + 2Rr_4).\end{aligned}$$

Using again the inequality $2r_i \leq R, i = 1, 2, 3, 4$, we obtain

$$T \geqslant \sqrt{(ab + 4r_1^2)(bc + 4r_2^2)(cd + 4r_3^2)(da + 4r_4^2)}.$$

□

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