Junior problems

J325. For positive real numbers a and b, define their *perfect mean* to be half of the sum of their arithmetic and geometric means. Find how many unordered pairs of integers (a, b) from the set $\{1, 2, \ldots, 2015\}$ have their perfect mean a perfect square.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J326. Let a, b, c be nonnegative real numbers. Prove that

$$\sqrt{2a^2+3b^2+4c^2}+\sqrt{3a^2+4b^2+2c^2}+\sqrt{4a^2+2b^2+3c^2} \geq (\sqrt{a}+\sqrt{b}+\sqrt{c})^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas

J327. A jeweler makes a circular necklace out of nine distinguishable gems: three sapphires, three rubies and three emeralds. No two gems of the same type can be adjacent to each other and necklaces obtained by rotation and reflection (flip) are considered to be identical. How many different necklaces can she make?

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J328. Let a, b, c be positive real numbers such that $a+b+c=2$. Prove that

$$\sqrt{a^2+bc}+\sqrt{b^2+ca}+\sqrt{c^2+ab} \leq 3.$$

Proposed by An Zhen-ping, Xianyang Normal University, China

J329. Let $a_1, a_2, \ldots, a_{2015}$ be positive integers such that

$$a_1+a_2+\cdots+a_{2015}=a_1\cdots a_{2015}.$$

Prove that among numbers $a_1, a_2, \ldots, a_{2015}$ at most nine are greater than 1.

Proposed by Titu Zvonaru, Comanești and Neculai Stanciu, Buzău, Romania

J330. Let $ABCD$ be a quadrilateral with centroid G, inscribed in a circle with center O, and diagonals intersecting at P. Prove that if O, G, P are collinear, then $ABCD$ is an isosceles trapezoid.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Senior problems

S325. Let S be the incenter of the triangle formed by the midpoints of the triangle ABC. Cevians AS, BS, CS intersect the sides BC, CA, AB in points M, N, P, respectively. Prove that if the perpendiculars to the sides of triangle ABC at points M, N, P are concurrent, then triangle ABC is isosceles.

Proposed by Roxana Mihaela Stanciu and Nela Ciceu, Bacău, Romania

S326. Let a, b, c be positive real numbers such that $a^3+b^3+c^3+abc=\frac{1}{3}$. Prove that

$$abc+9\left(\frac{a^5}{4b^2+bc+4c^2}+\frac{b^5}{4c^2+ca+4a^2}+\frac{c^5}{4a^2+ab+4b^2}\right)\geq\frac{1}{4(a+b+c)(ab+bc+ca)}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S327. Let H be the orthocenter of an acute triangle ABC and let D and E be the feet of the altitudes from vertices B and C, respectively. Line DE intersects the circumcircle of triangle ABC in points F and G (F lies on the smaller arc of AB and G lies on the smaller arc of AC). Denote by S the projection of H onto line DE. Prove that $EF+DS=DG+ES$.

Proposed by İlker Can Çiçek, Instanbul, Turkey

S328. Let k be an odd positive integer. There are k positive integers written on a circle that add up to $2k$. Prove that for any $1\leq m\leq k$, among the k given numbers, we can find one or more consecutively placed numbers that add up to $2m$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

S329. Given a quadrilateral $ABCD$ with $AB+AD=BC+DC$ let (I_1) be the incircle of triangle ABC which is tangent to AC at E, and let (I_2) be the incircle of triangle ADC which is tangent to AC at F. The diagonals AC and BD intersect at point P. Suppose BI_1 and DE intersect at point S and DI_2 and BF intersect at point T. Prove that S, P, T are collinear.

Proposed by Khakimboy Egamberganov, Tashkent, Uzbekistan

S330. Let $x_1,x_2,\ldots,x_n$ be real numbers, $n\geq 2$, such that $x_1^2+x_2^2+\ldots+x_n^2=1$. Prove that

$$\sum_{i=1}^{n}\sqrt{1-x_i^2}+k_n\cdot\sum_{1\leq j<k\leq n}x_jx_k\geq n-1$$

where $k_n=2-2\sqrt{1+\frac{1}{n-1}}$.

Proposed by Marius Stănean, Zalău, Romania

Undergraduate problems

U325. Let $A_1B_1C_1$ be a triangle with circumradius R_1. For each $n \geq 1$, the incircle of triangle $A_nB_nC_n$ is tangent to its sides at points A_{n+1}, B_{n+1}, C_{n+1}. The circumradius of triangle $A_{n+1}B_{n+1}C_{n+1}$, which is also the inradius of triangle $A_nB_nC_n$, is R_{n+1}. Find $\lim_{n\to\infty} \frac{R_{n+1}}{R_n}$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

U326. Find

$$\sum_{n=0}^{\infty} \frac{a_n + 2}{a_n^2 + a_n + 1},$$

where $a_0 > 1$ and $a_{n+1} = \frac{1}{3}\left(a_n^3 + 2\right)$ for all integers $n \geq 1$.

Proposed by Arkady Alt, San Jose, USA

U327. Let $(a_n)_{n\geq 0}$ be a sequence of real numbers with $a_0 = 1$ and

$$a_{n+1} = \frac{a_n}{n^2 a_n + {a_n}^2 + 1}.$$

Find the limit $\lim\limits_{n\to\infty} n^3 a_n$.

Proposed by Khakimboy Egamberganov, Tashkent, Uzbekistan

U328. Let $(a_n)_{n\geq 1}$ be an increasing sequence of positive real numbers such that $\lim\limits_{n\to\infty} a_n = \infty$ and the sequence $(a_{n+1} - a_n)_{n\geq 1}$ is monotonic. Evaluate

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n\sqrt{a_n}}.$$

Proposed by Mihai Piticari and Sorin Rădulescu, Romania

U329. Let $a_1 \leq \ldots \leq a_{\frac{n(n-1)}{2}}$ be the distances between n distinct points lying on the plane. Prove that there is a constant c such that for any n we can find indices i and j such that

$$\left|\frac{a_i}{a_j} - 1\right| < \frac{c \ln n}{n^2}.$$

Proposed by Nairi Sedrakyan, Yerevan, Armenia

U330. Find all functions $f : \mathbb{R} \to \mathbb{R}$ with the properties

(i) f is an antiderivative,

(ii) f is integrable on any compact interval,

(iii) $f(x)^2 = \int_0^x f(t)\,dt$ for all $x \in \mathbb{R}$.

Proposed by Mihai Piticari, Câmpulung Moldovenesc, Romania

Olympiad problems

O325. The *taxicab* distance between points $P_1 = (x_1, x_2)$ and $P_2(x_2, y_2)$ in a coordinate plane is

$$d(P_1P_2) = |x_1 - x_2| + |y_1 - y_2|.$$

The *taxicab* disk with center O and radius R is the set of points P such that $d(P, O) \leq R$. Given n points that are pairwise at most R taxicab distance apart, find the smallest constant c such that any such set of points can be covered by a taxicab disk of radius cR.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

O326. Let ABC be a non-isosceles triangle. Let D be a point on the ray $\overrightarrow{AB}$, but not on the segment AB, and let E be a point on the ray $\overrightarrow{AC}$, but not on the segment AC, such that $AB \cdot BD = AC \cdot CE$. The circumcircles of triangles ABE and ACD intersect in points A and F. Let O_1 and O_2 be the circumcenters of triangles ABC and ADE. Prove that lines AF, O_1O_2, and BC are concurrent.

Proposed by İlker Can Çiçek, Instanbul, Turkey

O327. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + abc = 4$. Prove that

$$a + b + c \leq \sqrt{2 - a} + \sqrt{2 - b} + \sqrt{2 - c}.$$

Proposed by An Zhen-ping, Xianyang Normal University, China

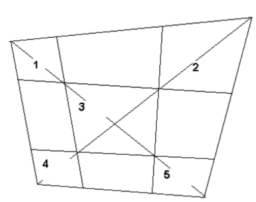
O328. The diagonals AD, BE, CF of the convex hexagon $ABCDEF$ intersect at point M. Triangles ABM, BCM, CDM, DEM, EFM, FAM are acute. Prove that circumcenters of these triangles are concyclic if and only if the areas of quadrilaterals $ABDE$, $BCEF$, $CDFA$ are equal.

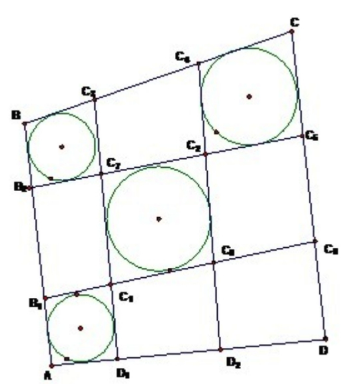
Proposed by Nairi Sedrakyan, Yerevan, Armenia

O329. For any positive integer r, denote by P_r the path on r vertices and by $\overline{P_r}$ its complement graph. For any positive integer g, prove that there exists a graph G with no cycles of length less than g such that G contains either P_r or $\overline{P_r}$.

Proposed by Cosmin Pohoaţă, Colombia University, USA

O330. Four segments divide a convex quadrilateral into nine quadrilaterals. The points of intersections of these segments lie on the diagonals of the quadrilateral (see figure below).





It is known that quadrilaterals Q_1, Q_2, Q_3, Q_4 admit inscribed circles. Prove that the quadrilateral Q_5 also has an inscribed circle.

Proposed by Nairi Sedrakyan, Yerevan, Armenia