

## Junior problems

J319. Let  $0 = a_0 < a_1 < \dots < a_n < a_{n+1} = 1$  such that  $a_1 + a_2 + \dots + a_n = 1$ . Prove that

$$\frac{a_1}{a_2 - a_0} + \frac{a_2}{a_3 - a_1} + \dots + \frac{a_n}{a_{n+1} - a_{n-1}} \geq \frac{1}{a_n}.$$

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

*Solution by Adnan Ali, Student in A.E.C.S-4, Mumbai, India*

The expression on the left-hand-side can be rewritten as

$$\frac{a_1^2}{a_1 a_2 - a_0 a_1} + \frac{a_2^2}{a_2 a_3 - a_1 a_2} + \dots + \frac{a_n^2}{a_n a_{n+1} - a_{n-1} a_n}$$

Now applying the Cauchy-Schwartz Inequality on the expression, we have

$$\begin{aligned} & \frac{a_1^2}{a_1 a_2 - a_0 a_1} + \frac{a_2^2}{a_2 a_3 - a_1 a_2} + \dots + \frac{a_n^2}{a_n a_{n+1} - a_{n-1} a_n} \\ & \geq \frac{(a_1 + a_2 + \dots + a_n)^2}{a_1 a_2 - a_0 a_1 + a_2 a_3 - a_1 a_2 + \dots + a_n a_{n+1} - a_{n-1} a_n} = \frac{1}{a_n a_{n+1} - a_0 a_1} = \frac{1}{a_n}. \end{aligned}$$

*Also solved by Daniel Lasoasa, Pamplona, Spain; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Alok Kumar, Delhi, India; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Arber Igrishita, Eqrem Qabej, Vushtrri, Kosovo; Arkady Alt, San Jose, California, USA; Bodhisattwa Bhowmik, RKMV, Agartala, Tripura, India; Daniel Văcaru, Pitești, Romania; David E. Manes, Oneonta, NY, USA; Ilyes Hamdi, Lycée Voltaire, Doha, Qatar; Farrukh Mukhammadiev, Academic Lyceum Nr1, Samarkand, Uzbekistan; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Sardor Bozorboyev, Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Shatlyk Mamedov, Dashoguz, Turkmenistan; Titu Zvonaru, Comănești, Romania and Neculai Stanciu, Buzău, Romania; Seung Hwan An, Taft School, Watertown, CT, USA; Chaeyeon Oh, Episcopal High School, Alexandria, VA, USA; Mehtaab Sawhney, USA; Yujin Kim, Stony Brook School, Stony Brook, NY, USA; Misiakos Panagiotis, Athens College (HAEF), Nea Penteli; William Kang, Bergen County Academies, Hackensack, NJ, USA; Ji Eun Kim, Tabor Academy, Marion, MA, USA; Timothy Chon, Horace Mann School, Bronx, NY, USA; Cody Johnson, USA; Jhiseung Daniel Hahn, Phillips Exeter Academy, Exeter, NH, USA; Michael Tang, Edina High School, MN, USA; Yong Xi Wang, East China Institute Of Technology, China; Yooree Ha, Ponte Vedra High School, Ponte Vedra, FL, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.*

J320. Find all positive integers  $n$  for which  $2014^n + 11^n$  is a perfect square.

*Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA*

*Solution by Prithwijit De, HBCSE, Mumbai, India*

Let  $p(n) = 2014^n + 11^n$  for  $n \geq 1$ . Then  $p(1) = 2025 = 45^2$ . Also  $p(n)$  is odd for all  $n$ . We will show that for no other value of  $n$  is  $p(n)$  a perfect square. For  $n$  even note that the last digit of  $p(n)$  is 7. Thus it cannot be a perfect square.

For  $n$  odd and  $n > 1$  note that  $p(n)$  leaves 3 as remainder when divided by 8. But the square of any odd integer leaves 1 as remainder when divided by 8. Thus  $p(n)$  cannot be a square for odd positive integers  $n$  greater than 1.

*Also solved by Daniel Lasaosa, Pamplona, Spain; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Seung Hwan An, Taft School, Watertown, CT, USA; Chaeyeon Oh, Episcopal High School, Alexandria, VA, USA; Mehtaab Sawhney, USA; Yujin Kim, Stony Brook School, Stony Brook, NY, USA; Misiakos Panagiotis, Athens College (HAEF), Nea Penteli; William Kang, Bergen County Academies, Hackensack, NJ, USA; Ji Eun Kim, Tabor Academy, Marion, MA, USA; Timothy Chon, Horace Mann School, Bronx, NY, USA; Cody Johnson, USA; Jhiseung Daniel Hahn, Phillips Exeter Academy, Exeter, NH, USA; Michael Tang, Edina High School, MN, USA; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Albert Stadler, Herrliberg, Switzerland; Alok Kumar, Delhi, India; Arber Avdullahu, Mehmet Akif College, Kosovo; Arkady Alt, San Jose, California, USA; Bodhisattwa Bhowmik, RKMV, Agartala, Tripura, India; Corneliu Mănescu-Avram, Transportation High School, Ploiești, Romania; David E. Manes, Oneonta, NY, USA; Ilyes Hamdi, Lycée Voltaire, Doha, Qatar; Jean Heibig, Paris, France; Farrukh Mukhammadiev, Academic Lyceum Nr1, Samarkand, Uzbekistan; Paul Revenant, Lycée Champollion, Grenoble, France; Sardor Bozorboyev, Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Titu Zvonaru, Comănești, Romania and Neculai Stanciu, Buzău, Romania; Yooree Ha, Ponte Vedra High School, Ponte Vedra, FL, USA.*

J321. Let  $x, y, z$  be positive real numbers such that  $xyz(x + y + z) = 3$ . Prove that

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} + \frac{54}{(x + y + z)^2} \geq 9.$$

*Proposed by Marius Stănean, Zalau, Romania*

*Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy*  
 Rewriting the inequality yields to

$$\frac{(xy)^2 + (yz)^2 + (zx)^2}{x^2y^2z^2} + \frac{54}{(x + y + z)^2} \geq 9$$

Define now the new variables

$$x + y + z = 3u, \quad xy + yz + zx = 3v^2, \quad xyz = w^3$$

and trivial AGM yields  $u \geq v \geq w$ . We also use the well known inequality

$$xy + yz + zx \geq \sqrt{3(xyz)(x + y + z)} \iff 3v^2 \geq 3$$

that is  $v \geq 1$ .

The inequality becomes

$$w^3u = 1 \implies \frac{9v^4 - 6uw^3}{w^6} + \frac{9}{u^2} \geq 9$$

that is

$$f(u) \doteq (3v^4 - 2)u^2 + \frac{2}{u^2} - 3 \geq 0,$$

$$f'(u) = 2u(3v^4 - 2) - \frac{4}{u^3} = 0 \iff u = u_0(v) = \sqrt[4]{\frac{2}{3v^2 - 2}}$$

If  $v \leq 2/\sqrt{3}$  then  $u_0(v) \leq 1$ , and this implies  $f(u) \geq f(1)$ .

$$f(1) = (3v^4 - 2)u^2 + \frac{2}{u^2} - 3 \geq 3v^4 - 2 + 2 - 3 \geq 3 \cdot 1 - 2 + 2 - 3 = 0$$

and this part of the proof is complete. Now let  $v > 2/\sqrt{3}$ .

$$f(u) = (3v^4 - 2)u^2 + \frac{2}{u^2} - 3 > (4 - 2)u^2 + \frac{2}{u^2} - 3 \geq 4 - 3 = 1$$

and also this part is complete.

*Also solved by Seung Hwan An, Taft School, Watertown, CT, USA; Chaeyeon Oh, Episcopal High School, Alexandria, VA, USA; Mehtaab Sawhney, USA; Yujin Kim, Stony Brook School, Stony Brook, NY, USA; Misiakos Panagiotis, Athens College (HAEF), Nea Penteli; Timothy Chon, Horace Mann School, Bronx, NY, USA; Cody Johnson, USA; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania.*

J322. Let  $ABC$  be a triangle with centroid  $G$ . The parallel lines through a point  $P$  situated in the plane of the triangle to the medians  $AA', BB', CC'$  intersect lines  $BC, CA, AB$  at  $A_1, B_1, C_1$ , respectively. Prove that

$$A'A_1 + B'B_1 + C'C_1 \geq \frac{3}{2}PG.$$

*Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania*

*Solution by Marius Stănean, Zalău, Romania*

Let  $(x : y : z)$  be the barycentric coordinates of  $P$  with respect to triangle  $ABC$  so that the three (signed) areas  $[PBC]$ ,  $[PCA]$ , and  $[PAB]$  are in the ratio  $x : y : z$ . We have

$$\begin{aligned} \frac{PA_1}{AA'} &= \frac{[PAB]}{[ABC]} = x \implies \\ \overrightarrow{A'A_1} &= \overrightarrow{PA_1} - \overrightarrow{PG} - \overrightarrow{GA'} = 3x \cdot \overrightarrow{GA'} - \overrightarrow{PG} - \overrightarrow{GA'} = \\ (3x - 1) &\left( -\frac{1}{3} \cdot \overrightarrow{A} + \left( \frac{1}{2} - \frac{1}{3} \right) \cdot \overrightarrow{B} + \left( \frac{1}{2} - \frac{1}{3} \right) \cdot \overrightarrow{C} \right) - \overrightarrow{PG} = \\ &\frac{3x - 1}{6} \cdot \left( -2 \cdot \overrightarrow{A} + \overrightarrow{B} + \overrightarrow{C} \right) - \overrightarrow{PG} \end{aligned}$$

Analog obtain

$$\begin{aligned} \overrightarrow{B'B_1} &= \frac{3y - 1}{6} \cdot \left( \overrightarrow{A} - 2 \cdot \overrightarrow{B} + \overrightarrow{C} \right) - \overrightarrow{PG} \\ \overrightarrow{C'C_1} &= \frac{3z - 1}{6} \cdot \left( \overrightarrow{A} + \overrightarrow{B} - 2 \cdot \overrightarrow{C} \right) - \overrightarrow{PG} \end{aligned}$$

Therefore

$$\begin{aligned} \overrightarrow{A'A_1} + \overrightarrow{B'B_1} + \overrightarrow{C'C_1} &= \frac{1 - 3x}{2} \cdot \overrightarrow{A} + \frac{1 - 3y}{2} \cdot \overrightarrow{B} + \frac{1 - 3z}{2} \cdot \overrightarrow{C} - 3 \cdot \overrightarrow{PG} = \\ \frac{3}{2} &\left( \left( \frac{1}{3} - x \right) \cdot \overrightarrow{A} + \left( \frac{1}{3} - y \right) \cdot \overrightarrow{B} + \left( \frac{1}{3} - z \right) \cdot \overrightarrow{C} \right) - 3 \cdot \overrightarrow{PG} = \\ &\frac{3}{2} \cdot \overrightarrow{PG} - 3 \cdot \overrightarrow{PG} = -\frac{3}{2} \cdot \overrightarrow{PG}. \end{aligned}$$

Considering this we have

$$\begin{aligned} \frac{3}{2}PG &= \left| -\frac{3}{2} \cdot \overrightarrow{PG} \right| = \left| \overrightarrow{A'A_1} + \overrightarrow{B'B_1} + \overrightarrow{C'C_1} \right| \leq \\ &\left| \overrightarrow{A'A_1} \right| + \left| \overrightarrow{B'B_1} \right| + \left| \overrightarrow{C'C_1} \right| = A'A_1 + B'B_1 + C'C_1. \end{aligned}$$

*Also solved by Daniel Lasaosa, Pamplona, Spain; Seung Hwan An, Taft School, Watertown, CT, USA; William Kang, Bergen County Academies, Hackensack, NJ, USA; Timothy Chon, Horace Mann School, Bronx, NY, USA.*

J323. In triangle  $ABC$ ,

$$\cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2} = \frac{\sqrt{5}-1}{2}.$$

Prove that  $\max(A, B, C) > 144^\circ$ .

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

*Solution by Henry Ricardo, New York Math Circle*

WLOG, assume that  $\max\{A, B, C\} = A \leq 162^\circ$ . Then  $\sin A \geq \sin 162^\circ = \sin 18^\circ = \frac{\sqrt{5}-1}{4}$ ,  $18^\circ \leq B + C < 180^\circ$ , and  $\sin(B + C) \geq \frac{\sqrt{5}-1}{4}$ .

Since  $\sin B + \sin C > \sin(B + C)$  for  $0 < B, C < 180^\circ$ , we have

$$\begin{aligned} \frac{\sqrt{5}-1}{2} &= \sin A + \sin B + \sin C > \sin A + \sin(B + C) \\ &> \frac{\sqrt{5}-1}{4} + \frac{\sqrt{5}-1}{4} = \frac{\sqrt{5}-1}{2}. \end{aligned}$$

This contradiction establishes that  $\max\{A, B, C\} > 162^\circ$ .

*Also solved by Daniel Lasaosa, Pamplona, Spain; Seung Hwan An, Taft School, Watertown, CT, USA; Chaeyeon Oh, Episcopal High School, Alexandria, VA, USA; Mehtaab Sawhney, USA; Yujin Kim, Stony Brook School, Stony Brook, NY, USA; William Kang, Bergen County Academies, Hackensack, NJ, USA; Ji Eun Kim, Tabor Academy, Marion, MA, USA; Timothy Chon, Horace Mann School, Bronx, NY, USA; Cody Johnson, USA; Jhiseung Daniel Hahn, Phillips Exeter Academy, Exeter, NH, USA; Arkady Alt, San Jose, California, USA; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Sardor Bozorboyev, Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Titu Zvonaru, Comănești, Romania and Neculai Stanciu, Buzău, Romania; Yooree Ha, Ponte Vedra High School, Ponte Vedra, FL, USA.*

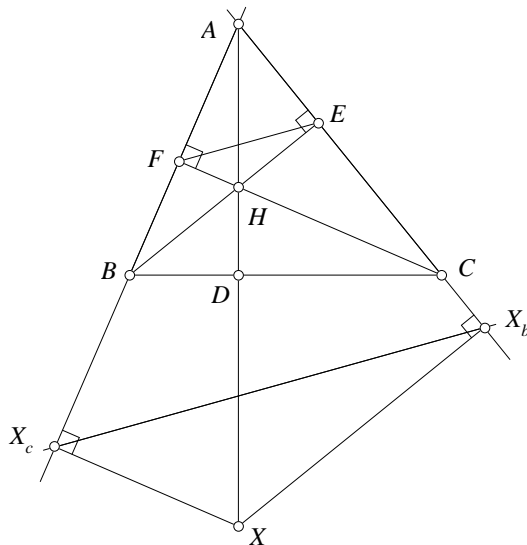
J324. Let  $ABC$  be a triangle and let  $X, Y, Z$  be the reflections of  $A, B, C$  in the opposite sides. Let  $X_b, X_c$  be the orthogonal projections of  $X$  on  $AC, AB$ ,  $Y_c, Y_a$  the orthogonal projections of  $Y$  on  $BA, BC$ , and  $Z_a, Z_b$  the orthogonal projections of  $Z$  on  $CB, CA$ , respectively. Prove that  $X_b, X_c, Y_c, Y_a, Z_a, Z_b$  are concyclic.

*Proposed by Cosmin Pohoata, Columbia University, USA*

*Solution by Ercole Suppa, Teramo, Italy*

We first prove the following claims:

**Claim 1.** The lines  $X_bX_c, Y_aY_c, Z_aZ_b$  are antiparallel to  $BC, AC, AB$  respectively.



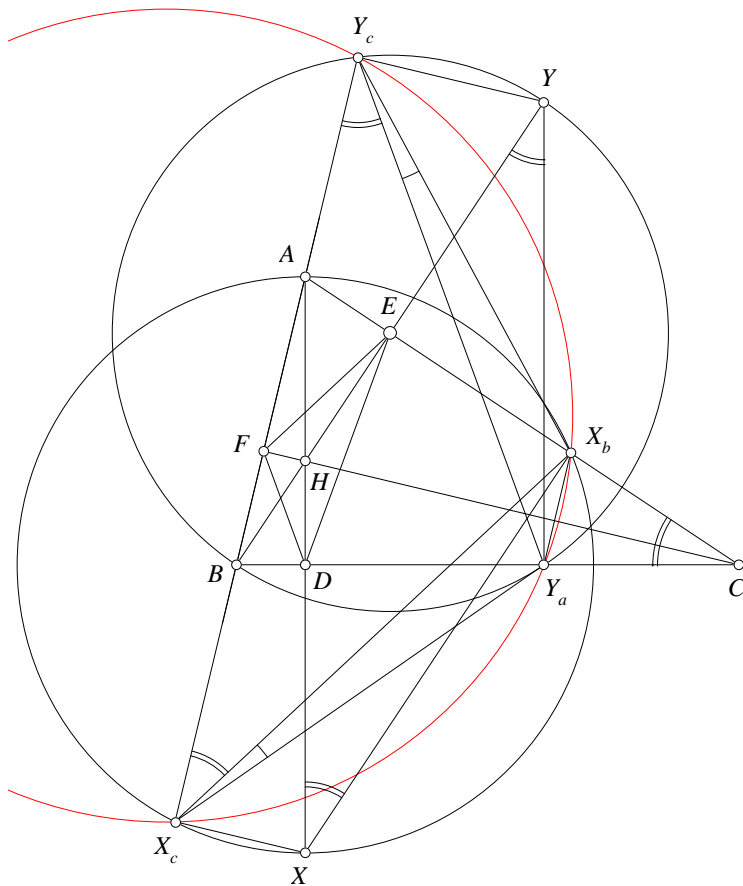
*Proof of Claim 1.* From the similar triangles  $\triangle AHE \sim \triangle AX X_b$  and  $\triangle AHF \sim \triangle AX X_c$  we get

$$AE : AX_b = AH : AX, \quad AF : AX_c = AH : AX \quad \Rightarrow \quad AE : AX_b = AF : AX_c$$

Therefore  $\triangle AFE$  and  $\triangle AX_cX_b$  are homotetic, so  $X_bX_c \parallel EF$ , i.e.  $X_bX_c$  is antiparallel to  $BC$ . A similar reasoning shows that  $Y_aY_c$  is antiparallel to  $AC$  and  $Z_aZ_b$  is antiparallel to  $AB$ , as claimed. ■

**Claim 2.** The lines  $X_bY_a, Y_cZ_b, X_cZ_a$  are parallel to  $AB, BC, AC$  respectively.

*Proof of Claim 2.* Let  $\alpha = \angle BAC, \beta = \angle ABC, \gamma = \angle ACB$  and let  $D, E, F$  be the orthogonal projections of  $A, B, C$  on  $BC, CA, AB$  respectively, as shown in figure.



From the cyclic quadrilaterals  $BY_aYY_c$  and  $AX_cXX_b$  we have

$$\begin{aligned} Y_aY_c &= BY \cdot \sin \beta = 2 \cdot AB \cdot \sin \alpha \sin \beta = AX \sin \alpha = X_bX_c \\ \angle X_cY_cY_a &= \angle BYY_a = \gamma = \angle AXX_b = \angle Y_cX_cX_b \end{aligned} \quad (1)$$

whence it follows that  $\triangle X_cY_cX_b$  and  $\triangle X_cY_cY_a$  are congruent (SAS). Therefore

$$\angle X_cX_bY_c = \angle X_cY_aY_c$$

and this implies that  $X_c, X_b, Y_c, Y_a$  are concyclic, so

$$\angle Y_aY_cX_b = \angle X_bX_cY_a \quad (2)$$

Adding (1) and (2) gives

$$\angle X_cY_cX_b = \angle X_cY_cY_a + \angle Y_aY_cX_b = \angle Y_cX_cX_b + \angle X_bX_cY_a = \angle Y_cX_cY_a$$

Therefore  $X_cY_aX_bY_c$  is an isosceles trapezoid. Thus  $X_cY_c \parallel Y_aX_b$ , i.e.  $AB \parallel X_bY_a$ . We can argue similarly to show that  $BC \parallel Y_cZ_b$  and  $AC \parallel X_cZ_a$  and the claim follows. ■

Returning to the original problem it suffices to notice that the above claims tell us that  $X_cX_bY_aY_cZ_bZ_a$  is a Tucker hexagon, so  $X_b, X_c, Y_c, Y_a, Z_a, Z_b$  are concyclic, which is what we wanted to prove.

*Also solved by Andrea Fanchini, Cantú, Italy; Mehtaab Sawhney, USA; Misiakos Panagiotis, Athens College (HAEF), Nea Penteli; Cody Johnson, USA; Bodhisattwa Bhownik, RKMV, Agartala, Tripura, India; Titu Zvonaru, Comănești, Romania and Neculai Stanciu, Buzău, Romania.*

## Senior problems

S319. Let  $a, b, c$  be positive real numbers such that  $a + b + c = 1$ . Prove that for any positive real number  $t$ ,

$$(at^2 + bt + c)(bt^2 + ct + a)(ct^2 + at + b) \geq t^3.$$

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

*First solution by the proposer*

By Hölder's inequality:

$$(at^2 + bt + c)(bt^2 + ct + a)(ct^2 + at + b) = (at^2 + bt + c)(a + bt^2 + ct)(at + b + ct^2) \geq (at + bt + ct)^3 = t^3.$$

*Second solution by Daniel Lasaosa, Pamplona, Spain*

After some algebra, the inequality rewrites as

$$\begin{aligned} abc(t^6 + 1) + (a^2b + b^2c + c^2a)(t^5 + 2t^2) + (ab^2 + bc^2 + ca^2)(2t^4 + t) + \\ + (a^3 + b^3 + c^3 + 4abc)t^3 \geq t^3. \end{aligned}$$

Now, by the AM-GM inequality, we have  $t^6 + 1 \geq 2t^3$ ,  $t^5 + 2t^2 \geq 3t^3$  and  $2t^4 + t = 3t^3$ , with equality iff  $t = 1$ . It therefore remains only to prove that

$$2abc + 3(a^2b + b^2c + c^2a) + 3(ab^2 + bc^2 + ca^2) + (a^3 + b^3 + c^3 + 4abc) \geq 1,$$

clearly true and with equality always because the LHS is nothing other than  $(a+b+c)^3 = 1^3$ . The conclusion follows, equality holds iff  $t = 1$ .

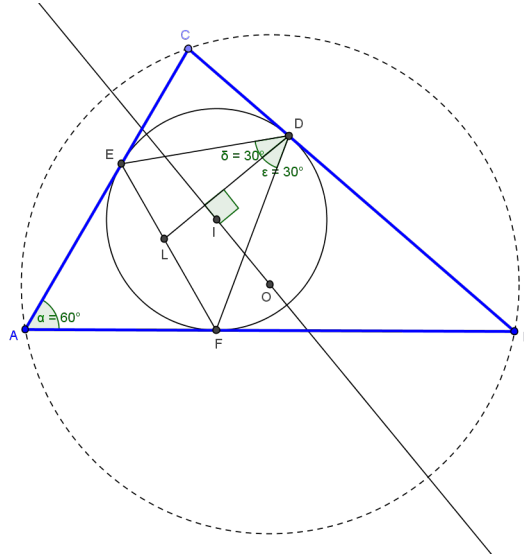
*Also solved by Ioan Viorel Codreanu, Satulung, Maramures, Romania; Seung Hwan An, Taft School, Watertown, CT, USA; Chaeyeon Oh, Episcopal High School, Alexandria, VA, USA; Mehtaab Sawhney, USA; Yujin Kim, Stony Brook School, Stony Brook, NY, USA; Misiakos Panagiotis, Athens College (HAEF), Nea Penteli; William Kang, Bergen County Academies, Hackensack, NJ, USA; Timothy Chon, Horace Mann School, Bronx, NY, USA; Jhiseung Daniel Hahn, Phillips Exeter Academy, Exeter, NH, USA; Yong Xi Wang, East China Institute Of Technology, China; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Albert Stadler, Herrliberg, Switzerland; Alok Kumar, Delhi, India; Arkady Alt, San Jose, California, USA; Bodhisattwa Bhowmik, RKMV, Agartala, Tripura, India; Marin Chirciu, Pitești, Romania; Farrukh Mukham-madiev, Academic Lyceum Nr1, Samarkand, Uzbekistan; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Samin Riasat, University of Waterloo, ON, Canada; Sardor Bozorboyev, Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Titu Zvonaru, Comănești, Romania and Neculai Stanciu, Buzău, Romania; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.*



S320. Let  $ABC$  be a triangle with circumcenter  $O$  and incenter  $I$ . Let  $D, E, F$  be the tangency points of the incircle with  $BC, CA, AB$ , respectively. Prove that line  $OI$  is perpendicular to angle bisector of  $\angle EDF$  if and only if  $\angle BAC = 60^\circ$ .

*Proposed by Marius Stănean, Zalau, Romania*

*Solution by Andrea Fanchini, Cantù, Italy*



Using barycentric coordinates and the Conway's notation, if  $\angle BAC = 60^\circ$ , then

$$a^2 = b^2 + c^2 - bc, \quad S_A = S_{60^\circ} = \frac{bc}{2}, \quad S_B = \frac{2c^2 - bc}{2}, \quad S_C = \frac{2b^2 - bc}{2}, \quad S = \frac{\sqrt{3}}{2}bc, \quad (1)$$

Now line  $DE$  have equation

$$\begin{vmatrix} 0 & s-c & s-b \\ s-c & 0 & s-a \\ x & y & z \end{vmatrix} = 0 \Rightarrow DE \equiv (s-a)x + (s-b)y - (s-c)z = 0$$

and line  $DF$  have equation

$$\begin{vmatrix} 0 & s-c & s-b \\ s-b & s-a & 0 \\ x & y & z \end{vmatrix} = 0 \Rightarrow DF \equiv (s-a)x - (s-b)y + (s-c)z = 0$$

Now, the formula to calculate the angle between two lines is

$$S_\theta = S \cot \theta = \frac{S_A(q_1 - r_1)(q_2 - r_2) + S_B(r_1 - p_1)(r_2 - p_2) + S_C(p_1 - q_1)(p_2 - q_2)}{\begin{vmatrix} 1 & 1 & 1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}$$

so applying this to the  $\angle EDF$  between the lines  $DE$  and  $DF$  we have

$$S_{EDF} = \frac{-a^2 S_A - b(a-c)S_B + c(b-a)S_C}{\begin{vmatrix} 1 & 1 & 1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ s-a & s-b & c-s \\ s-a & b-s & s-c \end{vmatrix}$$

and keeping in mind the (1), we have that

$$S_{EDF} = \frac{bc}{2} = S_{60^\circ}$$

so  $\angle EDF = 60^\circ$ .

Now we know that if  $\theta$  is the oriented angle between the line  $px + qy + rz = 0$  and a line  $d$ , the coordinates of the infinite point of this line are

$$(pa^2 + q(S_\theta - S_C) - r(S_\theta + S_B) : qb^2 + r(S_\theta - S_A) - p(S_\theta + S_C) : rc^2 + p(S_\theta - S_B) - q(S_\theta + S_A))$$

in our case, the angle between line  $DE \equiv (s-a)x + (s-b)y - (s-c)z = 0$  and the angle bisector  $DL$  is  $30^\circ$ , so the coordinates of the infinite point of  $DL$  are

$$\begin{aligned} DL_\infty(a^2(s-a) + (s-b)(\sqrt{3}S - S_C) + (s-c)(\sqrt{3}S + S_B) : \\ b^2(s-b) - (s-c)(\sqrt{3}S - S_A) - (s-a)(\sqrt{3}S + S_C) : \\ -c^2(s-c) + (s-a)(\sqrt{3}S - S_B) - (s-b)(\sqrt{3}S + S_A)) \end{aligned}$$

and using the (1) we obtain

$$DL_\infty(a(a-b+2c) : b(a-b-c) : c(2b-2a-c)).$$

Line  $OI$  have equation

$$\begin{vmatrix} a & b & c \\ a^2S_A & b^2S_B & c^2S_C \\ x & y & z \end{vmatrix} = 0 \Rightarrow OI \equiv bc(cS_C - bS_B)x - ac(cS_C - aS_A)y + ab(bS_B - aS_A)z = 0$$

so the infinite point of line  $OI$  using the (1) is

$$OI_\infty(a(ab - 4bc + ac + b^2 + c^2) : b(-ab - 2bc + 2ac - b^2 + 2c^2) : c(2ab - 2bc - ac + 2b^2 - c^2))$$

Now we know that two lines with infinite points  $(f : g : h)$  and  $(f' : g' : h')$  are perpendicular to each other if and only if

$$S_A f f' + S_B g g' + S_C h h' = 0$$

and this is the case of our two infinite points  $DL_\infty$  and  $OI_\infty$ , so line  $OI$  is perpendicular to angle bisector of  $\angle EDF$  and also the contrary is immediately proved.

*Also solved by Daniel Lasaosa, Pamplona, Spain; Mehtaab Sawhney, USA; Yujin Kim, Stony Brook School, Stony Brook, NY, USA; Misiakos Panagiotis, Athens College (HAEF), Nea Penteli; Ercole Suppa, Teramo, Italy; Bodhisattwa Bhowmik, RKMV, Agartala, Tripura, India; Sardor Bozorboyev, Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Titu Zvonaru, Comănești, Romania and Neculai Stanciu, Buzău, Romania.*

S321. Let  $x$  be a real number such that  $x^m(x+1)$  and  $x^n(x+1)$  are rational for some relatively prime positive integers  $m$  and  $n$ . Prove that  $x$  is rational.

*Proposed by Mihai Piticari, Campulung Moldovenesc, Romania*

*Solution by Yujin Kim, Stony Brook School, Stony Brook, NY, USA*

Suppose  $m > n$  and denote  $m - n = q > 0$ . One can assume  $x \neq 0, -1$ .

$$\frac{x^m(x+1)}{x^n(x+1)} \in \mathbb{Q} \rightarrow x^2 \in \mathbb{Q}$$

$$\frac{[x^n(x+1)]^m}{[x^m(x+1)]^n} \in \mathbb{Q} \rightarrow \frac{x^{mn}(x+1)^m}{x^{mn}(x+1)^n} \in \mathbb{Q} \rightarrow (x+1)^q \in \mathbb{Q}$$

We will show that if  $x^2, (x+1)^2$  are both rational and  $x$  is real. Thus  $x$  is rational.

Consider two polynomials:  $R(z) = z^2 - x^2$  and  $S(z) = (z+1)^2 - (x+1)^2$

Note that both  $R$  and  $S$  have rational coefficients and  $R(x) = S(x) = 0$ . Hence  $R$  and  $S$  have  $x$  as a common root. Suppose that  $x \notin \mathbb{Q}$ . Denote by  $m(z)$  the minimal polynomial of  $x$  that has the smallest degree polynomial with rational coefficients which has  $x$  as a root. Such polynomial exists since  $R(x) = S(x) = 0$ . Since  $x$  is irrational, it follows that degree  $m \geq 2$ .

On the other hand, both  $R(z)$  and  $S(z)$  are multiples of  $m(z)$ . This means that  $R$  and  $S$  have some other common root besides  $x$ . Denote such a root by  $y$ . We have  $y \neq x$ . Note that  $y$  could be a complex unreal number.

We have  $R(y) = S(y) = 0 \rightarrow y^\Sigma = x^\Sigma$  and  $(1+y)^\Sigma = (1+x)^\Sigma$ .

Consequently,  $|y|^\Sigma = |x|^\Sigma$  and  $|1+y|^\Sigma = |1+x|^\Sigma \rightarrow$

$$\begin{cases} |y| = |x| & \text{Recall that } x \in \mathbb{R} \\ |1+y| = |1+x| & \text{Denote } y = a + ib \end{cases}$$

Then,  $|a+ib| = |x| \rightarrow \sqrt{a^2+b^2} = |x| \rightarrow a^2+b^2 = x^2$  Also  $|1+a+ib| = |1+x| \rightarrow \sqrt{(1+a)^2+b^2} = |1+x| \rightarrow (1+a)^2+b^2 = (1+x)^2 \rightarrow 1+2a+a^2+b^2 = 1+2x+x^2 \rightarrow a = x$  and since  $a^2+b^2 = x^2$  this implies  $b = 0$ .

But then  $y = a+ib = x+i \cdot 0 = x$ , contradiction. The conclusion follows.

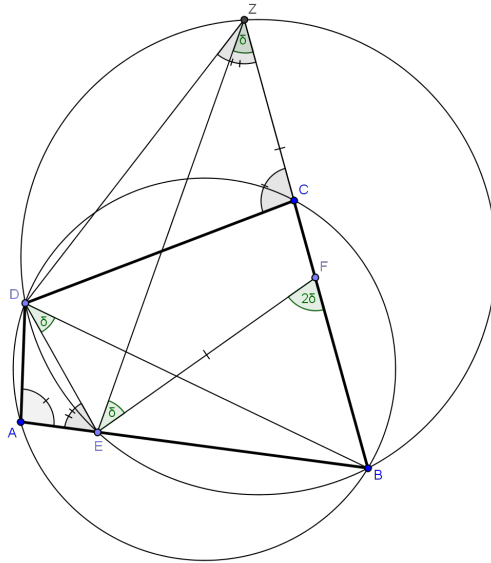
*Also solved by Seung Hwan An, Taft School, Watertown, CT, USA; William Kang, Bergen County Academies, Hackensack, NJ, USA; Timothy Chon, Horace Mann School, Bronx, NY, USA; Yooree Ha, Ponte Vedra High School, Ponte Vedra, FL, USA.*

S322. Let  $ABCD$  be a cyclic quadrilateral. Points  $E$  and  $F$  lie on the sides  $AB$  and  $BC$ , respectively, such that  $\angle BFE = 2\angle BDE$ . Prove that

$$\frac{EF}{AE} = \frac{FC}{AE} + \frac{CD}{AD}.$$

*Proposed by Nairi Sedrakyan, Yerevan, Armenia*

*Solution by Andrea Fanchini, Cantú, Italy*



Let  $Z$  be the intersection point between  $BC$  and the circumcircle of  $\triangle DEB$ .  
Then  $\angle BZE = \angle BDE = \delta$   
Now if we consider  $\triangle EFZ$ , we have

$$\angle FEZ + \angle EZF + \angle ZFE = \pi \Rightarrow \angle FEZ = \pi - \delta - (\pi - 2\delta) = \delta$$

Therefore  $\triangle EFZ$  is isosceles and then  $EF = FZ$ .

Now being  $ABCD$  a cyclic quadrilateral

$$\angle BAD + \angle BCD = \pi \Rightarrow \angle DCZ = \angle BAD$$

Furthermore being  $BEDZ$  a cyclic quadrilateral

$$\angle BED + \angle BZD = \pi \Rightarrow \angle AED = \angle BZD$$

So  $\triangle DCZ$  and  $\triangle DAE$  are similar, that is

$$\frac{CZ}{CD} = \frac{AE}{AD}$$

but  $CZ = FZ - FC$  or also, being  $EF = FZ$ , we have that  $CZ = EF - FC$ . Therefore

$$\frac{EF - FC}{CD} = \frac{AE}{AD} \Rightarrow \frac{EF}{AE} = \frac{FC}{AE} + \frac{CD}{AD}$$

so the equality is proved.

*Also solved by Yujin Kim, Stony Brook School, Stony Brook, NY, USA; William Kang, Bergen County Academies, Hackensack, NJ, USA; Jhiseung Daniel Hahn, Phillips Exeter Academy, Exeter, NH, USA; Bodhisattwa Bhowmik, RKMV, Agartala, Tripura, India; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan.*

S323. Solve in positive integers the equation

$$x + y + (x - y)^2 = xy.$$

*Proposed by Neculai Stanciu and Titu Zvonaru, Romania*

*Solution by Daniel Lasoasa, Pamplona, Spain*

Denote  $s = x + y$ ,  $d = x - y$ , and note that the proposed equation rewrites as

$$s^2 - d^2 = 4xy = 4x + 4y + 4(x - y)^2 = 4s + 4d^2, \quad (s - 2)^2 - 5d^2 = 4.$$

This is clearly a Pell-like equation  $c^2 - 5d^2 = 4$ , where  $s - 2 = c$  and  $d$  are integers of the same parity iff  $x, y$  are integers, and all of whose infinite solutions  $(c_n, d_n)$  may be found through the recurrent relations

$$c_{n+2} = 3c_{n+1} - c_n, \quad d_{n+2} = 3d_{n+1} - d_n, \quad n \geq 0,$$

with initial conditions  $(c_0, d_0) = (2, 0)$  and  $(c_1, d_1) = (3, 1)$ . Therefore, all solutions are of the form

$$s_n = \left( \frac{\sqrt{5} + 1}{2} \right)^{2n} + \left( \frac{\sqrt{5} - 1}{2} \right)^{2n} + 2, \quad d_n = \frac{1}{\sqrt{5}} \left( \frac{\sqrt{5} + 1}{2} \right)^{2n} - \frac{1}{\sqrt{5}} \left( \frac{\sqrt{5} - 1}{2} \right)^{2n},$$

or equivalently,

$$x_n = \frac{s_n + d_n}{2} = \frac{1}{\sqrt{5}} \left( \frac{\sqrt{5} + 1}{2} \right)^{2n+1} + \frac{1}{\sqrt{5}} \left( \frac{\sqrt{5} - 1}{2} \right)^{2n+1} + 1,$$

$$y_n = \frac{s_n - d_n}{2} = \frac{1}{\sqrt{5}} \left( \frac{\sqrt{5} + 1}{2} \right)^{2n-1} + \frac{1}{\sqrt{5}} \left( \frac{\sqrt{5} - 1}{2} \right)^{2n-1} + 1,$$

where  $n$  can take any non-negative integer value. We can readily check that indeed the following relation holds:

$$x_n y_n - x_n - y_n = \frac{1}{5} \left( \left( \frac{\sqrt{5} + 1}{2} \right)^{4n} + \left( \frac{\sqrt{5} - 1}{2} \right)^{4n} - 2 \right) = d_n^2 = (x_n - y_n)^2.$$

We conclude that all solutions of the proposed equation are those  $(x_n, y_n)$  already found, or the result of interchanging  $x_n, y_n$  by symmetry in the problem.

*Also solved by Brian Bradie, Christopher Newport University, Newport News, VA, USA; Chaeyeon Oh, Episcopal High School, Alexandria, VA, USA; Mehtaab Sawhney, USA; Cody Johnson, USA; Jhiseung Daniel Hahn, Phillips Exeter Academy, Exeter, NH, USA; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Albert Stadler, Herrliberg, Switzerland; Alok Kumar, Delhi, India; Arber Avdullahu, Mehmet Akif College, Kosovo; Bodhisattwa Bhowmik, RKMV, Agartala, Tripura, India; David E. Manes, Oneonta, NY, USA; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan.*

S324. Find all functions  $f : S \rightarrow S$  satisfying

$$f(x)f(y) + f(x) + f(y) = f(xy) + f(x + y)$$

for all  $x, y \in S$  when (i)  $S = \mathbb{Z}$ ; (ii)  $S = \mathbb{R}$ .

*Proposed by Prasanna Ramakrishnan, Port of Spain, Trinidad and Tobago*

*Solution by Adnan Ali, Student in A.E.C.S-4, Mumbai, India*

For the first part where  $S = \mathbb{Z}$ , putting  $x = y = 0$  we obtain  $f(0) = 0$ . Then, putting  $(x, y) = (1, -1)$ , we have

$$f(1) + f(-1) + f(1)f(-1) = f(0) + f(-1) \Rightarrow f(1)[f(-1) + 1] = 0.$$

(a)  $f(1) = 0$ , then,

$$f(x - 1) + f(1) + f(1)f(x - 1) = f(x) + f(x - 1) \Rightarrow \boxed{f(x) = 0} \quad \forall x \in \mathbb{Z}$$

(b)  $f(-1) = -1$ , then,

$$\begin{aligned} f(2) + f(-1) + f(-1)f(2) &= f(1) + f(-2) \Rightarrow f(1) + f(-2) = f(-1) \\ f(-2) + f(1) + f(1)f(-2) &= f(-1) + f(-2) \Rightarrow f(-1) = f(1) + f(1)f(-2) \end{aligned}$$

Combining the above two results,  $f(-2)[f(1) - 1] = 0$ .

(i)  $f(1) = 1$ , then,

$$f(x) + f(1) + f(x)f(1) = f(x) + f(x + 1) \Rightarrow f(x) + 1 = f(x + 1) \quad \forall x \in \mathbb{Z}$$

and due to induction,  $\boxed{f(x) = x} \quad \forall x \in \mathbb{Z}$ .

(ii)  $f(-2) = 0$ , then,  $f(1) + f(-2) = f(-1) = -1 \Rightarrow f(1) = -1$  and so,

$$f(x) + f(1) + f(x)f(1) = f(x) + f(x + 1) \Rightarrow f(x) + f(x + 1) + 1 = 0$$

and replacing  $x$  by  $x + 1$  in the above relation,

$$f(x + 1) + f(x + 2) + 1 = 0$$

and so,

$$(1.0) \cdots f(x) = f(x + 2),$$

thus applying induction, we have  $f(2n) = 0$  and  $f(2n - 1) = -1$  for all  $n \in \mathbb{Z}$  or

$$\boxed{f(x) = \begin{cases} 0, & \text{when } x \text{ is even} \\ -1, & \text{when } x \text{ is odd} \end{cases}}$$

Now we come to the second part where  $S = \mathbb{R}$ . We have the same conditions as before except that the domain is the real set. So,

(a)  $f(1) = 0$ , then,

$$f(x - 1) + f(1) + f(1)f(x - 1) = f(x) + f(x - 1) \Rightarrow \boxed{f(x) = 0} \quad \forall x \in \mathbb{R}$$

(b)  $f(-1) = -1$  and so,  $f(-2)[f(1) - 1] = 0$ .

(i)  $f(1) = 1$ , then,

$$f(x) + f(-1) + f(x)f(-1) = f(x - 1) + f(-x) \Rightarrow f(x - 1) = -f(-x) - 1.$$

And also, in

$$(1.1) \dots f(x+1) = f(x) + 1 \text{ (as it was in the previous part),}$$

replacing  $x+1$  by  $x$ , we have  $f(x) = f(x-1) + 1 = -f(-x)$ . Thus we conclude that  $-f(x) = f(-x) \forall x \in \mathbb{R}$ . And replacing  $x$  by  $x+1$  in (1.1),  $f(x+2) = f(x+1) + 1 = f(x) + 2$ . So,  $f(x+1) = f(x) + 1 \Rightarrow f(2) = f(1) + 1 = 2$ . Thus,

$$f(2) + f(x) + f(2)f(x) = f(2x) + f(x+2) \Rightarrow 2f(x) = f(2x) \forall x \in \mathbb{R}.$$

Now,

$$\begin{aligned} f(x+y) + f(x-y) + f(x+y)f(x-y) &= f(x^2 - y^2) + f(2x) \\ f(x+y) + f(y-x) + f(x+y)f(y-x) &= f(y^2 - x^2) + f(2y) \end{aligned}$$

adding up the above two relations and using  $f(-a) = -f(a) \forall a \in \mathbb{R}$ , we have  $f(x+y) = f(x) + f(y)$  implying  $f(x)f(y) = f(xy)$ , thus the function is both additive and multiplicative, implying that  $f(x) = x$   $\forall x \in \mathbb{R}$  and  $f(x) = 0 \forall x \in \mathbb{R}$ . It is easy to see that both of them satisfy the functional equation.

(ii)

$$f(x) = \begin{cases} 0, & \text{when } x \text{ is even } \forall x \in \mathbb{Z} \\ -1, & \text{when } x \text{ is odd } \forall x \in \mathbb{Z} \end{cases}$$

So, according to the previous part we have the result (1.0),

$$f(x+2) = f(x)$$

which here, holds true for all real  $x$ . So,  $x = \frac{1}{2}$  yields

$$f\left(\frac{1}{2}\right) = f\left(\frac{5}{2}\right) \dots (1.2)$$

Whereas,  $f\left(\frac{1}{2}\right) + f(2) + f\left(\frac{1}{2}\right)f(2) = f(1) + f\left(\frac{5}{2}\right)$  implying that

$$f\left(\frac{1}{2}\right) = -1 + f\left(\frac{5}{2}\right)$$

contradicting (1.2). Thus in the real domain,  $f(-1) = -1$  and  $f(-2) = 0$  cannot both be simultaneously true. Thus in summary, we have the functions as:

(1) When  $S = \mathbb{Z}$ ,

$$\begin{aligned} f(x) &= 0 \\ f(x) &= x \end{aligned}$$

$$f(x) = \begin{cases} 0, & \text{when } x \text{ is even} \\ -1, & \text{when } x \text{ is odd} \end{cases}$$

(2) When  $S = \mathbb{R}$ .

$$\begin{aligned} f(x) &= 0 \\ f(x) &= x \end{aligned}$$

*Also solved by Mehtaab Sawhney, USA; Misiakos Panagiotis, Athens College (HAEF), Nea Penteli; Cody Johnson, USA; Arber Avdullahu, Mehmet Akif College, Kosovo; Bodhisattwa Bhowmik, RKMV, Agartala, Tripura, India; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan.*

## Undergraduate problems

U319. Let  $A, B, C$  be the measured (in radians) of the angles of a triangle with circumradius  $R$  and inradius  $r$ . Prove that

$$\frac{A}{B} + \frac{B}{C} + \frac{C}{A} \leq \frac{2R}{r} - 1$$

*Proposed by Nermin Hodžić, Bosnia and Herzegovina and Salem Malikić, Canada*

*Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy*

We employ two known inequalities. The first one is

$$\frac{1}{ABC} \leq \frac{27}{2\pi^3} \frac{R}{r}$$

and can be found in problem 3757 of *Crux Mathematicorum*, vol.39–6, 2013. The second one is a known inequality

$$a^2c + b^2a + c^2b \leq \frac{4}{27}(a + b + c)^3 - abc$$

given that  $a, b, c \geq 0$ .

Therefore, we have

$$\begin{aligned} \frac{A}{B} + \frac{B}{C} + \frac{C}{A} &= \frac{A^2C + B^2A + C^2B}{ABC} \leq \\ &= \frac{1}{ABC} \left( \frac{4}{27}(A + B + C)^3 - ABC \right) = \\ &= \frac{1}{ABC} \frac{4}{27} \pi^3 - 1 \leq \frac{27}{2\pi^3} \frac{R}{r} \frac{4}{27} \pi^3 - 1 = 2 \frac{R}{r} - 1 \end{aligned}$$

*Also solved by Dragoljub Milošević, Gornji Milanovac, Serbia; Cody Johnson, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.*



U320. Evaluate

$$\sum_{n \geq 0} \frac{2^n}{2^{2^n} + 1}.$$

*Proposed by Titu Andreescu, University of Texas at Dallas*

*Solution by Daniel Lasasoa, Pamplona, Spain*

Denote for all non-negative integer  $N$ ,

$$S_N = \sum_{n=0}^N \frac{2^n}{2^{2^n} + 1}.$$

We show that for all positive integer  $N$ , we have

$$S_N = 1 - \frac{2^{N+1}}{2^{2^{N+1}} - 1}.$$

The result is clearly true for  $N = 0$  and  $N = 1$ , since

$$S_0 = \frac{1}{3} = 1 - \frac{2}{3} = 1 - \frac{2^1}{2^{2^1} - 1}, \quad S_1 = S_0 + \frac{2}{5} = 1 - \frac{4}{15} = 1 - \frac{2^2}{2^{2^2} - 1}.$$

If the result is true for  $N - 1$ , then for  $N$  we have

$$S_N = S_{N-1} + \frac{2^N}{2^{2^N} + 1} = 1 - 2^N \frac{(2^{2^N} + 1) - (2^{2^N} - 1)}{(2^{2^N} + 1)(2^{2^N} - 1)} = 1 - \frac{2^{N+1}}{2^{2 \cdot 2^N} - 1},$$

or the result is true by induction for all non-negative integer  $N$ . It follows that

$$\sum_{n \geq 0} \frac{2^n}{2^{2^n} + 1} = \lim_{N \rightarrow \infty} S_{N-1} = 1 - \lim_{N \rightarrow \infty} \frac{2^N}{2^{2^N} - 1} = 1 - \lim_{x \rightarrow \infty} \frac{x}{2^x - 1} = 1,$$

since as it is well known, the exponential function increases much more rapidly than the linear function, and where we have defined  $x = 2^N$ . The conclusion follows.

*Also solved by Reiner Martin, Bad Soden-Neuenhain, Germany; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; David E. Manes, Oneonta, NY, USA; Arkady Alt, San Jose, California, USA; Alok Kumar, Delhi, India; Albert Stadler, Herrliberg, Switzerland; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Yong Xi Wang, East China Institute Of Technology, China; Cody Johnson, USA; Ji Eun Kim, Tabor Academy, Marion, MA, USA; William Kang, Bergen County Academies, Hackensack, NJ, USA; Yujin Kim, Stony Brook School, Stony Brook, NY, USA; Mehtaab Sawhney, USA.*

U321. Consider the sequence of polynomials  $(P_s)_{s \geq 1}$  defined by

$$P_{k+1}(x) = (x^a - 1)P_k'(x) - (k + 1)P_k(x), k = 1, 2, \dots,$$

where  $P_1(x) = x^{a-1}$  and  $a$  is an integer greater than 1.

1. Find the degree of  $P_k$
2. Determine  $P_k(0)$

*Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania*

*Solution by the proposer*

1. We have  $\deg P_1 = a - 1$ , and from the recursive relation we obtain  $\deg P_2 = 2a - 2$ . Now, a simple inductive argument shows that  $\deg P_k = k(a - 1)$ .
2. Let us consider the function  $f : (-1, 1) \rightarrow \mathbb{R}$ , defined by  $f(x) = \frac{1}{x^a - 1}$ . Observe that

$$f'(x) = \frac{ax^{a-1}}{(x^a - 1)^2} = a \frac{P_1(x)}{(x^a - 1)^2}.$$

One can prove immediately by induction that

$$f^{(k)}(x) = a \frac{P_k(x)}{(x^a - 1)^{k+1}},$$

where we use the recursive relation in the definition of the sequence. This implies

$$P_k(0) = \frac{(-1)^{k+1}}{a} f^{(k)}(0),$$

and we reduce the problem of finding the value of  $P_k(0)$  to determine  $f^{(k)}(0)$ . In this respect, using the geometric series we have

$$f(x) = \frac{1}{x^a - 1} = - \sum_{s=0}^{\infty} x^{as},$$

and we obtain

$$f^{(k)}(0) = \begin{cases} -k! & \text{if } a|k \\ 0 & \text{otherwise} \end{cases}$$

Finally,

$$P_k(0) = \frac{(-1)^{k+1}}{a} f^{(k)}(0) = \begin{cases} \frac{(-1)^k k!}{a} & \text{if } a|k \\ 0 & \text{otherwise} \end{cases}$$

*Also solved by Bodhisattwa Bhowmik, RKMV, Agartala, Tripura, India.*

U322. Evaluate

$$\sum_{n=1}^{\infty} \frac{16n^2 - 12n + 1}{n(4n - 2)!}.$$

*Proposed by Titu Andreescu, USA and Oleg Mushkarov, Bulgaria*

*First solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA*

Note that

$$\begin{aligned} \frac{16n^2 - 12n + 1}{n(4n - 2)!} &= \frac{4n(4n - 3)}{n(4n - 2)!} + \frac{1}{n(4n - 2)!} \\ &= 4 \cdot \frac{4n - 3}{(4n - 2)!} + 4 \cdot \frac{4n - 1}{(4n)!} \\ &= 4 \left( \frac{1}{(4n - 3)!} - \frac{1}{(4n - 2)!} + \frac{1}{(4n - 1)!} - \frac{1}{(4n)!} \right). \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{16n^2 - 12n + 1}{n(4n - 2)!} &= 4 \sum_{n=1}^{\infty} \left( \frac{1}{(4n - 3)!} - \frac{1}{(4n - 2)!} + \frac{1}{(4n - 1)!} - \frac{1}{(4n)!} \right) \\ &= 4 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n!} = 4 \left( 1 - \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \right) \\ &= 4 \left( 1 - \frac{1}{e} \right). \end{aligned}$$

*Second solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA*

Write

$$16n^2 - 12n + 1 = (4n - 2)(4n - 3) + 2(4n - 2) - 1,$$

so that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{16n^2 - 12n + 1}{n(4n - 2)!} &= \sum_{n=1}^{\infty} \frac{1}{n(4n - 4)!} + 2 \sum_{n=1}^{\infty} \frac{1}{n(4n - 3)!} - \sum_{n=1}^{\infty} \frac{1}{n(4n - 2)!} \\ &= f''(1) + 2f'(1) - f(1), \end{aligned}$$

where

$$f(x) = \sum_{n=1}^{\infty} \frac{x^{4n-2}}{n(4n - 2)!}.$$

Next, consider the function

$$g(x) = \sum_{n=1}^{\infty} \frac{x^{4n-2}}{(4n - 2)!},$$

and note that  $g^{(4)}(x) - g(x) = 0$  subject to the initial conditions  $g(0) = 0$ ,  $g'(0) = 0$ ,  $g''(0) = 1$ , and  $g'''(0) = 0$ . Thus,

$$g(x) = \frac{1}{2}(\cosh x - \cos x).$$

To obtain  $f$ , multiply  $g$  by  $x$ , integrate term-by-term, determine the constant of integration using the initial condition  $f(0) = 0$ , multiply by 4, and divide by  $x^2$ . This yields

$$f(x) = \frac{2}{x^2}(2 + x \sinh x - \cosh x - x \sin x - \cos x).$$

Now,

$$\begin{aligned}f'(x) &= \frac{2}{x^2}(x \cosh x - x \cos x) - \frac{4}{x^3}(2 + x \sinh x - \cosh x - x \sin x - \cos x), \\f''(x) &= \frac{2}{x^2}(x \sinh x + \cosh x + x \sin x - \cos x) - \frac{8}{x^3}(x \cosh x - x \cos x) + \\&\quad \frac{12}{x^4}(2 + x \sinh x - \cosh x - x \sin x - \cos x),\end{aligned}$$

so that

$$\begin{aligned}f(1) &= 2 \left( 2 - \frac{1}{e} - \sin 1 - \cos 1 \right), \\f'(1) &= 2(\cosh 1 - \cos 1) - 4 \left( 2 - \frac{1}{e} - \sin 1 - \cos 1 \right), \\f''(1) &= 2(e + \sin 1 - \cos 1) - 8(\cosh 1 - \cos 1) + 12 \left( 2 - \frac{1}{e} - \sin 1 - \cos 1 \right),\end{aligned}$$

and

$$\sum_{n=1}^{\infty} \frac{16n^2 - 12n + 1}{n(4n - 2)!} = f''(1) + 2f'(1) - f(1) = 4 \left( 1 - \frac{1}{e} \right).$$

*Also solved by Daniel Lasoosa, Pamplona, Spain; Bodhisattwa Bhowmik, RKMV, Agartala, Tripura, India; Arkady Alt, San Jose, California, USA; Albert Stadler, Herrliberg, Switzerland; Cemal Kadirov, Istanbul University, Turkey; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Cody Johnson, USA; Ji Eun Kim, Tabor Academy, Marion, MA, USA; William Kang, Bergen County Academies, Hackensack, NJ, USA; Yujin Kim, Stony Brook School, Stony Brook, NY, USA; Mehtaab Sawhney, USA; Chaeyeon Oh, Episcopal High School, Alexandria, VA, USA.*

U323. Let  $X$  and  $Y$  be independent random variables following a uniform distribution

$$p_X(x) = \begin{cases} 1 & 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

What is the probability that inequality  $X^2 + Y^2 \geq 3XY$  is true?

*Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA*

*Solution by Daniel Lasaosa, Pamplona, Spain*

Note first that the proposed inequality can be written as

$$\left(Y - \frac{3 + \sqrt{5}}{2}X\right) \left(Y - \frac{3 - \sqrt{5}}{2}X\right) \geq 0,$$

ie since  $X, Y$  take only non-negative values, we must have either  $Y \geq \frac{3 + \sqrt{5}}{2}X \geq \frac{3 - \sqrt{5}}{2}X$  or  $Y \leq \frac{3 - \sqrt{5}}{2}X \leq \frac{3 + \sqrt{5}}{2}X$ . This defines inside the unit square with vertices  $(X, Y) = (0, 0), (0, 1), (1, 1), (1, 0)$  two triangles, one with vertices  $(X, Y) = (0, 0), (1, 0), \left(1, \frac{3 - \sqrt{5}}{2}\right)$ , and one with vertices  $(X, Y) = (0, 0), (0, 1), \left(\frac{3 - \sqrt{5}}{2}, 1\right)$ , inside which or on whose boundaries the inequality is true, being false in the rest of the unit square. Since both variables  $X, Y$  are independent and uniform, the probability of  $(X, Y)$  falling inside or on the boundary of one of these triangles equals the sum  $\frac{3 - \sqrt{5}}{2}$  of the areas of both triangles, divided by the area 1 of the unit square. The probability is therefore

$$\frac{3 - \sqrt{5}}{2}.$$

*Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Cody Johnson, USA; Ji Eun Kim, Tabor Academy, Marion, MA, USA; William Kang, Bergen County Academies, Hackensack, NJ, USA; Yujin Kim, Stony Brook School, Stony Brook, NY, USA.*

U324. Let  $f : [0, 1] \rightarrow \mathbb{R}$  be a differentiable function such that  $f(1) = 0$ . Prove that there is  $c \in (0, 1)$  such that  $|f(c)| \leq |f'(c)|$

*Proposed by Marius Cavachi, Constanta, Romania*

*Solution by G.R.A.20 Problem Solving Group, Roma, Italy*

We will prove that for all  $M > 0$  there is  $c \in (0, 1)$  such that  $M|f(c)| \leq |f'(c)|$ .

If there is a  $c \in (0, 1)$  such that  $f(c) = 0$  then the inequality is trivial. Otherwise by continuity  $f$  has constant sign and without loss of generality we may assume that  $f(x) > 0$  in  $(0, 1)$ . By the Mean Value Theorem, for all  $t \in (1/2, 1)$  there is  $c_t \in (1/2, t)$  such that

$$\frac{\ln(f(t)) - \ln(f(1/2))}{t - 1/2} = \frac{f'(c_t)}{f(c_t)}.$$

The condition  $f(1) = 0$  implies that the LHS tends to  $-\infty$  as  $t \rightarrow 1^-$ . Hence for all  $M > 0$  there is a  $c_t$  such that

$$\frac{f'(c_t)}{f(c_t)} \leq -M.$$

Therefore  $f'(c_t) < 0$  and

$$\frac{|f'(c_t)|}{|f(c_t)|} = \frac{-f'(c_t)}{f(c_t)} \geq M.$$

*Also solved by Reiner Martin, Bad Soden-Neuenhain, Germany; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Corneliu Mănescu- Avram, Transportation High School, Ploiești, Romania; Arkady Alt, San Jose, California, USA; Cody Johnson, USA.*

## Olympiad problems

O319. Let  $f(x)$  and  $g(x)$  be arbitrary functions defined for all  $x \in \mathbb{R}$ . Prove that there is a function  $h(x)$  such that  $(f(x) + h(x))^{2014} + (g(x) + h(x))^{2014}$  is an even function for all  $x \in \mathbb{R}$ .

*Proposed by Nairi Sedrakyan, Yerevan, Armenia*

*Solution by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain*

It is well-known that any function  $f$  may be written as  $f(x) = f_e(x) + f_o(x)$  where  $f_e$  and  $f_o(x)$  are respectively the even and odd part of function  $f$ . Also  $f_e(x) = \frac{f(x) + f(-x)}{2}$  while  $f_o(x) = \frac{f(x) - f(-x)}{2}$ .

Let us take  $h(x) = -f_o(x) - g_e(x)$ . Then,  $f(x) + h(x) = f_e(x) - g_e(x)$ , and  $g(x) + h(x) = -f_o(x) + g_o(x)$ . Therefore,

$$\begin{aligned} F(x) &= (f(x) + h(x))^{2014} + (g(x) + h(x))^{2014} \\ &= (f_e(x) - g_e(x))^{2014} + (-f_o(x) + g_o(x))^{2014}. \end{aligned}$$

Then,

$$\begin{aligned} F(-x) &= (f_e(-x) - g_e(-x))^{2014} + (-f_o(-x) + g_o(-x))^{2014} \\ &= (f_e(x) - g_e(x))^{2014} + (f_o(x) - g_e(x))^{2014} \\ &= F(x). \end{aligned}$$

*Also solved by Daniel Lasaosa, Pamplona, Spain; Yassine Hamdi, Lycee du Parc, Lyon, France; George - Petru Scărlătescu, Pitești, Romania; Misiakos Panagiotis, Athens College (HAEF), Nea Penteli; Mehtaab Sawhney, USA; Michael Tang, Edina High School, MN, USA.*

O320. Let  $n$  be a positive integer and let  $0 < y_i \leq x_i < 1$  for  $1 \leq i \leq n$ . Prove that

$$\frac{1 - x_1 \cdots x_n}{1 - y_1 \cdots y_n} \leq \frac{1 - x_1}{1 - y_1} + \cdots + \frac{1 - x_n}{1 - y_n}.$$

*Proposed by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain*

*Solution by Marius Stănean, Zalău, Romania*

We can prove this inequality by mathematical induction. For  $n = 1$  it's obvious. For  $n = 2$  we have

$$\frac{1 - x_1 x_2}{1 - y_1 y_2} \leq \frac{1 - x_1}{1 - y_1} + \frac{1 - x_2}{1 - y_2}.$$

Let  $a = \frac{y_1}{x_1} \leq 1$  and  $b = \frac{y_2}{x_2} \leq 1$ , so the inequality becomes

$$\frac{1 - x_1 x_2}{1 - abx_1 x_2} \leq \frac{1 - x_1}{1 - ax_1} + \frac{1 - x_2}{1 - bx_2} \iff$$

$$(1 - x_1)(1 - x_2) + ax_1 x_2(1 - x_1) + bx_1 x_2(1 - x_2) - 3abx_1 x_2 + abx_1^2 x_2^2 + abx_1 x_2(x_1 + x_2) + a^2 b x_1^2 x_2 + ab^2 x_1 x_2^2 - ab(a + b)x_1^2 x_2^2 \geq 0$$

dividing this inequality with  $x_1 x_2$  we obtain

$$\begin{aligned} & \frac{(1 - x_1)(1 - x_2)}{x_1 x_2} + ab(1 - x_1)(1 - x_2) + 2ab(x_1 + x_2 - 2) + \\ & a^2 x_1(1 - x_2) + ab^2 x_2(1 - x_1) + a(1 - x_1) + b(1 - x_2) \geq 0 \iff \\ & \frac{(1 - x_1)(1 - x_2)}{x_1 x_2} + a(1 - x_1)(b^2 x_2 - 2b + 1) + b(1 - x_2)(a^2 x_1 - 2a + 1) + ab(1 - x_1)(1 - x_2) \geq 0 \iff \\ & \frac{(1 - x_1)(1 - x_2)}{x_1 x_2} + ab(1 - x_1)(1 - x_2) + a(1 - x_1)(b - 1)^2 + \\ & b(1 - x_2)(a - 1)^2 - ab(a + b)(1 - x_1)(1 - x_2) \geq 0 \iff \\ & (1 - x_1)(1 - x_2) \left[ \frac{1}{x_1 x_2} + ab - ab(a + b) \right] + a(1 - x_1)(b - 1)^2 + b(1 - x_2)(a - 1)^2 \geq 0 \end{aligned}$$

which is true because

$$\frac{1}{x_1 x_2} + ab - ab(a + b) \geq 1 + ab - a - b = (1 - a)(1 - b) \geq 0.$$

Suppose that the original inequality holds for  $n - 1 \in \mathbb{N}$  and we want to prove it for  $n$ . Therefore we have

$$\frac{1 - x_1 \cdots x_{n-1}}{1 - y_1 \cdots y_{n-1}} \leq \frac{1 - x_1}{1 - y_1} + \cdots + \frac{1 - x_{n-1}}{1 - y_{n-1}}$$

but  $Y = y_1 y_2 \cdots y_{n-1} \leq x_1 x_2 \cdots x_{n-1} = X < 1$  and then applying the inequality for  $X, Y$ , respectively  $x_n, y_n$  we obtain

$$\begin{aligned} \frac{1 - x_1 \cdots x_n}{1 - y_1 \cdots y_n} &= \frac{1 - X \cdot x_n}{1 - Y \cdot y_n} \leq \frac{1 - X}{1 - Y} + \frac{1 - x_n}{1 - y_n} \leq \\ & \frac{1 - x_1}{1 - y_1} + \cdots + \frac{1 - x_{n-1}}{1 - y_{n-1}} + \frac{1 - x_n}{1 - y_n}. \end{aligned}$$

*Also solved by Daniel Lasaosa, Pamplona, Spain; Sardor Bozorboyev, Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Yassine Hamdi, Lycee du Parc, Lyon, France; Arkady Alt, San Jose, California, USA; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; William Kang, Bergen County Academics, Hackensack, NJ, USA; Misiakos Panagiotis, Athens College (HAEF), Nea Penteli; Yujin Kim, Stony Brook School, Stony Brook, NY, USA; Mehtaab Sawhney, USA.*

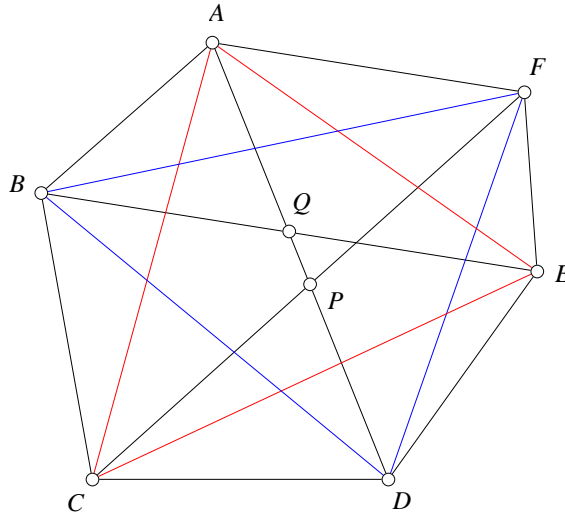


O321. Each of the diagonals  $AD, BE, CF$  of the convex hexagon  $ABCDEF$  divides its area in half. Prove that

$$AB^2 + CD^2 + EF^2 = BC^2 + DE^2 + FA^2.$$

*Proposed by Nairi Sedrakyan, Yerevan, Armenia*

*Solution by Ercole Suppa, Teramo, Italy*



We begin by proving the following preliminary results:

**Claim 1.** We have  $AC/DF = CE/FB = EA/BD$ .

*Proof of Claim 1.* Let  $P = AD \cap CF$ . Observe that the areas of triangles  $\triangle APF$  and  $\triangle DPC$  are equal because

$$[APF] = [ADEF] - [DEFP] = [CDEF] - [DEFP] = [DPC]$$

Therefore  $AP \cdot PF = CP \cdot PD$ , or  $AP/PD = CP/PF$ .

Thus  $\triangle APC \sim \triangle DPF$  (SAS) whence  $\angle CAP = \angle FDP$  and  $AC \parallel DF$ .

In a similar way, we can prove that  $BF \parallel CE$  and  $AE \parallel BD$ , so that  $\triangle ACE \sim \triangle DFB$  and the Claim 1 follows. ■

**Claim 2.** The three diagonals  $AD, BE, CF$  are concurrent.

*Proof of Claim 2.* Let  $P = AD \cap CF, Q = AD \cap BE$ . As proved in Claim 1 we have  $\triangle APC \sim \triangle DPF$ , so

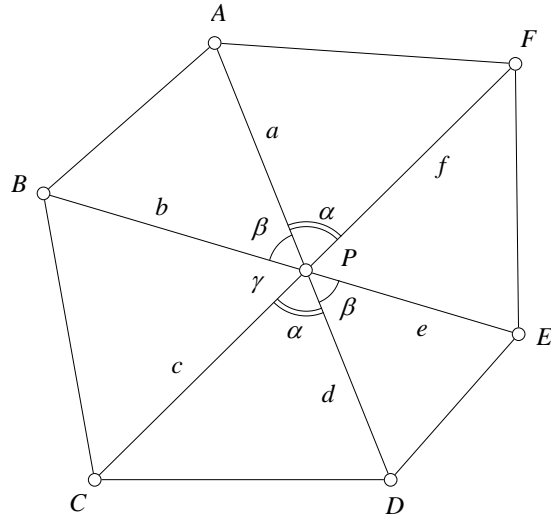
$$\frac{AP}{PD} = \frac{AC}{DF} \tag{1}$$

In a similar way, we can prove that

$$\frac{AQ}{QD} = \frac{EA}{BD} \tag{2}$$

From (1) and (2), taking into account of Claim 1, we have  $AP/PD = AQ/QD$  so that  $P = Q$ . Hence, the three diagonals  $AD, BE, CF$  concur in  $P$ , as claimed. ■

Coming back to the proposed problem, note first that the pairs of triangles  $\triangle FAP$  and  $\triangle CDP, \triangle ABP$  and  $\triangle DEP, \triangle BCP$  and  $\triangle EFP$  are equivalent.



Therefore denoting by  $a, b, c, d, e, f$  the lengths of  $PA, PB, PC, PD, PE, PF$  respectively, we have

$$a \cdot f = c \cdot d, \quad a \cdot b = d \cdot e, \quad b \cdot c = e \cdot f \quad (3)$$

Finally, putting  $\alpha = \angle FPA$ ,  $\beta = \angle APB$ ,  $\gamma = \angle BPC$ , the cosinus law yields

$$\begin{aligned} AB^2 + CD^2 + EF^2 &= a^2 + b^2 - 2ab \cos \beta + c^2 + d^2 - 2cd \cos \alpha + e^2 + f^2 - 2ef \cos \gamma \\ BC^2 + DE^2 + FA^2 &= b^2 + c^2 - 2bc \cos \gamma + d^2 + e^2 - 2de \cos \beta + f^2 + a^2 - 2fa \cos \alpha \end{aligned}$$

The above equalities and (3) gives

$$AB^2 + CD^2 + EF^2 = BC^2 + DE^2 + FA^2$$

which is precisely what we want to prove

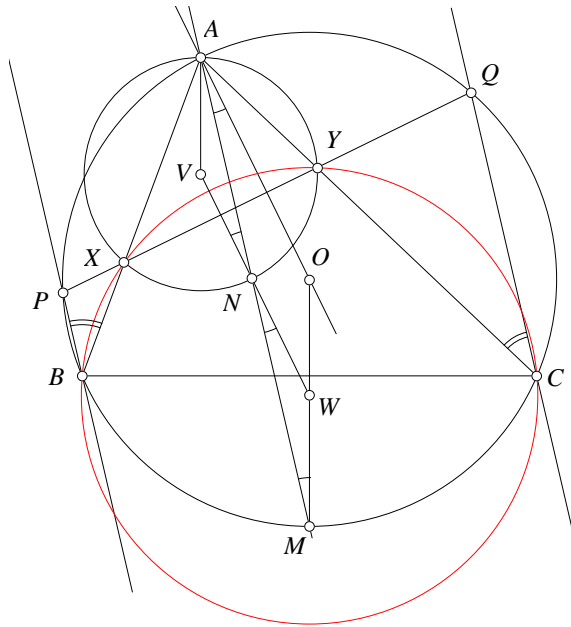
*Also solved by Daniel Lasaosa, Pamplona, Spain; Titu Zvonaru, Comănești, Romania and Neculai Stanciu, Buzău, Romania; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan; Farrukh Muhammadiev, Academic Lyceum Nr1, Samarkand, Uzbekistan; William Kang, Bergen County Academies, Hackensack, NJ, USA; Yujin Kim, Stony Brook School, Stony Brook, NY, USA.*

O322. Let  $ABC$  be a triangle with circumcircle  $\Gamma$  and let  $M$  be the midpoint of arc  $BC$  not containing  $A$ . Lines  $\ell_b$  and  $\ell_c$  passing through  $B$  and  $C$ , respectively, are parallel to  $AM$  and meet  $\Gamma$  at  $P \neq B$  and  $Q \neq C$ . Line  $PQ$  intersects  $AB$  and  $AC$  at  $X$  and  $Y$ , respectively, and the circumcircle of  $AXY$  intersects  $AM$  again at  $N$ .

Prove that the perpendicular bisectors of  $BC$ ,  $XY$ , and  $MN$  are concurrent.

*Proposed by Prasanna Ramakrishnan, Port of Spain, Trinidad and Tobago*

*Solution by Ercole Suppa, Teramo, Italy*



The parallelisms  $BP \parallel AM$  and  $CQ \parallel AM$  imply

$$\angle PBX = \frac{1}{2} \angle BAC = \angle QCY \quad (1)$$

The cyclic quadrilateral  $BCQP$  yields

$$\angle PBC + \angle PQC = 180^\circ \quad (2)$$

From (1) and (2) we obtain

$$\begin{aligned} \angle XBC + \angle XYC &= 180^\circ - \angle PQC - \angle PBX + \angle XYC = \\ &= 180^\circ - \angle PQC - \angle QCY + \angle XYC = \\ &= \angle QYC + \angle XYC = 180^\circ \end{aligned}$$

so  $BXYC$  is a cyclic quadrilateral.

Let  $V, W$  denote the centers of  $\odot(AXY)$  and  $\odot(BXYC)$  respectively. Observe that  $V, N, W$  are collinear since  $N$  is the midpoint of the arc  $XY$  and  $VW$  is the perpendicular bisector of  $XY$ .

Since  $\angle PBX = \angle QCY$  we have  $AP = AQ$ , so  $A$  lies on the perpendicular bisector of  $PQ$ . Therefore  $OA$  is the perpendicular bisector of  $PQ$  and this implies that  $OA \perp PQ$ .

Now, from  $VW \perp PQ$  and  $OA \perp PQ$ , it follows that  $OA \parallel VW$ .

Therefore we have

$$\angle WMN = \angle OAM = \angle ANV = \angle WNM \Rightarrow WN = WM$$

hence  $W$  belongs to the perpendicular bisector of  $MN$ .

Clearly  $W$  also belongs to the perpendicular bisectors of  $BC$  and  $XY$  so the proof is complete.

*Also solved by Daniel Lasaosa, Pamplona, Spain; Andrea Fanchini, Cantú, Italy; Sardor Bozorboyev, Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Farrukh Mukhammadiev, Academic Lyceum Nr1, Samarkand, Uzbekistan; Bodhisattwa Bhowmik, RKMV, Agartala, Tripura, India; Misiakos Panagiotis, Athens College (HAEF), Nea Penteli; Mehtaab Sawhney, USA.*

O323. Prove that the sequence  $2^{2^1} + 1, 2^{2^2} + 1, \dots, 2^{2^n} + 1, \dots$  and an arbitrary infinite increasing arithmetic sequence have either infinitely many terms in common or at most one term in common.

*Proposed by Nairi Sedrakyan, Yerevan, Armenia*

*Solution by Adnan Ali, Student in A.E.C.S-4, Mumbai, India*

Assume to the contrary that there exists an arbitrary infinite increasing arithmetic progression (or sequence)  $\{a_n\}_{n \geq 1}$  with common difference equal to  $d$ , such that it has  $a$  terms in common with the sequence  $\{F_n\}_{n \geq 1} = 2^{2^1} + 1, 2^{2^2} + 1, \dots, 2^{2^m} + 1, \dots$ , and  $2 \leq a < \infty$ . Then assume that the last two terms which are common to both the sequences, are  $a_k = 2^{2^j} + 1$  and  $a_\ell = 2^{2^m} + 1$  and  $k < \ell \Leftrightarrow j < m$ . Then,  $d|a_\ell - a_k \Leftrightarrow d|2^{2^j}(2^{2^m - 2^j} - 1)$ . If  $d = 2^b, b \leq 2^n$ , then we have  $d|2^{2^m}(2^{2^{m+1} - 2^m} - 1) \Leftrightarrow d|F_{m+1} - F_m$  and thus there  $\exists$  a  $u$  such that  $a_{\ell+u} = F_{m+1}$ . But this is a contradiction to the assumption that the sequences  $\{a_n\}$  and  $\{F_n\}$  share only  $a$  terms in common and  $2 \leq a < \infty$ . Thus if  $\{a_n\}$  and  $\{F_n\}$  share more than one term, then using the above method we can generate infinitely many common terms.

Next, let  $d$  have an odd factor say  $d_0$ . Then,  $2^{2^m - 2^j} \equiv 1 \pmod{d_0} \Rightarrow (2^{2^m - 2^j})^{2^{m-j}} \equiv 1 \pmod{d_0}$ . Thus  $d_0|2^{2^m}(2^{2^{2m-j} - 2^m} - 1) \Leftrightarrow d|F_{2m-j} - F_m$ . This shows the existence of a positive integer  $r$  such that  $a_{\ell+r} = F_{2m-j}$ . This again contradicts our assumption, thus there exist infinitely many common terms in  $\{a_n\}$  and  $\{F_n\}$  if they have more than one term in common, or they share at most one common term.

*Note:* An example of the arithmetic sequence which shares only one term with  $\{F_n\}_{n \geq 1}$  is the sequence  $5, 15, 25, 35, \dots$  which shares only 5 as a common term and none else.

By the Claim, the two describe sequences have either zero, one, or infinitely many terms in common. The conclusion follows.

*Note:* We may easily construct infinite increasing arithmetic sequences with no terms in common by taking sequences of non-integers, or of even integers. We may also easily construct infinite increasing arithmetic sequences with exactly one term in common, by taking one of its terms in common and an irrational difference, or by appropriately choosing the starting point and the difference. For example, defining  $s_m = 2^{2^1} + 1 + (m - 1)2^5$  for  $m = 1, 2, \dots$ , every term of this arithmetic sequence is congruent to 1 modulus  $2^{2^1} = 4$ , hence not of the form  $2^{2^n} + 1$ , except for  $n = 1$  when  $m = 1$ , which is the only common term, and there are no others.

*Also solved by Reiner Martin, Bad Soden-Neuenhain, Germany; Mehtaab Sawhney, USA; Misiakos Panagiotis, Athens College (HAEF), Nea Penteli; Daniel Lasasosa, Pamplona, Spain; Samin Riasat, University of Waterloo, ON, Canada; Sardor Bozorboyev, Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan.*

O324. Let  $a, b, c, d$  be nonnegative real numbers such that  $a^3 + b^3 + c^3 + d^3 + abcd = 5$ . Prove that

$$abc + bcd + cda + dab - abcd \leq 3$$

*Proposed by An Zhen-ping, Xianyang Normal University, China*

*Solution by Semchankau Aliaksei, Moscow Institute of Physics and Technology*

We will prove more general problem: let  $(a^3 + b^3 + c^3 + d^3)t + abcd = 5t^4, t > 0$ , prove that

$$(abc + bcd + cda + dab)t - abcd \leq 3t^4$$

In required inequality  $t = 1$ .

Let  $r = abc + bcd + cda + dab, S_3 = a^3 + b^3 + c^3 + d^3, s = abcd$ . Then we have  $S_3t + s = 5t^4$  and we have to prove that  $rt - s \leq 3t^4$ . Obviously  $t^4 = \frac{S_3t+s}{5}$ , so, our inequality transforms to

$$rt - s \leq \frac{3}{5}(S_3t + s) \Leftrightarrow 5rt - 5s \leq 3S_3t + 3s \Leftrightarrow (5r - 3S_3)t \leq 8s$$

If  $5r - 3S_3 \leq 0$ , then it is obvious, so we can conclude that  $5r - 3S_3 > 0$ . Let's suppose, that inequality isn't hold (and after that we will get a contradiction). So, now we conclude that

$$(5r - 3S_3)t > 8s \Leftrightarrow t > \frac{8s}{5r - 3S_3}.$$

Using it, we get:

$$S_3t + s = 5t^4 \Rightarrow$$

$$s = t(5t^3 - S_3) > \frac{8s}{5r - 3S_3}(5t^3 - S_3) \Rightarrow$$

$$\Rightarrow 5r - 3S_3 > 8(5t^3 - S_3) \Rightarrow 5r + 5S_3 > 40t^3 \Leftrightarrow r + S_3 > 8t^3$$

We know, that  $t > \frac{8s}{5r-3S_3}$ , so  $r + S_3 > 8\left(\frac{8s}{5r-3S_3}\right)^3$ , and it gives us that

$$(r + S_3)(5r - 3S_3)^3 > 8^4 s^3$$

$$(r + S_3)(5r - 3S_3) = (3r - S_3)^2 - (2r - 2S_3)^2 \leq (3r - S_3)^2$$

So, we can get that  $8^4 s^3 < (3r - S_3)^2 (5r - 3S_3)^2 \Rightarrow 8^2 s^{\frac{3}{2}} < (3r - S_3)(5r - 3S_3)$ .

$$(3r - S_3)(5r - 3S_3) = (4r - 2S_3)^2 - (r - S_3)^2 \leq (4r - 2S_3)^2,$$

so we get that

$$8s^{\frac{3}{4}} < 4r - 2S_3 \Leftrightarrow 4s^{\frac{3}{4}} < 2r - S_3 \Leftrightarrow 4a^{\frac{3}{4}}b^{\frac{3}{4}}c^{\frac{3}{4}}d^{\frac{3}{4}} \leq 2(abc + bcd + cda + dab) - (a^3 + b^3 + c^3 + d^3)$$

This is a contradiction because of the following lemma:

**Lemma.**

$$a^3 + b^3 + c^3 + d^3 + 4a^{\frac{3}{4}}b^{\frac{3}{4}}c^{\frac{3}{4}}d^{\frac{3}{4}} \geq 2(abc + bcd + cda + dab)$$

*Proof*

Let  $f(a, b, c, d) = a^3 + b^3 + c^3 + d^3 + 4a^{\frac{3}{4}}b^{\frac{3}{4}}c^{\frac{3}{4}}d^{\frac{3}{4}} - 2(abc + bcd + cda + dab)$ . Our goal is to prove that  $f(a, b, c, d) \geq 0$ . At first we will prove it in case  $a = b, c = d$ :

$$f(x, x, y, y) \geq 0 \Leftrightarrow 2(x^3 - x^2y - xy^2 + y^3) - 2xy(x - 2\sqrt{xy} + y) \geq 0 \Leftrightarrow (\sqrt{x} + \sqrt{y})^2(x + y) \geq xy$$

which is obvious.

Now we will find, when inequality  $f(a, b, c, d) \geq f(a, b, \sqrt{cd}, \sqrt{cd})$  holds:

$$\begin{aligned} f(a, b, c, d) \geq f(a, b, \sqrt{cd}, \sqrt{cd}) &\Leftrightarrow (c\sqrt{c} - d\sqrt{d})^2 \geq 2ab(\sqrt{c} - \sqrt{d})^2 \Leftrightarrow \\ &\Leftrightarrow (c + \sqrt{cd} + d)^2 \geq 2ab \Leftrightarrow (3\sqrt{cd})^2 \geq 2ab \Leftrightarrow 9cd \geq 2ab \Leftrightarrow 4cd \geq ab \end{aligned}$$

So, if  $4cd \geq ab$ , then we can change  $c, d$  to  $x, x$ , where  $x = \sqrt{cd}$ . We will try to do it with our numbers. Lets rearrange  $a, b, c, d$  in such way that  $a \geq b \geq c \geq d$ .

Obviously,  $4ab \geq cd$ , so we can change  $a, b$  to  $x, x$ . Now we need to prove, that  $f(x, x, c, d) \geq 0$ . If  $4cd \geq x^2$ , then we can replace  $c, d$  to  $y, y$  and we are done.

So, we will consider the case, when  $x^2 \geq 4cd$ . Let's take a look at pairs  $x, c$  and  $x, d$ . If  $4xc \geq xd$  and  $4xd \geq xc$ , then we can replace them to pairs  $l, l$  and  $r, r$ ,  $l = \sqrt{xc}, r = \sqrt{xd}$ .

$4xc \geq xd$  obviously holds, so we have to check  $4xd \geq xc$ . If it is true, then we are done. Else, let's suppose that  $4xd \leq xc \Leftrightarrow 4d < c$ .

$$f(x, x, c, d) \geq 0 \Leftrightarrow 2x^3 + c^3 + 4x^{\frac{6}{4}}c^{\frac{3}{4}}d^{\frac{3}{4}} \geq 2x^2(c + d) + 4xcd$$

$4x^{\frac{6}{4}}c^{\frac{3}{4}}d^{\frac{3}{4}} \geq 4xcd$  - obvious. It remains to prove, that

$$2x^3 + c^3 + d^3 \geq 2x^2(c + d)$$

$2x^3 + c^3 + d^3 \geq 2x^3 + c^3 = x^3 + x^3 + c^3 \geq 3x^2c$ .  $3x^2c \geq 2x^2(c + d) \Leftrightarrow 3c \geq 2c + 2d \Leftrightarrow c \geq 2d$ , but we already have  $c \geq 4d$ .

*Also solved by Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Arber Avdullahu, Mehmet Akif College, Kosovo; Adnan Ali, Student in A.E.C.S-4, Mumbai, India.*