On Fontene's Theorems

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Abstract

We prove the Fontene Theorems and then we solve some problems using them.

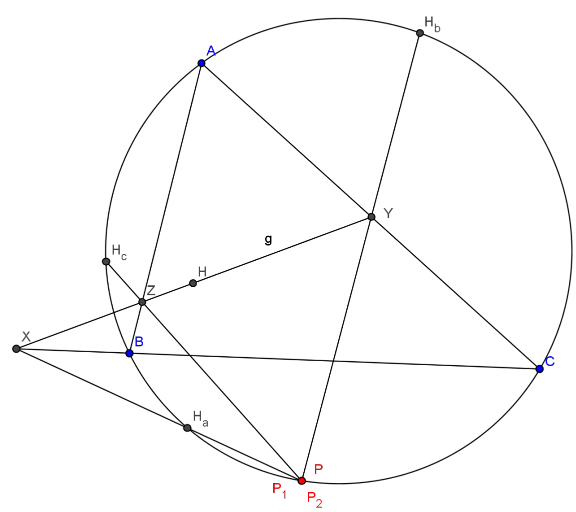
Theorem 1 (Fontene's first theorem) Given triangle $\triangle ABC$. Let P be an arbitrary point in the plane. A_1, B_1, C_1 are the midpoints of BC, CA, AB, $\triangle A_2B_2C_2$ is the pedal triangle of P with respect to triangle $\triangle ABC$. Let X, Y, Z be the intersections of B_1C_1 and B_2C_2; A_1C_1 and A_2C_2; A_1B_1 and A_2B_2. Then A_2X, B_2Y, C_2Z concur at the intersection of $(A_1B_1C_1)$ and $(A_2B_2C_2)$.

Theorem 2 (Fontene's second theorem) If a point P moves on the fixed line d which passes through the circumcenter O of triangle $\triangle ABC$ then the pedal circle of P with respect to triangle $\triangle ABC$ intersects the Nine-point circle of triangle $\triangle ABC$ at a fixed point.

Theorem 3 (Fontene's third theorem) Denote the isogonal conjugate of P with respect to triangle $\triangle ABC$ as P_0. Then the pedal circle of P is tangent to the Nine-point circle of triangle $\triangle ABC$ if and only if O, P, P_0 are collinear.

An useful result, which will be used later, is the following:

`(The Anti-Steiner point)` Given $\triangle ABC$, H its orthocenter, and g a line passing through H. The reflections of g in AB, BC, CA intersect at a point P lying on the circumcircle of $\triangle ABC$.



Proof. We denote X, Y, Z the intersections of g with BC, CA, AB and with H_a, H_b, H_c the reflections of H in BC, AC, AB. Denote P_1, P_2 the intersections of XH_a, ZH_c with Γ. It is well-known that H_a, H_b, H_c lie on the circumcircle of $\triangle ABC$. Our goal is to prove that XH_a, YH_b, ZH_c concur at a point on the circumcircle Γ of $\triangle ABC$. We clearly have

$$\widehat{BXZ} + \widehat{BZX} = \widehat{ABC} \Leftrightarrow \widehat{BXP_1} + \widehat{BZP_2} = \widehat{ABC}$$

We also know that

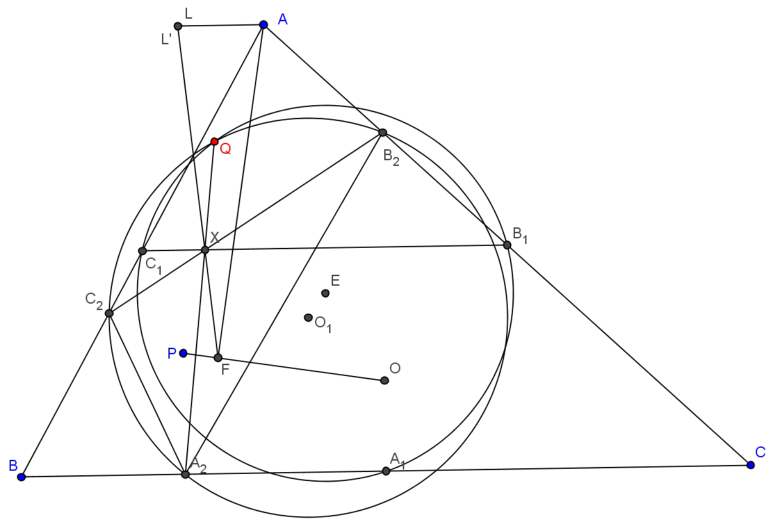
$$\widehat{BXP_1} = \widehat{CBP_1} - \widehat{BP_1H_a}, \ \widehat{BZP_2} = \widehat{BAP_2} + \widehat{AP_2H_c}$$

so $\widehat{P_1AC} + \widehat{BAP_2} + \widehat{ACH_c} - \widehat{BAH_a} = \widehat{ABC}$, therefore $\widehat{P_1AC} + \widehat{BAP_2} = \widehat{BAC}$, hence $P_1 = P_2$ and the conclusion follows. [1]

Another known configuration, whose proof consists almost entirely in easy angle chasing is the following

<u>Result.</u> Given a triangle $\triangle ABC$, P a point inside the triangle and P' its isogonal conjugate, let $\triangle DEF$ and $\triangle D'E'F'$ be the pedal triangles of P, P'. Prove that D, E, F, D', E', F' are concyclic on a circle whose center is the midpoint of PP'.

We can now prove **Fontene's first theorem**:



Proof. Let E be the center of $(A_1B_1C_1)$, O' be the center of $(A_2B_2C_2)$, F the projection of A on OP, l the reflection of A_2 with respect to B_1C_1. It's easy to notice that $AL \parallel BC$, so $\widehat{ALP} = 90°$

Because A, C_2, P, F, B_2 lie on (AP) and A, C_1, F, O, B_1 lie on (AO), we have $\widehat{FC_1X} = \widehat{FAB_1} = \widehat{B_2C_2F} = \widehat{XC_2F}$, so C_1, X, F, C_2 are concyclic.

Denote L' the intersection of FX and (AP). We have $AL'C_2F$ is a concyclic quadrilateral. We know that FXC_1C_2 is also cyclic, so $\widehat{AC_1X} = \widehat{C_2FL'} = \widehat{C_2AL'}$, therefore $AL' \parallel B_1C_1$, so $L' = L \Rightarrow L, X, F$ are collinear.

Denote Q the intersection of A_2X and the circumcircle of $\triangle A_1B_1C_1$. F' is the reflection of Q with respect to B_1C_1. Consider the symmetry $S_{B_1C_1} : (AO) \to (E)$ which clearly satisfies $B_1 \to B_1$, $C_1 \to C_1$, $A \to A_\perp$, where $A_\perp$ is the projection of A on BC, which lies on $(A_1B_1C_1)$ (Euler circle). Keeping in mind that $Q \in (E)$, we easily get $F' \in (AO)$.

On the other hand, $S_{B_1C_1}$ maps A_2 to L. Furthermore, A_2, X, Q are collinear, so L, X, F' are collinear, which is equivalent to $F = F'$. We deduce that A_2LQF is an isosceles trapezoid, so we have $XL \cdot XF = XQ \cdot XA_2 = XB_2 \cdot XC_2$, so $Q \in (O')$. Similarly, B_2Y, C_2Z also pass through Q, so we are done.

[1] If one wants to have all the relations symmetric, everything can be rewritten using oriented angles.

Now, a very easy application of this theorem [2]:

Problem 1. Given a triangle $\triangle ABC$, A_1, B_1, C_1 are the midpoints of BC, CA, AB, and D, E, F are the projections of A, B, C on BC, AC, AB. Let $\{X\} = EF \cap YZ, \{Y\} = DF \cap XZ, \{Z\} = DE \cap YZ$. Prove that:
a) DX, EY, FZ concur at W and W lies on the Euler circle of triangle $\triangle ABC$.
b) A_1X, B_1Y, C_1Z concur on the Euler circle of triangle $\triangle ABC$.

Proof. We will prove only *a)*, because *b)* can be solved in the same way.

From Fontene's first theorem, DX, EY, FZ concur at W, and W is the point of intersection of $(A_1B_1C_1)$ and (DEF), but these are both the Euler circle of triangle $\triangle ABC$, so W lies on the Euler circle of triangle $\triangle ABC$.

We can prove now **Fontene's Second Theorem**:

Proof. According to the proof of the first theorem, the point of contact Q of (E) and (O') is the reflection of a point F which lies on OP with respect to the line B_1C_1 .It's obvious that O is the orthocenter of triangle $\triangle A_1B_1C_1$, thus Q is the Anti-Steiner point of d. Therefore Q is fixed.

Let us generalize **Problem 1**:

Problem 1*. Given a triangle $\triangle ABC$, A_1, B_1, C_1 midpoints of BC, CA, AB, P an arbitrary point on the Euler line of triangle $\triangle ABC$, $\triangle DEF$ the pedal triangle of P, $\{X\} = EF \cap YZ, \{Y\} = DF \cap XZ, \{Z\} = DE \cap YZ$. Prove that DX, EY, FZ also pass through W, where W is the point defined in **Problem 1**.

Proof. We know that the Euler line of triangle $\triangle ABC$ passes through O, so all we have to do is to use Fontene's second theorem for this line.

You can see that **Problem 1*** is not obvious at all, and I encourage you to try to solve it without using any of the Fontene theorems.

Finally, let us prove our last theorem, namely **Fontene's third theorem**.

Proof. According to **Fontene's Second Theorem** and to the Result, we can prove that the second intersection point Q' of (O') and (E) is the Anti-Steiner point of P'. This means $Q \equiv Q'$ if and only if O, P, P' are collinear. [3]

[2] One can find a solution to this problem using trigonometry too.

[3] The Feuerbach point is a corollary of Fontene's third theorem when P coincides with the incenter of triangle $\triangle ABC$.

We will see now the power of these theorems in the next problems:

Problem 2. Given triangle $\triangle ABC$. Let A_1, B_1, C_1 be the midpoints of BC, CA, AB and P a point in the plane of triangle $\triangle ABC$. Let $\triangle A_2B_2C_2$ be the pedal triangle of triangle $\triangle ABC$. $A_1B_1 \cap A_2B_2 = \{Z\}, B_1C_1 \cap B_2C_2 = \{X\}, A_1C_1 \cap A_2C_2 = \{Y\}$, let O be the center of $(A_2B_2C_2)$. Prove that O is the orthocenter of triangle $\triangle XYZ$.

Proof. Using Fontene's first theorem, we know that A_2X, B_2Y, C_2Z are concurent in a point Q which lies on $(A_2B_2C_2)$. We look at the cyclic quadrilater $B_2A_2C_2Q$, and it's easy to see that the polar of X with respect to $(A_2B_2C_2)$ is YZ, so $XO \perp YZ$. In a similar way, we obtain $YO \perp XZ$, so O is the orthocenter of triangle $\triangle XYZ$.

Problem 3. Given a triangle $\triangle ABC$, O its circumcenter, l a line passing through O, $l \cap BC = \{X\}, l \cap AC = \{Y\}, l \cap AB = \{Z\}$. Prove that $(AX), (BY), (CZ)$ and the Euler circle of triangle $\triangle ABC$ are concurrent.

Proof. Using Fontene's second theorem in the particular cases $P = X, Y$ and Z, we get that all these circles pass through the Anti-Steiner point of l.

Problem 4. Given triangle $\triangle ABC$, P an arbitrary point in the plane. $\triangle A_1B_1C_1$ is the pedal triangle of P with respect to $\triangle ABC$. A_2, B_2, C_2 are the midpoints of BC, CA, AB, respectively. A_3, B_3, C_3 are the reflections of A_1, B_1, C_1 with respect to A_2, B_2, C_2, respectively. Prove that the three circles $(A_1B_1C_1), (A_2B_2C_2), (A_3B_3C_3)$ are concurrent.

Proof. Since $\triangle A_1B_1C_1$ is the pedal triangle of P with respect to triangle $\triangle ABC$, applying Carnot theorem we obtain: $BA_1^2 - CA_1^2 + CB_1^2 - AB_1^2 + AC_1^2 - BC_1^2 = 0$, so it's easy to see that $CA_3^2 - BA_3^2 + BC_3^2 - AC_3^2 + AB_3^2 - CB_3^2 = 0$.

We deduce that triangle $\triangle A_3B_3C_3$ is the pedal triangle of some point Q with respect to triangle $\triangle ABC$.

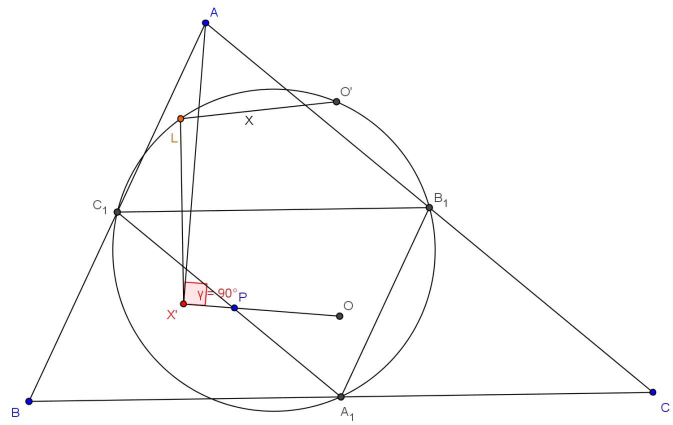
Now, it's easy to see that the perpendicular bisector of BC is the perpendicular bisector of A_1A_3 which passes through the midpoint of PQ, so the midpoint of PQ is O, the circumcenter of (ABC).

Denote Y the Anti-Steiner point of OP with respect to triangle $\triangle A_2B_2C_2$. From Fontene's second theorem we know that $Y \in (A_1B_1C_1)$. Moreover O, P, Q are collinear, so Y is the Anti-Steiner point of OQ with respect to triangle $\triangle A_2B_2C_2$, but triangle $\triangle A_3B_3C_3$ is the pedal triangle of Q with respect to triangle $\triangle ABC$, thus, according to the proof of Fontene theorem 2, $Y \in (A_3B_3C_3)$. The conclusion follows.

Problem 5 Given a triangle $\triangle ABC$, A_1, B_1, C_1 the midpoints of BC, AC, AB, O the circumcenter of triangle $\triangle ABC$, P an arbitrary point, L the Anti-Steiner point of P with respect to triangle $\triangle A_1B_1C_1$. Prove that the reflections of L in B_1C_1, A_1C_1, A_1B_1 are the projections of A, B, C on OP.

Proof. The lines PO and x (see the picture) are symmetrically placed with respect to the line B_1C_1. Therefore, as L lies on x, its reflection, X' in B_1C_1 must lie on OP.

For $\widehat{AB_1O} = \widehat{AC_1O} = 90°$, the points B_1, C_1 lie on the circle with diameter AO. The circles $(A_1B_1C_1)$ and (AB_1C_1) are congruent, hence, these circles are symmetrically placed with respect to the line B_1C_1. Since L lies on $(A_1B_1C_1)$, its reflection in B_1C_1, X' lies on (AB_1C_1), $AX' \perp PO$. In a similar way we obtain the other results.



In the end, we invite the readers to try their hands and solve the following problems.

Problem 6 For $\triangle ABC$, denote I the incenter, I_a, I_b, I_c excenters, N the Nagel point of $\triangle I_aI_bI_c$, O circumcircle of $\triangle I_aI_bI_c$, ON cuts this circle at P. Prove that the Simson line of P with respect to $\triangle I_aI_bI_c$ is parallel or perpendicular to NI.

Problem 7 Let F be the Feuerbach point of the triangle $\triangle ABC$ and P be the symmetric of I in F and Q be the isogonal conjugate of P with respect to triangle $\triangle ABC$. Prove that Q, O, I are collinear .

Problem 8 Given a triangle $\triangle ABC$, A_1, B_1, C_1 the midpoints of BC, AC, AB,H_a, H_b, H_c the projections of A, B, C on BC, CA, AB, O the circumcenter of triangle $\triangle ABC$, P an arbitrary point, L the Anti-Steiner point of P with respect to triangle $\triangle A_1B_1C_1$. Let X', Y', Z' be the symmetrics of L with respect to B_1C_1, A_1C_1, A_1B_1. Prove that $H_aL = AX', H_bL = BY', H_cL = AZ'$.