Junior problems

J325. For positive real numbers a and b, define their *perfect mean* to be half of the sum of their arithmetic and geometric means. Find how many unordered pairs of integers (a, b) from the set $\{1, 2, \ldots, 2015\}$ have their perfect mean a perfect square.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Alessandro Ventullo, Milan, Italy

We have

$$\frac{\frac{a+b}{2} + \sqrt{ab}}{2} = n^2 \qquad n \in \mathbb{N},$$

i.e.

$$\sqrt{a} + \sqrt{b} = 2n, \qquad n \in \mathbb{N}.$$

As $a, b \in \{1, 2, \ldots, 2015\}$, then a and b must be perfect squares and $n \leq 44$. Assume that $a \leq b$. If a is an odd perfect square, then b is an odd perfect square, and making a case by case analysis for $a \in \{1, 3^2, \ldots, 43^2\}$ we get $22 + 21 + \ldots + 2 + 1 = 253$ unordered pairs. Similarly, if a is an even perfect square, then b is an even perfect square and we get $22 + 21 + \ldots + 2 + 1 = 253$ unordered pairs. So, there are $253 + 253 = 506$ unordered pairs which satisfy the given conditions.

Also solved by Yujin Kim, The Stony Brook School, NY, USA; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Arber Igrishta, Eqrem Qabej, Vushtrri, Kosovo; Behzod Ibodullaev, Lyceum TCTU, Tashkent, Uzbekistan; David E. Manes, Oneonta, NY, USA; Dilshoda Ibragimova, Academic Lyceum Nr.2 under the SamIES, Samarkand, Uzbekistan; Farrukh Mukhammadiev, Academic Lyceum Nr.1 under the SamIES, Samarkand, Uzbekistan; Francesc Gispert Sanchez, CFIS, Universitat Politecnica de Catalunya, Barcelona, Spain; Haroun Meghaichi, University of Science and Technology Houari Boumediene, Algiers, Algeria; Joseph Lee, Loomis Chaffee School in Windsor, CT, USA; Luiz Ernesto, M.A.Leitão, Brasil; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan; Titu Zvonaru and Neculai Stanciu, Romania; Ji Eun Kim, Tabor Academy, MA, USA; Misiakos Panagiotis, Athens College(HAEF), Greece; Michael Tang, Edina High School, MN, USA; Kyoung A Lee, The Hotchkiss School, Lakeville, CT, USA; Myong Claire Yun, Seoul International School, South Korea.

J326. Let a, b, c be nonnegative integers. Prove that

$$\sqrt{2a^2+3b^2+4c^2}+\sqrt{3a^2+4b^2+2c^2}+\sqrt{4a^2+2b^2+3c^2}\geq(\sqrt{a}+\sqrt{b}+\sqrt{c})^2.$$

Proposed by by Titu Andreescu, University of Texas at Dallas, USA

Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA

Note that

$$\sqrt{2a^2+3b^2+4c^2}=3\sqrt{\frac{2a^2}{9}+\frac{3b^2}{9}+\frac{4c^2}{9}}\geq\frac{1}{3}(2a+3b+4c)$$

by Jensen's inequality. Similarly,

$$\sqrt{3a^2+4b^2+2c^2}\geq\frac{1}{3}(3a+4b+2c)$$

and

$$\sqrt{4a^2+2b^2+3c^2}\geq\frac{1}{3}(4a+2b+3c).$$

Therefore,

$$\begin{aligned}\sqrt{2a^2+3b^2+4c^2}+\sqrt{3a^2+4b^2+2c^2}+\sqrt{4a^2+2b^2+3c^2} &\geq 3(a+b+c)\\ &= 9\frac{(\sqrt{a})^2+(\sqrt{b})^2+(\sqrt{c})^2}{3}\\ &\geq 9\left(\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{3}\right)^2\\ &= (\sqrt{a}+\sqrt{b}+\sqrt{c})^2,\end{aligned}$$

where the final inequality follows from the quadratic mean-arithmetic mean inequality.

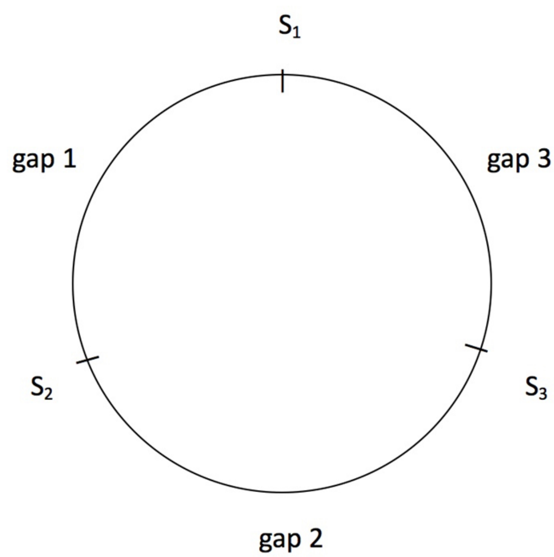
Also solved by George Gavrilopoulos, High School Of Nea Makri, Athens, Greece; Yong Xi Wang,East China Institute Of Technology, China; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Utsab Sarkar, Chennai Mathematical Institute, India; Titu Zvonaru and Neculai Stanciu, Romania; Stefan Petrevski, NOVA International School in Skopje, Macedonia; Albert Stadler, Herrliberg, Switzerland; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan; Rade Krenkov, SOU "Goce Delcev", Valandovo, Macedonia; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Paolo Perfetti, Universitá degli studi di Tor Vergata Roma, Roma, Italy; Odilbek Utamuratov, Uzbekistan; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Haroun Meghaichi, University of Science and Technology Houari Boumediene, Algiers, Algeria; Francesc Gispert Sanchez, CFIS, Universitat Politecnica de Catalunya, Barcelona, Spain; Farrukh Mukhammadiev, Academic Lyceum Nr.1 under the SamIES, Samarkand, Uzbekistan; Dilshoda Ibragimova, Academic Lyceum Nr.2 under the SamIES, Samarkand, Uzbekistan; Behzod Ibodullaev, Lyceum TCTU, Tashkent, Uzbekistan; David E. Manes, Oneonta, NY, USA; Arkady Alt, San Jose, CA, USA; Arber Igrishta, Eqrem Qabej, Vushtrri, Kosovo; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Myong Claire Yun, Seoul International School, South Korea; Kyoung A Lee, The Hotchkiss School, Lakeville, CT, USA; Michael Tang, Edina High School, MN, USA; Misiakos Panagiotis, Athens College(HAEF), Greece; Ji Eun Kim, Tabor Academy, MA, USA; Yujin Kim, The Stony Brook School, NY, USA.

J327. A jeweler makes a circular necklace out of nine distinguishable gems: three sapphires, three rubies and three emeralds. No two gems of the same type can be adjacent to each other and necklaces obtained by rotation and reflection (flip) are considered to be identical. How many different necklaces can she make?

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Myong Claire Yun, Seoul International School, South Korea

Start by placing the three sapphires S_1, S_2, S_3 as shown in the figure below. There will be three gaps in between to fill with emeralds and rubies.



The distribution of the mix jewels in the 3 gaps can be categorized as:

Type 1: $(4+1+1)$ or $(1+4+1)$ or $(1+1+4)$
Type 2: $(3+2+1)$ and the permutations
Type 3: $(2+2+2)$

Let's count separately the number of necklaces of each type:
Type 1: $(4+1+1)$: We need 2 rubies and 2 emeralds to fill in gap 1.

$$\binom{3}{2}\binom{3}{2} = 9$$

ways to select these 4 jewels.

They can be arranged in $2 \cdot 2 \cdot 2 = 8$ ways. The remaining two gaps can be filled out in 2 ways. Hence these are $9 \cdot 8 \cdot 2 = 144$ necklaces of type $(4+1+1)$.

Type 2: $(3+2+1)$: We need 2 rubies and 1 emerald or 1 ruby and 2 emeralds to fill in gap 1.

$$2\binom{3}{2}\binom{3}{1} = 18$$

ways.

Arranging can be done in 2 ways. Gap 3 can be gilled out in 2 ways. We are left with one ruby abd one emerald to place in gap 2 $\rightarrow$ 2 ways.

Therefore there is $18 \cdot 2 \cdot 2 \cdot 2 = 144$ necklaces of type $(3+2+1)$.

Type 3: $(2+2+2)$:

$$2\binom{3}{1}\binom{3}{1} = 18$$

ways to fill gap 1.

Once the first gap is taken care of, there is

$$\binom{2}{1}\binom{3}{1}2 = 8$$

ways to fill out gap 2. Therefore, there are $18 \cdot 8 \cdot 2 = 288$ necklaces of type $(2+2+2)$.

In conclusion, there is $3 \cdot 144 + 6 \cdot 144 + 1 \cdot 288 = 1584$ different necklaces.

Also solved by Yujin Kim, The Stony Brook School, NY, USA; Ji Eun Kim, Tabor Academy, MA, USA; Misiakos Panagiotis, Athens College(HAEF), Greece; Kyoung A Lee, The Hotchkiss School, Lakeville, CT, USA; Arber Igrishta, Eqrem Qabej, Vushtrri, Kosovo; Joseph Lee, Loomis Chaffee School in Windsor, CT, USA.

J328. Let a, b, cbe postive real numbers such that $a+b+c=2$,Prove that

$$\sqrt{a^2+bc}+\sqrt{b^2+ca}+\sqrt{c^2+ab}\le 3$$

Proposed by An Zhen-ping, Xianyang Normal University, China

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

Without loss of generality, assume that $a \ge b \ge c$. By the AM-GM inequality, we have

$$\begin{aligned}\sqrt{a^2+bc} &= \sqrt{\frac{a^2+bc}{a+c}\cdot(a+c)} \le \frac{1}{2}\left(\frac{a^2+bc}{a+c}+(a+c)\right)\\ &\le \frac{1}{2}\left(\frac{a^2+ac}{a+c}+(a+c)\right)=\frac{1}{2}(2a+c),\end{aligned}$$

and

$$\sqrt{b^2+ca} = \sqrt{\frac{b^2+ca}{b+c}\cdot(b+c)} \le \frac{1}{2}\left(\frac{b^2+ca}{b+c}+(b+c)\right),$$

$$\sqrt{c^2+ab} = \sqrt{\frac{c^2+ab}{b+c}\cdot(b+c)} \le \frac{1}{2}\left(\frac{c^2+ab}{b+c}+(b+c)\right).$$

Summing up the above inequalities and using $c \le b$, we have

$$\begin{aligned}\sqrt{a^2+bc}+\sqrt{b^2+ca}+\sqrt{c^2+ab} &\le \frac{1}{2}\left(2a+2b+3c+\frac{(b^2+c^2)+a(b+c)}{b+c}\right)\\ &\le \frac{1}{2}(3a+2b+3c+\frac{b^2+bc}{b+c})\\ &= \frac{3}{2}(a+b+c)=3.\end{aligned}$$

Equality if and only if $\{a, b, c\} = \{1, 1, 0\}$.

Also solved by Yujin Kim, The Stony Brook School, NY, USA; Ji Eun Kim, Tabor Academy, MA, USA; Misiakos Panagiotis, Athens College(HAEF), Greece; Kyoung A Lee, The Hotchkiss School, Lakeville, CT, USA; Myong Claire Yun, Seoul International School, South Korea; Yong Xi Wang,East China Institute Of Technology, China; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Arkady Alt, San Jose, CA, USA; Farrukh Mukhammadiev, Academic Lyceum Nr.1 under the SamIES, Samarkand, Uzbekistan; Francesc Gispert Sanchez, CFIS, Universitat Politecnica de Catalunya, Barcelona, Spain; Jean Heibig, Lycée Stanislas, Paris, France; Joseph Lee, Loomis Chaffee School in Windsor, CT, USA; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Paolo Perfetti, Universitá degli studi di Tor Vergata Roma, Roma, Italy; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan; Titu Zvonaru and Neculai Stanciu, Romania; Utsab Sarkar, Chennai Mathematical Institute, India.

J329. Let $a_1, a_2, \ldots, a_{2015}$ be positive integers such that

$$a_1 + a_2 + \cdots + a_{2015} = a_1 a_2 \cdots a_{2015}.$$

Prove that among numbers $a_1, a_2, \ldots, a_{2015}$ at most nine are greater than 1.

Proposed by Titu Zvonaru, Comaneşti and Neculai Stanciu, Buzău, Romania

Solution by G.R.A.20 Problem Solving Group, Roma, Italy

We show that at most *six* are greater than 1.

Let us assume that $a_1 \geq a_2 \geq \cdots \geq a_d \geq 2$ and $a_{d+1} = \cdots = a_{2015} = 1$ where $d \geq 2$ is the maximal number of terms greater than 1. Then

$$2015a_1 \geq a_1 2^{d-1}$$

which implies that $d \leq 1 + \lfloor \log_2(2015) \rfloor = 11$. Moreover

$$da_1 + 2015 - d \geq a_1 a_2 2^{d-2}$$

which implies that given $a_1 \geq 2$ and $2 \leq d \leq 11$, then a_2 should satisfies

$$2 \leq a_2 \leq \min(F(a_1, d), a_1) \quad \text{where} \quad F(a_1, d) := \left\lfloor \frac{1}{2^{d-2}} \left(\frac{2015 - d}{a_1} + d \right) \right\rfloor.$$

Note that F is decreasing (not strictly) with respect to a_1 and we can easily solve the problem by considering several cases.
1) $d < 11$ because $F(2, 11) = 1$.
2) $d < 10$ because $F(3, 10) = 2$ and $F(4, 10) = 1$ implies that then $a_1 \leq 3$, $a_2 = \cdots = a_{10} = 2$ and

$$a_1 + 2022 = a_1 + 9 \cdot 2 + 2015 - 10 \neq a_1 \cdot 2^9 \leq 1536.$$

3) $d < 9$ because $F(8, 9) = 2$ and $F(9, 9) = 1$ implies that $a_1 \leq 8$ and by examining all the finite cases we have no solutions.
4) $d < 8$ because $F(16, 8) = 2$ and $F(17, 8) = 1$ implies that $a_1 \leq 16$ and by examining all the finite cases we have no solutions.
5) $d < 7$ because $F(35, 7) = 2$ and $F(36, 7) = 1$ implies that $a_1 \leq 35$ and by examining all the finite cases we have no solutions.
6) $d = 6$ because $F(77, 6) = 2$ and $F(78, 6) = 1$ implies that $a_1 \leq 77$ and by examining all the finite cases we have precisely one solution

$$a_1 = 17,\ a_2 = 5,\ a_3 = 3,\ a_4 = a_5 = a_6 = 2,\ a_7 = \cdots = a_{2015} = 1$$

and

$$17 + 5 + 3 + 2 + 2 + 2 + 2015 - 6 = 17 \cdot 5 \cdot 3 \cdot 2^3 = 2040.$$

Also solved by Yujin Kim, The Stony Brook School, NY, USA; Ji Eun Kim, Tabor Academy, MA, USA; Michael Tang, Edina High School, MN, USA; Kyoung A Lee, The Hotchkiss School, Lakeville, CT, USA; Myong Claire Yun, Seoul International School, South Korea; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Farrukh Mukhammadiev, Academic Lyceum Nr.1 under the SamIES, Samarkand, Uzbekistan; George Gavrilopoulos, High School Of Nea Makri, Athens, Greece; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan.

J330. Let $ABCD$ be a quadrilateral with centroid G, inscribed in a circle with center O, and diagonals intersecting at P. Prove that if O, G, P are collinear, then either $ABCD$ is an isosceles trapezoid or an orthogonal quadrilateral.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan

Let M and N be the midpoints of AC and BD. Observe that G is midpoint of MN and quadrilateral $OMNP$ is cyclic. If $AC \perp BD$, then $OMNP$ is a rectangle. The diagonals of $OMNP$ bisect each other, so G lies on OP.

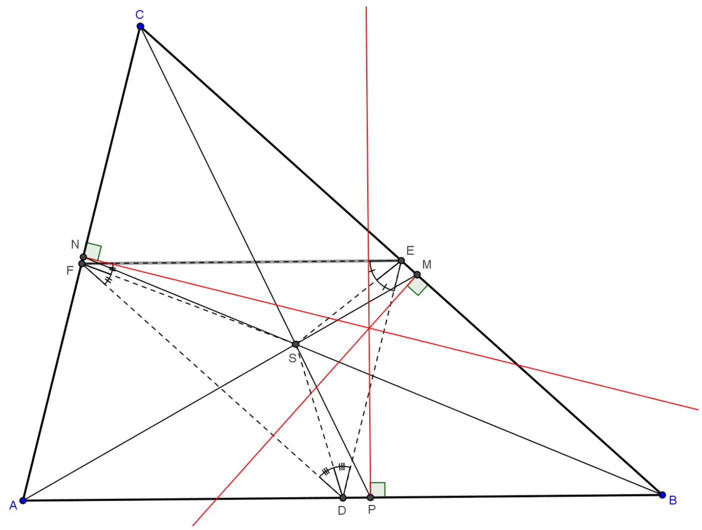
Assume that $ABCD$ is not orthogonal, then consider Ω, the center of the circumcircle of $OMNP$. Observe that $\Omega \neq G$ and Ω lies on OP. Then $\Omega M = \Omega N$, $\Omega G \perp MN$ and $MO = ON$. It follows that triangles AMO is congruent to triangle BNO, so $AC = BD$. Therefore $\angle CDA = \angle BAD$ and $ABCD$ is an isosceles trapezoid.

Also solved by Farrukh Mukhammadiev, Academic Lyceum Nr.1 under the SamIES, Samarkand, Uzbekistan; Titu Zvonaru and Neculai Stanciu, Romania.

S325. Let S be the incenter of the triangle formed by the midpoints of the triangle ABC. Cevians AS, BS, CS intersect the sides BC, CA, AB in points M, N, P, respectively. Prove that if the perpendiculars to the sides of triangle ABC at points M, N, P are concurrent, then triangle ABC is isosceles.

Proposed by Roxana Mihaela Stanciu and Nela Ciceu, Bacău, Romania

Solution by Andrea Fanchini, Cantú, Italy



The incenter of the triangle DEF formed by the midpoints of the triangle ABC is known as Spieker center and it have barycentric coordinates $S = X_{10}(b+c : c+a : a+b)$.

So the line AS have equation $AS \equiv (a+b)y-(c+a)z = 0$ and then point $M = AS \cap \underbrace{BC}_{x=0}$, have coordinates $M(0 : c+a : a+b)$.

Line BS have equation $BS \equiv (a+b)x-(b+c)z = 0$ and then point $N = BS \cap \underbrace{CA}_{y=0}$, so it have coordinates $N(b+c : 0 : a+b)$.

Line CS have equation $CS \equiv (c+a)x-(b+c)y = 0$ and then point $P = CS \cap \underbrace{AB}_{z=0}$, so it have coordinates $P(b+c : c+a : 0)$.

The infinite point of the perpendicular to the side BC is $BC_{\infty\perp}(-a^2 : S_C : S_B)$, so the perpendicular to the side BC at point M have equation $MBC_{\infty\perp} \equiv [(c+a)S_B-(a+b)S_C]x-a^2(a+b)y+a^2(c+a)z = 0$

The infinite point of the perpendicular to the side CA is $CA_{\infty\perp}(S_C : -b^2 : S_A)$, so the perpendicular to the side CA at point N have equation $NCA_{\infty\perp} \equiv b^2(a+b)x-[(b+c)S_A-(a+b)S_C]y-b^2(b+c)z = 0$

The infinite point of the perpendicular to the side AB is $AB_{\infty\perp}(S_B : S_A : -c^2)$, so the perpendicular to the side AB at point P have equation $PAB_{\infty\perp} \equiv -c^2(c+a)x+c^2(b+c)y+[(b+c)S_A-(c+a)S_B]z = 0$

Now three lines $p_ix+q_iy+r_iz=0$, with $i=1,2,3$, are concurrent if and only if

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0$$

In our case the perpendiculars to the sides of triangle ABC at points M, N, P are concurrent if and only if

$$\begin{vmatrix} (c+a)S_B-(a+b)S_C & -a^2(a+b) & a^2(c+a) \\ b^2(a+b) & (a+b)S_C-(b+c)S_A & -b^2(b+c) \\ -c^2(c+a) & c^2(b+c) & (b+c)S_A-(c+a)S_B \end{vmatrix} = 0$$

It's easy to prove that this is true if and only if $a=b$ or $b=c$ or $c=a$ and that's when triangle ABC is isosceles, q.e.d.
For example when $a=b \quad \Rightarrow \quad S_A=S_B=\frac{c^2}{2}, \quad S_C=a^2-\frac{c^2}{2}$ and so we have

$$\begin{vmatrix} (c+3a)S_A-2a^3 & -2a^3 & a^2(c+a) \\ 2a^3 & 2a^3-(c+3a)S_A & -a^2(c+a) \\ -c^2(c+a) & c^2(c+a) & 0 \end{vmatrix}$$

summing the first two rows we obtain

$$\begin{vmatrix} (c+3a)S_A-2a^3 & -2a^3 & a^2(c+a) \\ (c+3a)S_A & -(c+3a)S_A & 0 \\ -c^2(c+a) & c^2(c+a) & 0 \end{vmatrix} = 0$$

similarly for the other two cases.

Also solved by Farrukh Mukhammadiev, Academic Lyceum Nr.1 under the SamIES, Samarkand, Uzbekistan; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan.

S326. Let a, b, c be positive reals such that, $a^3+b^3+c^3+abc=\frac{1}{3}$. Prove that,

$$abc+9\sum_{cyc}\frac{a^5}{4b^2+bc+4c^2}\geq\frac{1}{4\left(a+b+c\right)\left(ab+bc+ca\right)}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

Applying the Cauchy-Schwarz inequality for the sequences

$$\begin{array}{ll} x_1=\sqrt{abc}, & y_1=\sqrt{9abc} \\ x_2=\sqrt{\frac{9a^5}{4b^2+bc+4c^2}}, & y_2=\sqrt{a(4b^2+bc+4c^2)} \\ x_3=\sqrt{\frac{9b^5}{4c^2+ca+4a^2}}, & y_3=\sqrt{b(4c^2+ca+4a^2)} \\ x_4=\sqrt{\frac{9c^5}{4a^2+ab+4b^2}}, & y_4=\sqrt{c(4a^2+ab+4b^2)} \end{array}$$

we obtain

$$\begin{aligned} & abc+9\left(\frac{a^5}{4b^2+bc+4c^2}+\frac{b^5}{4c^2+ca+4a^2}+\frac{c^5}{4a^2+ab+4b^2}\right) \\ & \geq\frac{(3abc+3a^3+3b^3+3c^3)^2}{9abc+4(ab^2+bc^2+ca^2)+3abc+4(ac^2+ba^2+cb^2)} \\ & =\frac{1}{4[(ab^2+abc+ba^2)+(bc^2+abc+c^2b)+(ca^2+abc+ac^2)]} \\ & =\frac{1}{4(a+b+c)(ab+bc+ca)}. \end{aligned}$$

Equality if and only if $a=b=c=\sqrt[3]{\frac{1}{12}}$.

Also solved by Utsab Sarkar, Chennai Mathematical Institute, India; Arkady Alt, San Jose, CA, USA; Behzod Ibodullaev, Lyceum TCTU, Tashkent, Uzbekistan; Farrukh Mukhammadiev, Academic Lyceum Nr.1 under the SamIES, Samarkand, Uzbekistan; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Odilbek Utamuratov, Uzbekistan; Paolo Perfetti, Universitá degli studi di Tor Vergata Roma, Roma, Italy; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan; Titu Zvonaru and Neculai Stanciu, Romania.

S327. Let H be the orthocenter of an acute triangle ABC and let D and E be the feet of the altitudes from vertices B and C, respectively. Line DE intersects the circumcircle of triangle ABC in points F and G (F lies on the smaller arc of AB and G lies on the smaller arc of AC). Denote by S the projection of H onto line DE. Prove that $EF + DS = DG + ES$.

Proposed by İlker Can Çiçek, Instanbul, Turkey

First solution by Prithwijit De, HBCSE, Mumbai, India

Assume $\angle B \geq \angle C$. Let $HS = x$ and $DE = y$. Angle chasing gives us $\angle DHS = \angle B$ and $\angle EHS = \angle C$. Also $x = 2R\cos A\cos B\cos C$, where R is the circumradius of triangle ABC and $y = a\cos A$. Thus

$$DS - ES = x(\tan B - \tan C) = 2R\cos A\sin(B - C). \tag{1}$$

By the intersecting chord theorem it follows that $DG \cdot DF = AD \cdot DC$ and $EG \cdot EF = AE \cdot EB$. Subtracting one from the other leads to

$$DG - EF = \frac{AD \cdot DC - AE \cdot EB}{y} = \frac{a\cos A(c\cos C - b\cos B)}{a\cos A} = 2R\cos A\sin(B - C). \tag{2}$$

Hence $DS - ES = DG - EF$ and the result follows.

Second solution by Ivan Borsenco, Massachusetts Institute of Technology, USA

Let M be the midpoint of FG and N be the midpoint of DE. Denote by O and Ω the circumcenter and the center of the nine-point circle of triangle ABC. Observe that M is the projection of O, N is the projection of Ω, and S is the projection of H onto FG. Because $O\Omega = H\Omega$, we have $MN = NS$, $DS = EM$ and $ES = DM$. From the fact that $FM = MG$, we have $EF + EM = DG + DM$, which is equivalent to $EF + DS = DG + ES$, as desired.

Also solved by Misiakos Panagiotis, Athens College(HAEF), Greece; Kyoung A Lee, The Hotchkiss School, Lakeville, CT, USA; George Gavrilopoulos, High School Of Nea Makri, Athens, Greece; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Andrea Fanchini, Cantú, Italy; Arkady Alt, San Jose, CA, USA; Behzod Ibodullaev, Lyceum TCTU, Tashkent, Uzbekistan; Farrukh Mukhammadiev, Academic Lyceum Nr.1 under the SamIES, Samarkand, Uzbekistan; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan; Saturnino Campo Ruiz, Salamanca, Spain; Titu Zvonaru and Neculai Stanciu, Romania.

S328. Let k be an odd positive integer. There are k positive integers written on a circle that add up to $2k$. Prove that for any $1 \leq m \leq k$, among the k given numbers, we can find one or more consecutively placed numbers that add up to $2m$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by the author

Let $a_1, \ldots, a_k$ be numbers written on a circle. Consider k sums of $2m$ consecutive numbers on a circle: $s_i = a_i + a_{i+1} + \cdots + a_{i+2m-1}$, where $i \in \{1, \ldots, k\}$ and the indices are taken modulo k. We have

$$s_1 + s_2 + \cdots + s_k = 2m(a_1 + a_2 + \cdots + a_k) = 4km.$$

If all s_i are equal to $4m$, then $a_i = a_{2m+i}$ for all $i \in \{1, \ldots, k\}$. Observe that because k is odd, we have $d = \gcd(k, 2m) \mid m$. We find that in this case our k numbers on a circle, consist of $\frac{k}{d}$ congruent blocks of d numbers. The sum of numbers in each block is $2d$, so if we consider $\frac{m}{d}$ consecutive blocks, their sum is $2m$, as desired.

If not all s_i are equal, then there exists s_j such that $s_j \leq 4m - 1$. Consider $2m$ sums:

$$a_j, \ a_j + a_{j+1}, \ \ldots, \ a_j + a_{j+1} + \cdots + a_{j+2m-1}.$$

If any of these sums is equal to $2m$, we are done. Otherwise, each of these sums is less than $4m$ and by the pigeonhole principle, two of these sums give the same residue module $2m$. Taking their difference, we find a sum of several consecutive numbers on circle that is equal to $2m$.

S329. Given a quadrilateral $ABCD$ with $AB + AD = BC + DC$ let (I_1) be the incircle of triangle ABC which is tangent to AC at E, and let (I_2) be the incircle of triangle ADC which is tangent to AC at F. The diagonals AC and BD intersect at point P. Suppose BI_1 and DE intersect at point S and DI_2 and BF intersect at point T. Prove that S, P, T are collinear.

Proposed by Khakimboy Egamberganov, Tashkent, Uzbekistan

Solution by Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan

At first we use these following lemmas:

Lemma 1: Let $ABCD$ be a quadrilateral. Prove that $AB + AD = BC + BD$ if and only if $ABCD$ is $B-$ extangential or $D-$ extangential.

Proof: Assume that $ABCD$ is B or $D-$ extangential. It is enough to prove in case of $ABCD$ is $B-$ extangential.

Let B excircle of $ABCD$ tangents to the sides BA, CD, AD, BC at points X, Y, Z, T, respectively. We have $BX = BT$ or $BA + AD + DZ = BC + CD + DY$ and $DZ = DY$, so $BA + AD = BC + CD$

Assume that $BA + AD = BC + CD$ **(1)** and let Ω be the circle tangent to lines BA, CD, AD. It is enough to prove that CB is tangent to Ω.Let tangent line from C to Ω intersects BA at B_0

Then quadrilateral AB_0CD - B_0 - extangencial. $\implies AB_0 + AD = CD + CD_0$ **(2)**. We have from (1) and (2) that $B = B_0$, hence *Lemma1* is proved.

Lemma 2: The lines I_1I_2, AC, BD are concurrent.

Proof: From *Lemma 1* $ABCD$ is $B-$ extangential. Let $I_1I_2 \cap AC = P_0$ is insimilicenter of (I_1) and (I_2).

However B is exsimilicenter of (I_1) and Ω, and D is insimilicenter of (I_2) and Ω.

From Monge-D'Alembert cirle theorem we have B, D, P_0 are collinear, so $P_0 = P$.

Lemma 3: Let Γ_b is $B-$ excircle of $\triangle ABC$ and Γ_d be is $D-$ excircle of $\triangle ADC$. Then AC is tangent to Γ_b at F and is also tangent to Γ_d at E.

*Proof:*It is obviously, we have $\frac{CA+CB-AB}{2} = \frac{AD+AC-DC}{2}$.

Hence F is exsimilicenter of (I_2) and Γ_b, E is exsimilicenter of (I_1) and Γ_d.

We prove the problem by applying abovementioned lemmas: Let T_0 be the exsimilicenter of (I_2) and Ω.

We have by Monge-D'Alembert cirle theorem:

(1) For $\{(I_1), (I_2), \Gamma_b\}$ we obtain T_0, F, B are collinear , so $T_0 \in BF$.

(2) For $\{(I_2), \Omega, \Gamma_b\}$ we get T_0, I, I_2 are collinear , so $T_0 \in DI_2$.

From (1) and (2) we have $T = T_0$ or T is exsimilicenter of (I_2) and Ω.

Similarly we have S is insimilicenter of (I_1) and Ω.

Finally applying by Monge-D'Alembert cirle theorem for $(I_1), (I_2), \Omega$ we obtain that T, S, P are collinear.

S330. Let $x_1, x_2, \ldots, x_n$ be real numbers, $n \geq 2$, such that $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. Prove that,

$$\sum_{i=1}^{n} \sqrt{1 - x_i^2} + k_n \cdot \sum_{1 \leq j < k \leq n} x_j x_k \geq n - 1$$

Where $k_n = 2 - 2\sqrt{1 + \frac{1}{n-1}}$.

Proposed by Marius Stănean, Zalău, Romania

Solution by the author

We have two cases:

1. if $\sum\limits_{1 \leq j < k \leq n} x_j x_k \leq 0$ then

$$\sum_{i=1}^{n} \sqrt{1 - x_i^2} + k_n \cdot \sum_{1 \leq j < k \leq n} x_j x_k \geq \sum_{i=1}^{n} 1 - x_i^2 = n - 1;$$

2. if $\sum\limits_{1 \leq j < k \leq n} x_j x_k \geq 0$ then noting $q = \sum\limits_{1 \leq j < k \leq n} x_j x_k \geq 0$ the inequality we can write as

$$\sum_{i=1}^{n} \sqrt{1 - x_i^2} \geq n - 1 - k_n \cdot q.$$

By squaring both sides we get

$$\sum_{i=1}^{n} (1 - x_i^2) + 2 \sum_{1 \leq j < k \leq n} \sqrt{1 - x_j^2} \cdot \sqrt{1 - x_k^2} \geq (n-1)^2 - 2(n-1)k_n q + k_n^2 q^2 \iff$$

$$2 \sum_{1 \leq j < k \leq n} \sqrt{1 - x_j^2} \cdot \sqrt{1 - x_k^2} \geq (n-1)(n-2) - 2(n-1)k_n q + k_n^2 q^2.$$

Using Cauchy-Schwarz inequality we have for $1 \leq j < k \leq n$

$$\sqrt{1 - x_j^2} \cdot \sqrt{1 - x_k^2} = \sqrt{x_1^2 + \ldots + x_{j-1}^2 + x_{j+1}^2 + \ldots + x_n^2} \cdot \sqrt{x_1^2 + \ldots + x_{k-1}^2 + x_{k+1}^2 + \ldots + x_n^2} \geq$$

$$| \, x_1^2 + \ldots + x_{j-1}^2 + x_{j+1}^2 + \ldots + x_{k-1}^2 + x_{k+1}^2 + \ldots + x_n^2 + x_j x_k \, |$$

and summing this for all j, k such that $1 \leq j < k \leq n$ we get

$$\sum_{1 \leq j < k \leq n} \sqrt{1 - x_j^2} \cdot \sqrt{1 - x_k^2} \geq \sum_{1 \leq j < k \leq n} | \, 1 - x_i^2 - x_k^2 + x_j x_k \, | \geq$$

$$\left| \binom{n}{2} - (n-1)(x_1^2 + x_2^2 + \ldots + x_n^2) + q \right| = \frac{(n-1)(n-2)}{2} + q$$

Therefore to prove the inequality (1) it is enough to show that

$$(n-1)(n-2) + 2q \geq (n-1)(n-2) - 2(n-1)k_n q + k_n^2 q^2 \iff$$

$$q(2 + 2(n-1)k_n - k_n^2 q) \geq 0.$$

But k_n is a root of the quadratic equation $x^2 - 4x - \frac{4}{n-1} = 0 \iff (n-1)x^2 - 4(n-1)x - 4 = 0$, so $\frac{2+2(n-1)k_n}{k_n^2} = \frac{n-1}{2}$ therefore (2) is equivalent with the following inequality

$$q\left(\frac{n-1}{2} - q\right) \geq 0$$

which is true because $q \geq 0$ and $q \leq \frac{n-1}{2}$. The last inequality follows from

$$\sum_{1 \leq j < k \leq n} (x_j - x_k)^2 \geq 0 \Longleftrightarrow (n-1)(x_1^2 + x_2^2 + \ldots + x_n^2) \geq 2 \sum_{1 \leq j < k \leq n} x_j x_k.$$

The equality hold when $q = 0$ which means that one of the numbers $x_1, x_2, \ldots, x_n$ is ± 1 and the other 0 or $q = \frac{n-1}{2}$ which means that $x_1 = x_2 = \ldots = x_n = \pm\sqrt{\frac{1}{n}}$.

Also solved by Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan, Utsab Sarkar, Chennai Mathematical Institute, India

U325. Let $A_1B_1C_1$ be a triangle with circumradius R_1. For each $n \geq 1$, the incircle of triangle $A_nB_nC_n$ is tangent to its sides at points A_{n+1}, B_{n+1}, C_{n+1}. The circumradius of triangle $A_{n+1}B_{n+1}C_{n+1}$, which is also the inradius of triangle $A_nB_nC_n$ is R_{n+1}. Find $\lim_{n\to\infty} \dfrac{R_{n+1}}{R_n}$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

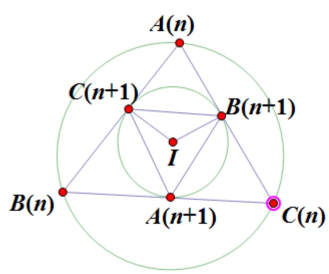
Solution by Yong Xi Wang,East China Institute Of Technology, China

Suppose $\Delta A_nB_nC_n$ has an incircle with radius r_n,then we have $R_{n+1} = r_n$ and use well known

$$\frac{r_n}{R_n} = 4\sin\frac{A_n}{2}\sin\frac{B_n}{2}\sin\frac{C_n}{2}$$

on the other hand,we have

$$\angle A_{n+1} = \frac{\pi}{2} - \frac{\angle A_n}{2}, \angle B_{n+1} = \frac{\pi}{2} - \frac{\angle B_n}{2} \angle C_{n+1} = \frac{\pi}{2} - \frac{\angle C_n}{2}$$



Because

$$\angle A_n + \angle B_{n+1}IC_{n+1} = \pi, \angle C_{n+1}A_{n+1}B_{n+1} = 2\angle B_{n+1}IC_{n+1}$$

so we have

$$\angle A_{n+1} - \frac{\pi}{3} = -\frac{1}{2}\left(\angle A_n - \frac{\pi}{3}\right) \Longrightarrow \angle A_n = \frac{\pi}{3} + \left(\angle A_1 - \frac{\pi}{3}\right)\left(-\frac{1}{2}\right)^{n-1}$$

simaler we have

$$\angle B_n = \frac{\pi}{3} + \left(\angle B_1 - \frac{\pi}{3}\right)\left(-\frac{1}{2}\right)^{n-1}, \angle C_n = \frac{\pi}{3} + \left(\angle C_1 - \frac{\pi}{3}\right)\left(-\frac{1}{2}\right)^{n-1}$$

so we have

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n = \frac{\pi}{3}$$

so

$$\lim_{n\to\infty}\frac{R_{n+1}}{R_n} = \lim_{n\to\infty}\frac{r_n}{R_n} = \lim_{n\to\infty} 4\sin\frac{\pi}{6}\sin\frac{\pi}{6}\sin\frac{\pi}{6} = \frac{1}{2}$$

Also solved by Michael Tang, Edina High School, MN, USA; Ji Eun Kim, Tabor Academy, MA, USA; Kyoung A Lee, The Hotchkiss School, Lakeville, CT, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Arkady Alt, San Jose, CA, USA; Saturnino Campo Ruiz, Salamanca, Spain; Titu Zvonaru and Neculai Stanciu, Romania.

U326. Find

$$\sum_{n=0}^{\infty} \frac{a_n + 2}{a_n^2 + a_n + 1}$$

where $a_0 > 1$ and $3a_{n+1} = a_n^3 + 2$ for all integers $n \geq 1$.

Proposed by Arkady Alt, San Jose, CA, USA

Solution by G. C. Greubel, Newport News, VA , USA

Consider the series

$$\sum_{n=0}^{\infty} \frac{a_n + 2}{a_n^2 + a_n + 1}$$

where $3a_{n+1} = a_n^3 + 2$. The reduction of the series is as follows.

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \frac{a_n + 2}{a_n^2 + a_n + 1} \\ &= \sum_{n=0}^{\infty} \frac{(a_n - 1)(a_n + 2)}{a_n^3 - 1} \end{aligned}$$

where $3a_{n+1} = a_n^3 + 2$ was used. Now,

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \frac{a_n^2 + a_n - 2}{3(a_{n+1} - 1)} \\ &= \sum_{n=0}^{\infty} \frac{(a_n - 1)(a_n^2 + a_n - 2)}{3(a_n - 1)(a_{n+1} - 1)} \\ &= \sum_{n=0}^{\infty} \frac{a_{n+1} - a_n}{(a_n - 1)(a_{n+1} - 1)} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{a_n - 1} - \frac{1}{a_{n+1} - 1} \right). \end{aligned}$$

The collapse of this telescopic series leads to the desired result, namely,

$$\sum_{n=0}^{\infty} \frac{a_n + 2}{a_n^2 + a_n + 1} = \frac{1}{a_0 - 1}.$$

Also solved by Ji Eun Kim, Tabor Academy, MA, USA; Michael Tang, Edina High School, MN, USA; Kyoung A Lee, The Hotchkiss School, Lakeville, CT, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Yong Xi Wang,East China Institute Of Technology, China; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Cailan Li; Haroun Meghaichi, University of Science and Technology Houari Boumediene, Algiers, Algeria; Moubinool Omarjee Lycée Henri IV, Paris, France; Paolo Perfetti, Universitá degli studi di Tor Vergata Roma, Roma, Italy; Prithwijit De, HBCSE, Mumbai, India; Albert Stadler, Herrliberg, Switzerland; Alessandro Ventullo, Milan, Italy.

U327. Let $(a_n)_{n\geq 0}$ be a sequence of real numbers with $a_0 = 1$ and

$$a_{n+1} = \frac{a_n}{n^2 a_n + a_n^2 + 1}.$$

Find the limit $\lim\limits_{n\to\infty} n^3 a_n$.

Proposed by Khakimboy Egamberganov, Tashkent, Uzbekistan

Solution by Alessandro Ventullo, Milan, Italy

Observe that $a_{n+1} < \dfrac{a_n}{n^2 a_n} = \dfrac{1}{n^2}$, so $\sum\limits_{n=0}^{\infty} a_n$ converges by the Comparison Test. Let

$$\sum_{n=0}^{\infty} a_n = c \in \mathbb{R}.$$

We have

$$\frac{1}{a_{n+1}} = n^2 + a_n + \frac{1}{a_n}.$$

So,

$$\begin{aligned}
\frac{1}{a_1} &= 0^2 + a_0 + \frac{1}{a_0} \\
\frac{1}{a_2} &= 1^2 + a_1 + \frac{1}{a_1} \\
\vdots \quad &\vdots \quad \vdots \\
\frac{1}{a_n} &= (n-1)^2 + a_{n-1} + \frac{1}{a_{n-1}},
\end{aligned}$$

and summing the two columns, we get

$$\frac{1}{a_n} = \frac{(n-1)n(2n-1)}{6} + \sum_{k=0}^{n-1} a_k + 1.$$

Hence,

$$\frac{1}{n^3 a_n} = \frac{(n-1)n(2n-1)}{6n^3} + \frac{\sum_{k=0}^{n-1} a_k + 1}{n^3}.$$

Since

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} a_k + 1}{n^3} = \lim_{n\to\infty} \frac{c+1}{n^3} = 0,$$

we get

$$\lim_{n\to\infty} \frac{1}{n^3 a_n} = \lim_{n\to\infty} \frac{(n-1)n(2n-1)}{6n^3} = \frac{1}{3},$$

so

$$\lim_{n\to\infty} n^3 a_n = 3.$$

Also solved by Arkady Alt, San Jose, CA, USA; Ji Eun Kim, Tabor Academy, MA, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Yong Xi Wang,East China Institute Of Technology, China; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Haroun Meghaichi, University of Science and Technology Houari Boumediene, Algiers, Algeria; Moubinool Omarjee Lycée Henri IV, Paris, France; Paolo Perfetti, Universitá degli studi di Tor Vergata Roma, Roma, Italy; Albert Stadler, Herrliberg, Switzerland.

U328. Let $(a_n)_{n\geq 1}$ be an increasing sequence of positive real numbers such that $\lim_{n\to\infty} a_n = \infty$ and the sequence $(a_{n+1} - a_n)_{n\geq 1}$ is monotonic. Evaluate

$$\lim_{n\to\infty} \frac{a_1 + \ldots + a_n}{n\sqrt{a_n}}$$

Proposed by Mihai Piticari and Sorin Rădulescu, Romania

Solution by Paolo Perfetti, Universitá degli studi di Tor Vergata Roma, Roma, Italy

The sequence $(a_{n+1} - a_n)_{n\geq 1}$ converges to a limit L because of its monotonicity. Suppose $L > 0$ is finite. By Cesaro–Stolz we have

$$L = \lim_{n\to\infty} (a_{n+1} - a_n) \implies L = \lim_{n\to\infty} \frac{a_n}{n}$$

Moreover

$$\lim_{n\to\infty} (\sqrt{a_{n+1}} - \sqrt{a_n}) = \lim_{n\to\infty} \frac{a_{n+1} - a_n}{\sqrt{a_{n+1}} + \sqrt{a_n}} = \frac{L}{\infty} = 0$$

Also by Cesaro–Stolz we consider

$$\lim_{n\to\infty} \frac{(a_1 + \ldots + a_{n+1}) - (a_1 + \ldots + a_n)}{(n+1)\sqrt{a_{n+1}} - n\sqrt{a_n}} = \lim_{n\to\infty} \frac{a_{n+1}}{(n+1)\sqrt{a_{n+1}} - n\sqrt{a_n}}$$

The reciprocal gives

$$\lim_{n\to\infty} \frac{(n+1)\sqrt{a_{n+1}} - n\sqrt{a_n}}{a_{n+1}} = \lim_{n\to\infty} \left(\frac{n+1}{a_{n+1}} (\sqrt{a_{n+1}} - \sqrt{a_n}) + \frac{\sqrt{a_n}}{a_{n+1}} \right)$$

and

$$\lim_{n\to\infty} \frac{n+1}{a_{n+1}} (\sqrt{a_{n+1}} - \sqrt{a_n}) = \frac{1}{L} \cdot 0 = 0, \qquad \lim_{n\to\infty} \frac{\sqrt{a_n}}{a_{n+1}} = 0$$

The conclusion is

$$\lim_{n\to\infty} \frac{a_{n+1}}{(n+1)\sqrt{a_{n+1}} - n\sqrt{a_n}} = \infty$$

because $\dfrac{a_{n+1}}{(n+1)\sqrt{a_{n+1}} - n\sqrt{a_n}}$ is positive via the increasing monotonicity of (a_n) and $a_n > 0$.
Now let $L = \lim_{n\to\infty} (a_{n+1} - a_n) = 0$ and then $0 = \lim_{n\to\infty} \dfrac{a_n}{n}$
by Cesaro–Stolz. As above we consider

$$\lim_{n\to\infty} \frac{(a_1 + \ldots + a_{n+1}) - (a_1 + \ldots + a_n)}{(n+1)\sqrt{a_{n+1}} - n\sqrt{a_n}} = \lim_{n\to\infty} \frac{a_{n+1}}{(n+1)\sqrt{a_{n+1}} - n\sqrt{a_n}}$$

that is

$$\lim_{n\to\infty} \frac{a_{n+1}}{\sqrt{a_{n+1}} \left(n \frac{\sqrt{a_{n+1}} - \sqrt{a_n}}{\sqrt{a_{n+1}}} + 1 \right)}$$

Now we prove that
$n\dfrac{\sqrt{a_{n+1}} - \sqrt{a_n}}{\sqrt{a_{n+1}}}$ is bounded above. We write

$$\frac{n}{\sqrt{a_{n+1}}} \frac{a_{n+1} - a_n}{\sqrt{a_{n+1}} + \sqrt{a_n}} \leq \frac{n(a_{n+1} - a_n)}{2a_n}$$

Lemma: Under the assumptions of the statement of the problem, the positive sequence $\frac{n(a_{n+1}-a_n)}{2a_n}$ is bounded above.

Proof: Let's argue by contradiction supposing that for any positive p there exists n_p such that $\frac{n_P(a_{n_p+1}-a_{n_p})}{2a_{n_p}}$ or $a_{n_p+1}-a_{n_p} \geq \frac{pa_{n_p}}{n_p}$. The monotonicity of $a_{n+1}-a_n$ yields

$$a_{n+1}-a_n \geq a_{n_p+1}-a_{n_p} \geq \frac{pa_{n_p}}{n_p} \; \forall \; 1 \leq n \leq n_p - 1$$

This implies

$$a_1 = a_{n_p} + \sum_{k=1}^{n_p}(a_k - a_{k+1}) < a_{n_p} - n_p\frac{pa_{n_p}}{n_p} = n_p(1-p) < 0$$

and this is a contradiction with the positivity of the sequence $\{a_n\}$.

The consequence of the Lemma is the existence of a constant C such that $0 < \frac{n(a_{n+1}-a_n)}{2a_n} \leq C$. This implies

$$\lim_{n\to\infty} \frac{a_{n+1}}{\sqrt{a_{n+1}}\left(n\frac{\sqrt{a_{n+1}}-\sqrt{a_n}}{\sqrt{a_{n+1}}}+1\right)} \geq \frac{a_{n+1}}{\sqrt{a_{n+1}}\,(C+1)} = \infty$$

The last step is the case $L = \infty$.
This means that for all $p > 0$ there exists n_p such that $n > n_p$ implies $a_{n+1} - a_n > p$. It follows

$$\begin{aligned} a_n &= a_1 + \sum_{k=1}^{n_p-1}(a_{k+1}-a_k) + \sum_{k=n_p}^{n}(a_{k+1}-a_k) \geq \\ &\geq a_1 + (n_p-1)\min_{1\leq k\leq n_p-1}(a_{k+1}-a_k) + p(n-n_p+1) \\ &= a_1 + (n_p-1)(a_2-a_1) + p(n-n_p+1) \geq p(n-n_p+1) \geq p\frac{n}{2} \end{aligned}$$

if $n \geq 2(n_p-1)$ The monotonicity increasing of $\{a_n\}$ has been used.
Based on this we can write

$$\begin{aligned} &a_1 + \ldots + a_{2n_p-3} + \left(a_{2n_p-2} + \ldots + a_n\right) \geq (2n_p-3)a_1 + (n-2n_p+3)p\frac{n}{2} \geq \\ &\geq (n-2n_p+3)p\frac{n}{2} \geq p\frac{n^2}{4} \end{aligned}$$

for any $n \geq 4n_p + 6$. Finally we put the above lover bounds in $\frac{a_1+\ldots+a_n}{n\sqrt{a_n}}$ and write

$$\frac{a_1+\ldots+a_n}{n\sqrt{a_n}} = \frac{\sqrt{a_1+\ldots+a_n}}{n}\frac{\sqrt{a_1+\ldots+a_n}}{\sqrt{a_n}} \geq \frac{\sqrt{a_1+\ldots+a_n}}{n} \geq \sqrt{p}\frac{n}{2}\frac{1}{n} = \sqrt{p}$$

Since p can be as large as we want, this shows that

$$\lim_{n\to\infty}\frac{a_1+\ldots+a_n}{n\sqrt{a_n}} = \infty$$

also when $L = \infty$.

Also solved by Moubinool Omarjee Lycée Henri IV, Paris, France.

U329. Let $a_1 \le a_2 \le \ldots \le a_{\frac{n(n-1)}{2}}$ be the distances between n distinct points lying on the plane. Prove that there is a constant c such that for any n we can find indices i and j such that

$$\left|\frac{a_i}{a_j} - 1\right| < \frac{c \ln n}{n^2}.$$

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Solution by Ivan Borsenco, Massachusetts Institute of Technology, USA

Assume that for any pair of indices (i,j) we have $\left|\frac{a_i}{a_j} - 1\right| \ge d$ and d is the greatest such number. Then

$$a_{\frac{n(n-1)}{2}} \ge (1+d)a_{\frac{n(n-1)}{2}-1} \ge \ldots \ge (1+d)^{\frac{n(n-1)}{2}-1}a_1.$$

Let $AB = a_{\frac{n(n-1)}{2}}$ and $CD = a_1$. From the fact that $AC + BC > AB$, we can assume without loss of generality that $AC \ge \frac{AB}{2}$. Then

$$d \le \left|\frac{\min(AC,AD)}{\max(AC,AD)} - 1\right| = \frac{CD}{\max(AC,AD)} \le \frac{CD}{AC} \le \frac{2CD}{AB} = \frac{2}{(1+d)^{\frac{n(n-1)}{2}-1}}$$

and $d(1+d)^{\frac{n(n-1)}{2}-1} \le 2$.

If in our configuration $AB = a_{\frac{n(n-1)}{2}}$ and $AC = a_1$, then

$$d \le \left|\frac{AC}{AB} - 1\right| \le \frac{BC}{AB} = \frac{1}{(1+d)^{\frac{n(n-1)}{2}-1}}.$$

In any case, we obtain $d(1+d)^{\frac{n(n-1)}{2}-1} \le 2$.

Assume that $n \ge 4$ and consider $f(x) = x(1+x)^{\frac{n(n-1)}{2}-1}$. Observe that

$$f\left(\frac{100\ln n}{n^2}\right) = \frac{100\ln n}{n^2}\left(1 + \frac{1}{\frac{n^2}{100\ln n}}\right)^{\frac{n(n-1)}{2}-1}.$$

Because

$$\left(1 + \frac{1}{\frac{n^2}{100\ln n}}\right)^{\frac{n^2}{100\ln n}+1} \ge e,$$

we get

$$1 + \frac{1}{\frac{n^2}{100\ln n}} \ge e^{\frac{100\ln n}{n^2+100\ln n}} > e^{\frac{10\ln n}{n^2}}.$$

Hence

$$f\left(\frac{100\ln n}{n^2}\right) \ge \frac{100\ln n}{n^2} \cdot e^{\frac{10\ln n}{n^2}\left(\frac{n(n-1)}{2}-1\right)} \ge \frac{100\ln n}{n^2} \cdot e^{3\ln n} = 100\ln n \cdot n > 2.$$

Therefore for $c = 100$, we always have $d < \frac{c\ln n}{n^2}$.

U330. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ with the properties

$$\begin{array}{ll} \text{(i)} & f \text{ is an antiderivative} \\ \text{(ii)} & f \text{ is integrable on any compact interval} \\ \text{(iii)} & f^2(x) = \displaystyle\int_0^x f(t)dt \text{ for all } x \in \mathbb{R} \end{array}$$

Proposed by Mihai Piticari, Câmpulung Moldovenesc, Romania

Solution by Paolo Perfetti, Universitá degli studi di Tor Vergata Roma, Roma, Italy

We claim that $f_1(x) \equiv 0$ and $f_2(x) = x/2$ are solutions. We are going to prove that there are no other solutions. Clearly $f(0) = 0$. If f is an antiderivative, it is differentiable so by (iii)

$$2f(x)f'(x) = f(x)$$

It follows that if $f(x_0) \neq 0$ then $f'(x_0) = 1/2$. Suppose now that there exists a function $g(x)$ satisfying (i)–(iii) but $g(x) \neq f_1$ and $g \neq f_2$. This means that there exists a point $\xi > 0$ (or $\xi < 0$) such that $g(\xi) \neq 0$ and $g(\xi) \neq \xi/2$.

By the mean value theorem

$$\frac{g(\xi) - g(0)}{\xi - 0} = g'(\eta),$$

If $g(\eta) \neq 0$ we have $g'(\eta) = 1/2$ and then $g(\xi) = \xi/2$ which is impossible so we are forced to sat that $g(\eta) = 0$.

Now, consider the following two Lemmas.

Lemma 1: If f satisfying (i)–(iii) is such that $f(x_1) = 0$, $x_1 \neq 0$, (say $x_1 > 0$) then $f(x) = 0$ for any $x \in [0, x_1]$.

Proof: Suppose $f(x_2) \neq 0$, $0 < x_2 < x_1$. By the mean value theorem it would follow that a value x_3 would exist such that $f(x_3) \neq 0$ and $f'(x_3) < 0$. but this is impossible.

Lemma 2 : If f satisfies (i)–(iii) and $f(x) \equiv 0$, for $x \in [a, b]$ with $a \leq 0 \leq b$, $a \neq b$, the $f(x) = 0$ for any $x \in \mathbb{R}$.

Proof: Suppose indeed $f(x_2) \neq 0$ and $x_2 > b$. By $f'(x_2) = 1/2$ and the fact that any derivative is a "Darboux–function", it follows that $f'(x)$ assumes all the value between the value 0 and 1/2 but this is impossible.

The conclusion is that the function $g(x)$ described above, does not exist.

O325. The *taxicab distance* between points $P_1 = (x_1, y_1)$ and $P_2(x_2, y_2)$ in a coordinate plane is given by

$$d(P_1P_2) = |x_1 - x_2| + |y_1 - y_2|.$$

The *taxicab disk* with center O and radius R is the set of points P such that $d(P, O) \leq R$. Given n points that are pairwise at most R taxicab distance apart, find the smallest constant c such that any such set of points can be covered by a taxicab disk of radius cR.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by the author

Assume that points are at most a unit taxicab distance apart. We are looking for the smallest c such that any such set of points can be covered by the taxicab disk of radius c. Consider four points with coordinates $(\pm\frac{1}{4}, \pm\frac{1}{4})$. Then they are at most one taxicab unit apart and only can be covered by a taxicab disk with radius at least $\frac{1}{2}$. We prove that $c = \frac{1}{2}$ is the smallest constant.

Consider a rotation of the plane with center at the origin and angle 45°. Any taxicab disk becomes a square and points $P_i = (x_i, y_i)$ become $P_i' = (x_i', y_i') = \left(\frac{1}{\sqrt{2}}(x_i + y_i), \frac{1}{\sqrt{2}}(x_i - y_i)\right)$. We are looking for the least square that can cover our rotated set. Note that x-coordinates of any two points in the rotated set are at most $\frac{1}{\sqrt{2}}$ apart:

$$\frac{1}{\sqrt{2}}|x_i + y_i - x_j - y_j| \leq \frac{1}{\sqrt{2}}\left(|x_i - x_j| + |y_i - y_j|\right) \leq \frac{1}{\sqrt{2}}.$$

Similarly, any two y-coordinates are at most $\frac{1}{\sqrt{2}}$ apart. We have $\max(x_i') - \min(x_i') \leq \frac{1}{\sqrt{2}}$ and $\max(y_i') - \min(y_i') \leq \frac{1}{\sqrt{2}}$. Thus rotated set can be covered by a $\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}$ square, which initially corresponds to a taxicab disk of radius $\frac{1}{2}$.

O326. Let ABC be a non-isosceles triangle. Let D be a point on the ray $\overrightarrow{AB}$, but not on the segment AB, and let E be a point on the ray $\overrightarrow{AC}$, but not on the segment AC, such that $AB \cdot BD = AC \cdot CE$. The circumcircles of triangles ABE and ACD intersect in points A and F. Let O_1 and O_2 be the circumcenters of triangles ABC and ADE. Prove that lines AF, O_1O_2, and BC are concurrent.

Proposed by İlker Can Çiçek, Instanbul, Turkey

Solution by Adnan Ali, Student in A.E.C.S-4, Mumbai, India

We first prove two Lemmas:

Lemma 1: O_1O_2 is the perpendicular bisector of BC.

Proof: It is sufficient to show that O_2 lies on the perpendicular bisector of BC. Let r be the radius of $\odot(ADE)$. Then, power of B, C w.r.t to $\odot(ADE)$ is expressed as

$$AB \cdot BD = r^2 - O_2B^2 = AC \cdot CE = r^2 - O_2C^2$$

Thus $O_2B = O_2C$, proving that O_2 lies on the perpendicular bisector of BC.

Lemma 2: AF bisects BC.

Proof: Let $M = AF \cap BC$, $\overrightarrow{BC} \cap \odot(ABE) = Q$, $\overrightarrow{CB} \cap \odot(ADC) = P$. Then, power of C w.r.t to $\odot(ABE)$ and power of B w.r.t $\odot(ADC)$ can be expressed as respectively:

$$AC \cdot CE = QC \cdot CB$$

$$= AB \cdot BD = PB \cdot BC$$

thus showing that $PB = QC$ (1).

Now expressing the power of M w.r.t to $\odot(ABE)$

$$AM \cdot MF = QM \cdot MB$$

and power of M w.r.t to $\odot(ADC)$ as

$$AM \cdot MF = PM \cdot MC.$$

Hence

$$PM \cdot MC = QM \cdot MB \Rightarrow PB \cdot MC + BM \cdot MC = QC \cdot MB + MB \cdot MC$$

$$\Rightarrow PB \cdot MC = QC \cdot MB \Rightarrow MB = MC \quad \text{(from (1))}.$$

Hence AF bisects BC.

Now from Lemma 1 and Lemma 2 we conclude that AF, O_1O_2, BC concur at M, the midpoint of BC.

Also solved by Misiakos Panagiotis, Athens College(HAEF), Greece; Andrea Fanchini, Cantú, Italy; Farrukh Mukhammadiev, Academic Lyceum Nr.1 under the SamIES, Samarkand, Uzbekistan; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan; Titu Zvonaru and Neculai Stanciu, Romania.

O327. Let a, b, c be positive reals such that $a^2+b^2+c^2+abc=4$. Prove that,

$$a+b+c \leq \sqrt{2-a}+\sqrt{2-b}+\sqrt{2-c}$$

Proposed by An Zhen-ping, Xianyang Normal University, China

Solution by Utsab Sarkar, Chennai Mathematical Institute, India

Notice that $a, b, c \in (0,2)$ and hence we'll make the substitution for $a = 2\cos A, b = 2\cos B, c = 2\cos C$. Therefore our constraint becomes to,

$$\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1 \implies A+B+C=\pi$$

Thus for a triangle ΔABC we need only to prove,

$$\sum_{cyc}\left(\sqrt{2-a}-a\right) \geq 0 \iff \sum_{cyc}\left(\sin\frac{A}{2}-\cos A\right) \geq 0$$

$$\iff \sum_{cyc}\sqrt{\frac{(a+b-c)(a+c-b)}{bc}} \geq \sum_{cyc}\frac{b^2+c^2-a^2}{bc} \iff \sum_{cyc} a\sqrt{bc(a+b-c)(a+c-b)} \geq \sum_{cyc}(a^2b+a^2c-a^3)$$

Now using Ravi Transformation we see, $a = y+z, b = z+x, c = x+y$ which provides an equivalent inequality to prove,

$$\sum_{cyc}(y+z)\sqrt{(x+y)(x+z)yz} \geq \sum_{cyc}(y+z)^2 x$$

Where x, y, z are positive reals.

By Cauchy–Schwarz Inequality we get,

$$(x+y)(x+z) \geq (x+\sqrt{yz})^2$$

Now,it only remains to show,

$$\sum_{cyc}(y+z)\sqrt{yz}\,(x+\sqrt{yz}) \geq \sum_{cyc}(y+z)^2 x \iff \sum_{cyc} x(y+z)\sqrt{yz} \geq 6xyz$$

And the last inequality is true by AM-GM, since

$$\sum_{cyc} x(y+z)\sqrt{yz} \geq \sum_{cyc} x\cdot 2\sqrt{yz}\cdot\sqrt{yz} = \sum_{cyc} 2xyz = 6xyz$$

.

Also solved by Ji Eun Kim, Tabor Academy, MA, USA; Misiakos Panagiotis, Athens College(HAEF), Greece; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Yong Xi Wang,East China Institute Of Technology, China; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Arkady Alt, San Jose, CA, USA; Behzod Ibodullaev, Lyceum TCTU, Tashkent, Uzbekistan; Farrukh Mukhammadiev, Academic Lyceum Nr.1 under the SamIES, Samarkand, Uzbekistan; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Paolo Perfetti, Universitá degli studi di Tor Vergata Roma, Roma, Italy; Piriyakorn, Triam Udom Suksa School, Thailand; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan; Shatlyk Mamedov, School Nr. 21, Dashoguz, Turkmenistan; Titu Zvonaru and Neculai Stanciu, Romania.

O328. The diagonals AD, BE, CF of the convex hexagon $ABCDEF$ intersect at point M. Triangles ABM, BCM, CDM, DEM, EFM, FAM are acute. Prove that circumcenters of these triangles are concyclic if and only if the areas of quadrilaterals $ABDE$, $BCEF$, $CDFA$ are equal.

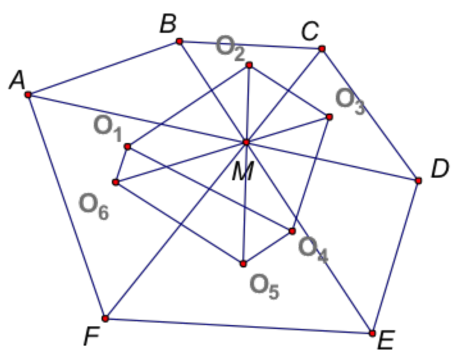
Proposed by Nairi Sedrakyan, Yerevan, Armenia

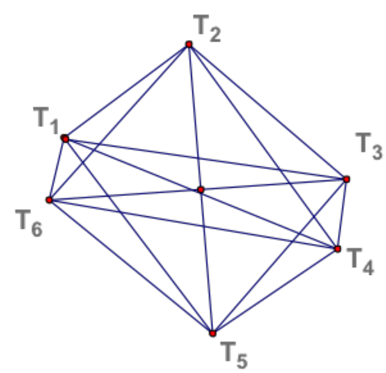
Solution by the author

Proof of necessity: Let O_1, O_2, O_3, O_4, O_5 and O_6 be the centers of the circumcircles of triangles ABM, BCM, CDM, DEM, EFM and FAM, respectively. Notice that O_1, O_2 and O_4, O_5 are laying on the perpendicular bisectors of BM and EM, respectively.

Therefore, $O_1O_2 \perp BE$ and $O_4O_5 \perp BE \Rightarrow O_1O_2 \parallel O_4O_5$. Moreover, the projection of O_1O_4 onto BE has length $\frac{1}{2}BE$. Since O_1, O_2, O_4, O_5 are concyclic by condition of the problem and $O_1O_2 \parallel O_4O_5 \Rightarrow O_1O_4 = O_2O_5$, and BE forms equal angles with O_1O_4 and O_2O_5. By analogy we get $O_1O_4 = O_3O_6$.

Consider T_1T_4, T_2T_5, T_3T_6 sharing the same midpoint and $T_1T_4 = O_1O_4, T_2T_5 = O_2O_5, T_3T_6 = O_3O_6$. Note that $T_1T_2T_4T_5$ is a rectangle, therefore, $T_1T_5 \parallel BE$ and T_1T_5 is the projection of T_1T_4 onto $T_1T_5 \Rightarrow T_1T_5 = \frac{1}{2}BE$.





Similarly we get $T_1T_3 \parallel AD, T_1T_3 = \frac{1}{2}AD$. Finally,

$$S_{ABDE} = \frac{1}{2}AD \cdot BE \sin\angle AMB = \frac{1}{2} \cdot 2T_1T_3 \cdot 2T_1T_5 \sin\angle T_3T_1T_5 = 4S_{T_1T_3T_5}.$$

Proof of sufficiency: Again, let O_1, O_2, O_3, O_4, O_5 and O_6 be the centers of the circumcircles of triangles ABM, BCM, CDM, DEM, EFM and FAM, respectively. Notice that O_1, O_2 and O_4, O_5 are laying on the perpendicular bisectors of BM and EM, respectively.

Therefore, $O_1O_2 \perp BE$ and $O_4O_5 \perp BE \Rightarrow O_1O_2 \parallel O_4O_5$. Let $AD = 2a, BE = 2b, CF = 2c, \angle AMB = \gamma, \angle BMC = \alpha, \angle CMD = \beta$. Since

$$\alpha + \beta + \gamma = 180^o$$

$$S_{ABDE} = 2ab\sin\gamma, S_{BCEF} = 2bs\sin\alpha, S_{CDFA} = 2ac\sin\beta$$

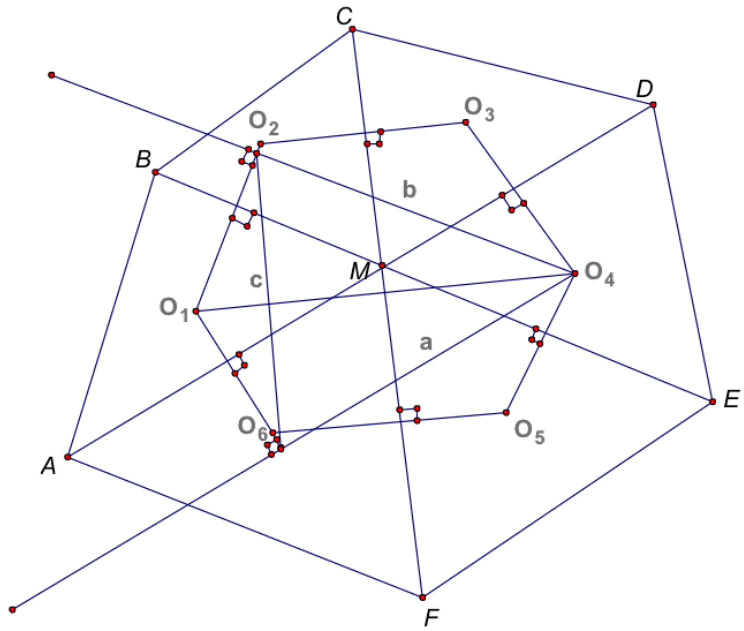
and

$$S_{ABDE} = S_{BCEF} = S_{CDFA} \Rightarrow$$

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}.$$

We deduce that α, β, γ are angles of a triangle with sides a, b and c. The lengths of projections of O_1O_4 onto BE and AD are equal to $\frac{1}{2}BE$ and $\frac{1}{2}AD$, respectively.

Hence, $O_1O_4 = 2R$, where R - radius of the circumcircle of the triagle with sidelengths a, b, c.



By analogy we get that $O_2O_5 = 2R, O_3O_6 = 2R \Rightarrow O_1O_4 = O_2O_5 = O_3O_6$.
Finally,

$$O_1O_2 \parallel O_4O_5, O_2O_3 \parallel O_5O_6, O_3O_4 \parallel O_6O_1$$

and

$$O_1O_4 = O_2O_5 = O_3O_6$$

yields that by condition of the problem, O_1, O_2, O_3, O_4, O_5 and O_6 are concyclic.

Also solved by Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan.

O329. For any positive integer r, denote by P_r the path on r vertices. For any positive integer g, prove that there exists a graph G with no cycles of length less than g with the following property: any two coloring of the vertices of G contains a monochromatic copy of P_r.

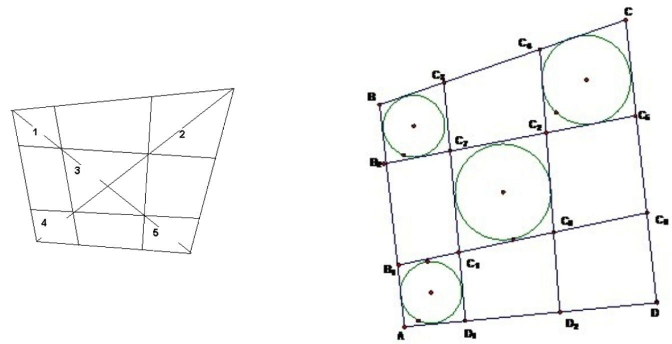
Proposed by Cosmin Pohoata, Columbia University, USA

Solution by the author

Recall the celebrated result by Erdös that there exists graphs with arbitrarily large girth and arbitrarily large chromatic number. More precisely, for any positive integers k and g, there is some graph G which has no cycle of length less than g and for which $\chi(G) \geq k$. This was one of the first applications of the probabilistic method in combinatorics. For the proof, see for example Lecture 11 from Daniel Spielman's graph theory class at Yale: http://www.cs.yale.edu/homes/spielman/462/2007/.

For our problem, we choose $k = 4r + 1$. Next, notice that if some vertex $x \in V(G)$ has degree $< 4r$, then we can use induction to argue that we can in fact color G using $4r$ colors. Indeed, remove x from G and notice that any $4n$-coloring of the new graph can be augmented to a coloring of G (since we can assign x one of the $4n$ colors, provided that $\deg x < 4r$). This contradicts the fact that $\chi(G) \geq 4r + 1$. Consequently, every graph G with chromatic number at least $4n + 1$ has a subgraph with a minimum degree vertex at least $4r$. It follows that any two coloring of G contains a monochromatic subgraph of minimum degree at least r. Within this monochromatic subgraph, it is easy to see that we have a path of length r; hence we found a monochromatic copy of P_r.

O330. Four segments divide a convex quadrilateral into nine quadrilaterals. The points of intersections of these segments lie on the diagonals of the quadrilateral (see figure below).



It is known that quadrilaterals Q_1, Q_2, Q_3, Q_4 admit inscribed circles. Prove that the quadrilateral Q_5 also has an inscribed circle.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Solution by Cosmin Pohoata, Columbia University, USA

The following is an exposition of the proof from Cosmin Pohoata and Titu Andreescu, 110 Geometry Problems for the International Mathematical Olympiad, XYZ Press, 2014.

The key idea is to make use of the following result due to Marius Iosifescu repeatedly.

Iosifescu's Theorem. A convex quadrilateral $ABCD$ admits an inscribed circle if and only if

$$\tan\frac{x}{2}\cdot\tan\frac{z}{2}=\tan\frac{y}{2}\cdot\tan\frac{w}{2},$$

where $x=\angle ABD$, $y=\angle ADB$, $z=\angle BDC$ and $w=\angle DBC$.

Proof. Using the trigonometric formula

$$\tan^2\frac{u}{2}=\frac{1-\cos u}{1+\cos u},$$

we get that the equality in the theory is equivalent to

$$(1-\cos x)(1-\cos z)(1+\cos y)(1+\cos w)$$

$$=(1-\cos y)(1-\cos w)(1+\cos x)(1+\cos z).$$

Let $a=AB$, $b=BC$, $c=CD$, $d=DA$ and $q=BD$. From the Law of Cosines, we have

$$\cos x=\frac{a^2+q^2-d^2}{2aq},$$

so that

$$1-\cos x=\frac{d^2-(a-q)^2}{2aq}=\frac{(d+a-q)(d-a+q)}{2aq}$$

and

$$1+\cos x=\frac{(a+q)^2-d^2}{2aq}=\frac{(a+q+d)(a+q-d)}{2aq}.$$

In the same way

$$1-\cos y=\frac{(a+d-q)(a-d+q)}{2dq},\quad 1+\cos y=\frac{(d+q+a)(d+q-a)}{2dq},$$

$$1-\cos z=\frac{(b+c-q)(b-c+q)}{2cq},\quad 1+\cos z=\frac{(c+q+b)(c+q-b)}{2cq},$$
$$1-\cos w=\frac{(c+b-q)(c-b+q)}{2bq},\quad 1+\cos w=\frac{(b+q+c)(b+q-c)}{2bq}.$$

Thus, after some rearranging, the identity we started with becomes

$$P\left((d-a+q)^2(b-c+q)^2-(a-d+q)^2(c-b+q)^2\right)=0,$$

where

$$P=\frac{d+a-q)(b+c-q)(d+q+a)(b+q+c)}{16abcdq^4}$$

is a positive expression according to the triangle inequality applied in triangles ABD and BCD. Factoring the above and cancelling out terms in the parantheses yields

$$4qP(b+d-a-c)\left((d-a)(b-c)+q^2\right)=0,$$

where the expression in the second paranthesis can never be equal to zero - since, by the triangle inequality, $q>a-d$ and $q>b-c$, so q^2 is always either (strictly) less or greater than $(a-d)(b-c)$. It thus follows that

$$\tan\frac{x}{2}\cdot\tan\frac{z}{2}=\tan\frac{y}{2}\cdot\tan\frac{w}{2}$$

holds if and only if $b+d-a-c=0$, i.e. $b+d=a+c$. By Pithot's theorem, we therefore conclude that this happens if and only if $ABCD$ admits an inscribed circle.

□

In order to use Iosifescu's theorem for quadrilateral 5, we aim to prove that

$$\tan\frac{\angle FHG}{2}\cdot\tan\frac{\angle BDC}{2}=\tan\frac{\angle ADB}{2}\cdot\tan\frac{\angle EHF}{2}.$$

However, quadrilateral 3 has an incircle, so

$$\tan\frac{\angle EFH}{2}\cdot\tan\frac{\angle FHG}{2}=\tan\frac{\angle EHF}{2}\cdot\tan\frac{\angle HFG}{2}.$$

It thus suffices to show that

$$\tan\frac{\angle EFH}{2}\cdot\tan\frac{\angle ADB}{2}=\tan\frac{\angle HFG}{2}\cdot\tan\frac{\angle BDC}{2}.$$

However, by applying Iosifescu's lemma once again to quadrilateral 1, the above happens if and only if

$$\tan\frac{\angle ABD}{2}\cdot\tan\frac{\angle BDC}{2}=\tan\frac{\angle ADB}{2}\cdot\tan\frac{\angle DBC}{2},$$

which we know is true if and only if $ABCD$ has inscribed circle.

This is where the magic comes in. We show that $ABCD$ has an inscribed circle by using Iosifescu's theorem for the other diagonal AC of $ABCD$. Namely, it suffices to show that

$$\tan\frac{\angle BAC}{2}\cdot\tan\frac{\angle ACD}{2}=\tan\frac{\angle BCA}{2}\cdot\tan\frac{\angle CAD}{2}.$$

This is however easy to prove in the same style as above. Indeed, by writing out Iosifescu's lemma for quadrilaterals 2, 3 and 4, and keeping in mind the vertical angles from the vertices of quadrilateral 3, we get precisely the above statement. We leave the details for the reader.

Also solved by Adnan Ali, Mumbai, India, AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Farrukh Mukhammadiev, Academic Lyceum Nr.1 under the SamIES, Samarkand, Uzbekistan; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan; Shatlyk Mamedov, School Nr. 21, Dashoguz, Turkmenistan; Titu Zvonaru and Neculai Stanciu, Romania.