Junior problems

J331. Determine all positive integers n such that

$$n!\left(1+\frac{1}{2}+\ldots+\frac{1}{n}+\frac{1}{n!}\right)$$

is divisible by n.

Proposed by Alessandro Ventullo, Milan, Italy

J332. Let $n \geq 3$ and $0 = a_0 < a_1 < ... < a_{n+1}$ such that $a_1a_2 + a_2a_3 + ... + a_{n-1}a_n = a_na_{n+1}$. Prove that

$$\frac{1}{a_3^2-a_0^2}+\frac{1}{a_4^2-a_1^2}+\ldots+\frac{1}{a_{n+1}^2-a_{n-2}^2} \geq \frac{1}{a_{n-1}^2}.$$

Proposed by Titu Andreescu, University of Texas at Dallas

J333. Consider an equiangular hexagon $ABCDEF$. Prove that

$$AC^2 + CE^2 + EA^2 = BD^2 + DF^2 + FB^2.$$

Proposed by Nairi Sedrakyan, Armenia

J334. Let ABC be a triangle with $\angle A \geq 60^\circ$, and let D and E be points on lines AB and AC, respectively. Prove that

$$\frac{BC}{\min\left(BD, DE, EC\right)} \geq \sqrt{5-4\cos A}.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J335. Prove that for any $a, b > -1$,

$$\max\left\{(a+3)(b^2+3), (a^2+3)(b+3)\right\} \geq 2(a+b+2)^{\frac{3}{2}}.$$

Proposed by Titu Andreescu, University of Texas at Dallas

J336. Let ABC be a right triangle with altitude AD, and let T be an arbitrary point on segment BD. Lines through T tangent to the circumcircle of triangle ADC intersect line AB at X, Y, respectively. Prove that $AX = BY$.

Proposed by Josef Tkadlec, Charles University, Czech Republic

Senior problems

S331. Find the minimum value of

$$E(x_1,\ldots,x_n)=x_1^2+\ldots+x_n^2+x_1x_2+\ldots x_1x_n+\ldots+x_{n-1}x_n+x_1+\ldots+x_n,$$

when $x_1,\ldots,x_n\in\mathbb{R}$.

Proposed by Dorin Andrica, Babeş-Bolyai University, Romania

S332. Prove that in any triangle with side lengths a, b, c and median lengths m_a, m_b, m_c,

$$4(m_a+m_b+m_c)\leq\sqrt{8a^2+(b+c)^2}+\sqrt{8b^2+(c+a)^2}+\sqrt{8c^2+(a+b)^2}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S333. Let $x>1$ and let $(a_n)_{n\geq 1}$ be the sequence defined by $a_n=[x^n]$ for every positive integer n. Prove that if $(a_n)_{n\geq 1}$ is a geometric progression, then x is an integer.

Proposed by Marius Cavachi, Constanta, Romania

S334. Let $a_0\geq 0$ and $a_{n+1}=a_0\cdot\ldots\cdot a_n+4$ for $n\geq 0$. Prove that

$$a_n-\sqrt[4]{(a_{n+1}+1)(a_n^2+1)-4}=1$$

for all $n\geq 1$.

Proposed by Titu Andreescu, University of Texas at Dallas

S335. Let ABC be a triangle. Let D be a point on the ray BA, which is not on the side AB, and let E be a point on the side AC, which is different from A And C. Let X be the reflection of D with respect to B and let Y be the reflection of E with respect to C. Suppose $4BC^2+DE^2=XY^2$. Prove that $BE\perp CD$ if and only if $\angle BAC=90^\circ$.

Proposed by İlker Can Çiçek, Instanbul, Turkey

S336. Let M be a point inside the triangle ABC and let K be a point symmetric to point M with respect to AC. Line BM intersects AC in point N and line BK intersects AC in point P. Prove that if $\angle AMP=\angle CMN$, then $\angle ABP=\angle CBN$.

Proposed by Nairi Sedrakyan, Armenia

Undergraduate problems

U331. Find all positive integers $a > b \geq 2$ such that

$$a^b - a = b^a - b.$$

Proposed by Mircea Becheanu, Bucharest, Romania

U332. Find $\inf_{(x,y)\in D}(x+1)(y+1)$, where $D = \{(x,y) \mid x, y \in \mathbb{R}_+, x \neq y, \text{and } x^y = y^x\}$.

Proposed by Arkady Alt, San Jose, USA

U333. Evaluate

$$\prod_{n\geq 0}\left(1 - \frac{2^{2^n}}{2^{2^{n+1}}+1}\right).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U334. Prove that if $x \in \mathbb{R}$ with $|x| \geq e$ then, $e^{|x|} > \left(\frac{e^2+x^2}{2e}\right)^e$, while the inequality is reversed if $|x| \leq e$.

Proposed by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

U335. Let p, $a_1, \dots, a_n$, $b_1, \dots, b_n$ be positive real numbers. Prove that

$$a_1\left(\frac{a_1}{b_1}\right)^p + \cdots + a_n\left(\frac{a_1+\cdots+a_n}{b_1+\cdots+b_n}\right)^p < \left(\frac{p+1}{p}\right)^p\left(\frac{a_1^{p+1}}{b_1^p} + \cdots + \frac{a_n^{p+1}}{b_n^p}\right).$$

Proposed by Nairi Sedrakyan, Armenia

U336. Find a closed form for the sum $E_n = \sum_{k=0}^{n}\binom{n}{2k+1}3^k$. Deduce the value of constant c such that $0 < \lim_{n\to\infty}\frac{E_n}{c^n} < \infty$, and the limit itself.

Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

Olympiad problems

O331. Let ABC be a triangle, let m_a, m_b, m_c be the lengths of its medians, and let p_0 be the semiperimeter of its orthic triangle. Prove that

$$\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\le\frac{3\sqrt{3}}{2p_0}.$$

Proposed by Mircea Lascu, Zalau, Romania

O332. Let ABC be a triangle. Prove that

$$\frac{\cos^4 A}{\sin^4 B+\sin^4 C}+\frac{\cos^4 B}{\sin^4 C+\sin^4 A}+\frac{\cos^4 C}{\sin^4 A+\sin^4 B}\ge\frac{1}{6}.$$

Proposed by An Zhen-ping, Xianyang Normal University, China

O333. Let ABC be a scalene acute triangle and denote by O, I, H its circumcenter, incenter, and orthocenter, respectively. Prove that if the circumcircle of triangle OIH passes through one of the vertices of triangle ABC then it also passes through one other vertex.

Proposed by Josef Tkadlec, Charles University, Czech Republic

O334. Let a, b, c be positive real numbers such that $a,b,c\ge 1$. Prove that

$$\frac{1}{(a^3+1)^2}+\frac{1}{(b^3+1)^2}+\frac{1}{(c^3+1)^2}\ge\frac{3}{2(a^2b^2c^2+1)}.$$

Proposed by İlker Can Çiçek, Instanbul, Turkey

O335. Determine all positive integers n such that $f_n(x,y,z)=x^{2n}+y^{2n}+z^{2n}-xy-yz-zx$ divides $g_n(x,y,z)=(x-y)^{5n}+(y-z)^{5n}+(z-x)^{5n}$, as polynomials in x, y, z with integer coefficients.

Proposed by Dorin Andrica, Babeş-Bolyai University, Romania

O336. Let a,b,c be positive distinct real numbers and let u,v,w be positive real numbers such that $a+b+c=u+v+w$ and

$$(a^2-bc)r+(b^2-ac)s+(c^2-ab)t\ge 0$$

for $(r,s,t)=(u,v,w)$, $(r,s,t)=(v,w,u)$, $(r,s,t)=(w,u,v)$. Prove that

$$x^ay^bz^c+x^by^cz^a+x^cy^az^b\ge x^uy^vz^w+x^wy^uz^v+x^vy^wz^u$$

for all nonnegative numbers x,y,z.

Proposed by Albert Stadler, Switzerland