Junior problems

J337. Prove that for each integer $n \geq 0$, $16^n + 8^n + 4^{n+1} + 2^{n+1} + 4$ is the product of two integers greater than 4^n.

Proposed by Titu Andreescu, University of Texas at Dallas

J338. Consider lattice points $P_1, P_2, \ldots, P_{n^2}$ with coordinates (u, v), where $1 \leq u, v \leq n$. For points $P_i = (u_i, v_i)$ and $P_j(u_j, v_j)$, define $d(P_iP_j) = |u_i - u_j| + |v_i - v_j|$. Evaluate

$$\sum_{1 \leq i < j \leq n^2} d(P_iP_j).$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J339. Solve in positive integers the equation

$$\frac{x-1}{y+1} + \frac{y-1}{z+1} + \frac{z-1}{x+1} = 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas

J340. Let ABC be a triangle with incircle ω and circumcircle Γ. Let I be the incenter of ABC, D the tangency point of ω with BC, M the midpoint of ID, and N the antipode of A with respect to Γ. Let A' be the tangency point of Γ with the circle tangent to rays AB, AC, and externally to Γ. Prove that lines $A'D$ and MN intersect on Γ.

Proposed by Marius Stănean, Zalău, Romania

J341. Let ABC be a triangle and let P be a point in its interior. Let $PA = x$, $PB = y$, $PC = z$. Prove that

$$\frac{(x+y+z)^9}{xyz} \geq 729a^2b^2c^2.$$

Proposed by Marcel Chiriță, Bucharest, Romania

J342. Solve in positive real numbers the system of equations

$$\frac{x^3}{4} + y^2 + \frac{1}{z} = \frac{y^3}{4} + z^2 + \frac{1}{x} = \frac{z^3}{4} + x^2 + \frac{1}{y} = 2.$$

Proposed by Titu Andreescu, University of Texas at Dallas

Senior problems

S337. Let a, b, c be positive real numbers such that $ab + bc + ca = 3$. Prove that

$$(a^3 + 1)(b^3 + 1)(c^3 + 1) \geq 8.$$

Proposed by Titu Andreescu, University of Texas at Dallas

S338. Let ABC be a triangle with $\angle A$ being the largest angle. Let D, E, F be points on sides BC, CA, AB, such that AD is the altitude from A, DE is the internal angle bisector of $\angle ADC$, and DF is the internal angle bisector of $\angle ADB$. If $AE = AF$, prove that ABC is isosceles or right.

Proposed by Titu Zvonaru and Neculai Stanciu, Romania

S339. Let p be a prime congruent to 2 mod 7. Solve in nonnegative integers the system of equations

$$7(x + y + z)(xy + yz + zx) = p(2p^2 - 1)$$

$$70xyz + 21(x - y)(y - z)(z - x) = 2p(p^2 - 4).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S340. Prove that in a triangle with sides a, b, c and semiperimeter s,

$$\sum_{cyc} \frac{a}{\sqrt{(s-b)(s-c)}} + \frac{5}{4} \sum_{cyc} \frac{(a-b)^2}{(s-a)(s-b)} \geq \frac{3R}{r}.$$

Proposed by Mircea Lascu and Titu Zvonaru, Romania

S341. Let a, b, c, d be nonnegative real numbers such that $a + b + c + d = 6$. Find the maximum value of $4a + 2ab + 2abc + abcd$.

Proposed by Marius Stănean, Zalău, Romania

S342. Let $a \geq b \geq c$ be positive real numbers. Prove that for all $t \in [0, \frac{\pi}{4}]$,

$$\frac{a-b}{a\sin t + b\cos t} + \frac{b-c}{b\sin t + c\cos t} + \frac{c-a}{c\sin t + a\cos t} \geq 0.$$

Proposed by Titu Andreescu, University of Texas at Dallas

Undergraduate problems

U337. Let n be a positive integer. Find all real polynomials f and g such that

$$(x^2+x+1)^n f(x^2-x+1) = (x^2-x+1)^n g(x^2+x+1),$$

for all real numbers x.

Proposed by Marcel Chiriță, Bucharest, Romania

U338. Determine the number of pairs (a, b) such that the equation $ax = b$ is solvable in the ring $(\mathbb{Z}_{2015}, +, \cdot)$.

Proposed by Dorin Andrica, Cluj-Napoca, Romania

U339. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that (i) f is continuous on $\mathbb{R}$ and (ii) $f(x) < f(x + \frac{1}{n})$ for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}^+$. Prove that f is increasing over $\mathbb{R}$.

Proposed by Mihai Piticari, Câmpulung Moldovenesc, Romania

U340. Define

$$A_n = \frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \cdots + \frac{n}{n^2+n^2}.$$

Find

$$\lim_{n\to\infty} n\left(n\left(\frac{\pi}{4} - A_n\right) - \frac{1}{4}\right).$$

Proposed by Yong Xi Wang, China

U341. Let $f : [0,1] \to \mathbb{R}$ be a non-decreasing function such that the sequence $\{I_n\}_{n\geq 1}$ defined by

$$I_n = \int_0^1 f(x)e^{nx}dx$$

is bounded. Prove that $f(x) = 0$ for all $x \in (0,1)$.

Proposed by Mihai Piticari, Câmpulung Moldovenesc, Romania

U342. Let A be an $m \times n$ matrix with real entries, and let $w_1, \ldots, w_n$ be nonnegative real numbers associated with the columns of A. For every $j = 1, \ldots, n$, we say that w_j represents the *weight* of column A_j. Describe a polynomial time algorithm for finding a subset of linearly independent columns of A whose sum of weights is maximum.

Proposed by Cosmin Pohoață, Columbia University, USA

Olympiad problems

O337. Does there exist an irreducible polynomial $P(x)$ with integer coefficients such that $P(n)$ is a perfect power greater than 1 for all $n \in \mathbb{N}$?

Proposed by Oleksiy Klurman, Université de Montréal, Canada

O338. Let P be a point inside equilateral triangle ABC. Denote by A_1, B_1, C_1 the projections of P onto triangle's sides and denote by A_2, B_2, C_2 the midpoints of PA_1, PB_1, PC_1, respectively. Prove that AA_1, BB_1, CC_1 are concurrent if an only if AA_2, BB_2, CC_2 are concurrent.

Proposed by Nairi Sedrakyan, Armenia

O339. Let a, b, c be positive real numbers satisfying

$$\frac{1}{a^3+b^3+1}+\frac{1}{b^3+c^3+1}+\frac{1}{c^3+a^3+1} \geq 1.$$

Prove that

$$(a+b)(b+c)(c+a) \leq 6+\frac{2}{3}(a^3+b^3+c^3).$$

Proposed by Nguyen Viet Hung, Hanoi, Vietnam

O340. Let ABC be a triangle and let BB' and CC' be the altitudes from B and C, respectively. Ray $C'B'$ intersects the circumcircle of triangle ABC at B''. Let α_A be the measure of angle ABB''. Similarly, define angles α_B and α_C. Prove that

$$\sin\alpha_A \sin\alpha_B \sin\alpha_C \leq \frac{3\sqrt{6}}{32}.$$

Proposed by Dorin Andrica, Cluj-Napoca, Romania

O341. Let a be a positive integer. Find all nonzero integer polynomials $P(X)$, $Q(X)$ such that

$$P(X)^2+aP(X)Q(X)+Q(X)^2=P(X).$$

Proposed by Mircea Becheanu, Bucharest, Romania

O342. Let $n \geq 2015$ be a positive integer and let $A \subset \{1,\dots,n\}$ such that $|A| \geq \left\lfloor \frac{n+1}{2} \right\rfloor$. Prove that A contains a three term arithmetic progression.

Proposed by Oleksiy Klurman, Université de Montréal, Canada