Junior problems

J337. Prove that for each integer $n \geq 0$, $16^n + 8^n + 4^{n+1} + 2^{n+1} + 4$ is the product of two numbers greater than 4^n.

Proposed by Titu Andreescu, University of Texas at Dallas

Solution by Moubinool Omarjee, Lycée Henri IV, Paris, France

We have

$$16^n + 8^n + 4^{n+1} + 2^{n+1} + 4 = (4^n + 2)^2 + 2^n(4^n + 2) =$$
$$= (4^n + 2)(4^n + 2^n + 2).$$

and the conclusion follows.

Also solved by Daniel Lasaosa, Pamplona, Spain; Kwon Il Kobe Ko, Cushing Academy, Ashburnham, MA, USA; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Albert Stadler, Herrliberg, Switzerland; Alessandro Ventullo, Milan, Italy; Apratim Dey, Indian Statistical Institute, Kolkata, India; Arber Avdullahu, Mehmet Akif College, Kosovo; Corneliu Mănescu-Avram, Technological Transportation High School, Ploieşti, Romania; Daniel López-Aguayo, Centro de Ciencias Matemáticas UNAM, Morelia, Mexico; David E. Manes, Oneonta, NY, USA; George Gavrilopoulos, Nea Makri High School, Athens, Greece; Haimoshri Das, South Point High School, India; Henry Ricardo, New York Math Circle; Joel Schlosberg Bayside, NY, USA; José Hernández Santiago,México; Nermin Hodzic, University of Tuzla at Tuzla, Bosnia and Herzegovina; Samriddhi Singha Roy, Kolkata, India; Neculai Stanciu and Titu Zvonaru, Romania; Polyahedra, Polk State College, FL, USA; Phillip Ahn, Concord Academy, Concord, MA, USA; Thomas Choi, Phillips Academy Andover, Andover, MA, USA; Michael Tang, Edina High School, MN, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Sonny An; S.Viswanathan, Center for Scientific Learning, Nagpur, Maharashtra, India; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Nick Iliopoulos, Music Junior HS, Trikala, Greece; Rade Krenkov, SOU Goce Delcev, Valandovo, Macedonia; Paul Revenant, Lycée Champollion, Grenoble, France.

J338. Consider lattice points $P_1, P_2, \ldots, P_{n^2}$ with coordinates (u, v), where $1 \le u, v \le n$. For points $P_i = (u_i, v_i)$ and $P_j(u_j, v_j)$, define $d(P_iP_j) = |u_i - u_j| + |v_i - v_j|$. Evaluate

$$\sum_{1 \le i < j \le n^2} d\left(P_iP_j\right).$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Note that each combination $u_i \neq u_j$ appears exactly n^2 times in the sum (as many as possible combinations of values v_i, v_j not necessarily distinct), whereas each combination $u_i = u_j$ appeats exactly $\binom{n}{2}$ times in the sum (as many as possible combinations of values $v_i \neq v_j$), for a total of $\binom{n}{2}n^2 + n\binom{n}{2} = \binom{n^2}{2}$ elements in the sum, as expected. Considering that the terms with $u_i = u_j$ do not contribute to the sum, we can readily see that a value u_i contributes with positive sign to the sum $n^2(u_i - 1)$ times (as many times as it appears together with any of the $u_j < u_i$), and with negative sign $n^2(n - u_i)$ times (as many times as it appears together with any of the $u_j > u_i$). Therefore, the net contribution to the sum of a given $k \in \{1, 2, \ldots, n\}$ is $kn^2(2k - n - 1)$, ie the total contribution to the sum of the $|u_i - u_j|$ is

$$\sum_{k=1}^{n} kn^2(2k - n - 1) = 2n^2 \sum_{k=1}^{n} k^2 - n^2(n+1) \sum_{k=1}^{n} k =$$

$$= 2n^2 \frac{n(n+1)(2n+1)}{6} - n^2(n+1)\frac{n(n+1)}{2} = \frac{n^3(n+1)(n-1)}{6}.$$

Since the contribution to the sum is the same for the $|v_i - v_j|$ by symmetry, we conclude that

$$\sum_{1 \le i < j \le n^2} d\left(P_iP_j\right) = \frac{n^3(n+1)(n-1)}{3}.$$

Also solved by Nermin Hodzic, University of Tuzla at Tuzla, Bosnia and Herzegovina; Kwon Il Kobe Ko, Cushing Academy, Ashburnham, MA, USA; Polyahedra, Polk State College, FL, USA; Phillip Ahn, Concord Academy, Concord, MA, USA; Thomas Choi, Phillips Academy Andover, Andover, MA, USA; Michael Tang, Edina High School, MN, USA; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Alberto Espuny Díaz, CFIS, Universitat Politècnica de Catalunya, Barcelona, Spain; Arber Avdullahu, Mehmet Akif College, Kosovo; Arkady Alt, San Jose, CA, USA; Haimoshri Das, South Point High School, India.

J339. Solve in positive integers the equation

$$\frac{x-1}{y+1}+\frac{y-1}{z+1}+\frac{z-1}{x+1}=1.$$

Proposed by Titu Andreescu, University of Texas at Dallas

Solution by Michael Tang, Edina High School, MN, USA

Make the substitutions $a=x+1$, $b=y+1$, $c=z+1$, so that $a,b,c\geq 2$. Then we have

$$\frac{a-2}{b}+\frac{b-2}{c}+\frac{c-2}{a}=1 \qquad (*)$$

or

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=1+2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

By the AM-GM inequality, $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3$, and equality holds iff $\frac{a}{b}=\frac{b}{c}=\frac{c}{a}$, or $a=b=c$. If $a=b=c$, then we get $3\cdot(a-2)/a=1$, so $a=b=c=3$, or $x=y=z=2$, which a solution to (*). Otherwise, the inequality is strict, so

$$1+2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=\frac{a}{b}+\frac{b}{c}+\frac{c}{a}>3 \implies \frac{1}{a}+\frac{1}{b}+\frac{1}{c}>1.$$

We determine all possible triples (a,b,c) with $a,b,c\geq 2$ that satisfy this inequality, and then check them against (*). Without loss of generality, let $a\leq b\leq c$, as our inequality is symmetric in a,b,c. If $a\geq 3$, then

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\leq\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=1$$

a contradiction, so $a=2$. Then we must have $\frac{1}{b}+\frac{1}{c}>\frac{1}{2}$. If $4\leq b\leq c$, then

$$\frac{1}{b}+\frac{1}{c}\leq\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$

a contradiction, so $b\leq 3$. If $b=2$, then $\frac{1}{c}>0$, so c can be any positive integer at least 2. Therefore, we have $(a,b,c)=(2,2,t)$ for some $t\geq 2$, as well as its permutations. If $(a,b,c)=(2,2,t)$, then (*) becomes $(t-2)/2=1$, so $t=4$, and $(a,b,c)=(2,2,4)\iff(x,y,z)=(1,1,3)$ satisfies the equation. Similarly, $(x,y,z)=(1,3,1)$ and $(x,y,z)=(3,1,1)$ satisfy the equation.

If $b=3$, then

$$\frac{1}{c}>\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$$

so $c\leq 5$, and the triples $(a,b,c)=(2,3,3),(2,3,4),(2,3,5)$ and their permutations satisfy our inequality. However, it turns out that none of these triples satisfy (*), so we find no new solutions.

Therefore, the solutions are $(x,y,z)=(2,2,2),(1,1,3),(1,3,1),(3,1,1)$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Kwon Il Kobe Ko, Cushing Academy, Ashburnham, MA, USA; Polyahedra, Polk State College, FL, USA; Phillip Ahn, Concord Academy, Concord, MA, USA; Thomas Choi, Phillips Academy Andover, Andover, MA, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Sonny An; S.Viswanathan, Center for Scientific Learning, Nagpur, Maharashtra, India; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Alessandro Ventullo, Milan, Italy; Arber Avdullahu, Mehmet Akif College, Kosovo; Haimoshri Das, South Point High School, India; José Hernández Santiago,México; Moubinool Omarjee Lycée Henri IV , Paris, France; Nick Iliopoulos, Music Junior HS, Trikala, Greece; Paul Revenant, Lycée Champollion, Grenoble, France; Prithwijit De, HBCSE, Mumbai, India; Samriddhi Singha Roy, Kolkata, India; Simantiris Ioannis, Moraitis School, Athens, Greece; Neculai Stanciu and Titu Zvonaru, Romania; Nermin Hodzic, University of Tuzla at Tuzla, Bosnia and Herzegovina; Rade Krenkov, SOU Goce Delcev, Valandovo, Macedonia.

J340. Let ABC be a triangle with incircle ω and circumcircle Γ. Let I be the incenter of ABC, D the tangency point of ω with BC, M the midpoint of ID, and N the antipode of A with respect to Γ. Let A' be the tangency point of Γ with the circle tangent to rays AB, AC, and externally to Γ. Prove that lines $A'D$ and MN intersect on Γ.

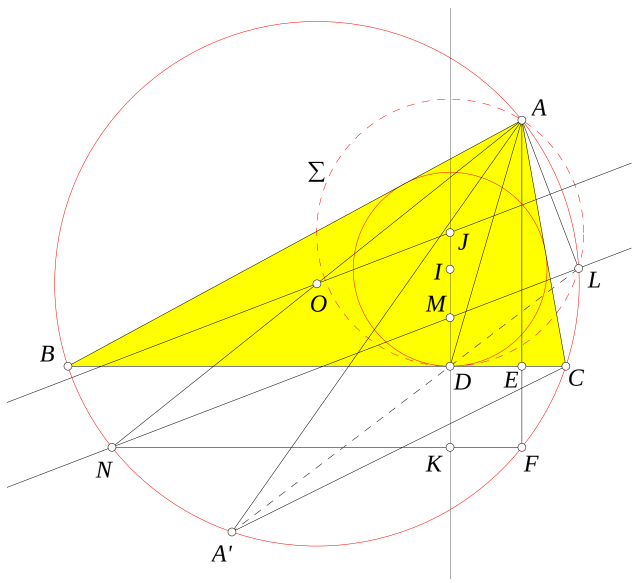
Proposed by Marius Stănean, Zalău, Romania

Solution by Polyahedra, Polk State College, FL, USA

Suppose that NM intersects Γ at L. We first prove a key lemma,

Lemma. The circumcircle Σ of $\triangle ADL$ is tangent to BC at D.

Proof. Let O be the center of Γ. Suppose that the line through O and parallel to NM intersects ID at J. The lemma follows from the claim that J is the center of Σ. Let E be the foot of the altitude on BC and $h = AE$. It suffices to show that $AD^2 = 2hJD$.



Referring to the figure, we see that $JD = MD + JM$ and $JM = \frac{AL \cdot NM}{2NK}$. Also,

$$AL = AN \sin(\angle ANF - \angle MNK) = AF \cdot \frac{NK}{NM} - NF \cdot \frac{MK}{NM}.$$

Hence

$$JM = \frac{1}{2}\left[AF - \frac{(NK + DE)MK}{NK}\right] = \frac{1}{2}\left(h - \frac{r}{2} - \frac{DE \cdot MK}{NK}\right), \quad JD = \frac{1}{2}\left(h + \frac{r}{2} - \frac{DE \cdot MK}{NK}\right),$$

and it suffices to show that

$$DE^2 = h\left(\frac{r}{2} - \frac{DE}{NK} \cdot MK\right).$$

Now

$$DE = s - c - b\cos C = \frac{a+b-c}{2} - \frac{a^2+b^2-c^2}{2a} = \frac{(c-b)(s-a)}{a},$$

$$\frac{1}{2}hr = \frac{\Delta^2}{as} = \frac{(s-a)(s-b)(s-c)}{a},$$

$$NK = NF - DE = 2R\sin(C-B) - DE = \frac{c^2+a^2-b^2}{2a} - \frac{a^2+b^2-c^2}{2a} - \frac{(c-b)(s-a)}{a} = \frac{(c-b)s}{a},$$

$$\begin{aligned} hMK &= h(MD+EF) = \frac{1}{2}hr + 2hR\cos B\cos C \\ &= \frac{(s-a)(s-b)(s-c)}{a} + \frac{(c^2+a^2-b^2)(a^2+b^2-c^2)}{4a^2}. \end{aligned}$$

Substituting in these expressions we finally get

$$\begin{aligned} \frac{1}{2}hr - \frac{DE}{NK}\cdot hMK &= (s-a)\left[\frac{(s-b)(s-c)}{s} - \frac{(c^2+a^2-b^2)(a^2+b^2-c^2)}{4a^2 s}\right] \\ &= \frac{(s-a)[a^2(a^2-b^2-c^2+2bc)-(a^4-b^4-c^4+2b^2c^2)]}{4a^2 s} \\ &= \frac{(c-b)^2(s-a)^2}{4a^2} = DE^2, \end{aligned}$$

establishing the lemma.

It is well known and easy to prove, by an inversion with center A and power $AB\cdot AC$, that AA' is the reflection of AD across AI. For details, see the solution to O306 in Issue **4** (2014). The lemma then implies that

$$\angle CDL = \angle LAD = \angle LAC + \angle CAD = \angle LA'C + \angle BAA' = \angle LA'C + \angle BCA',$$

which forces the collinearity of A', D, L.

Also solved by Daniel Lasaosa, Pamplona, Spain; George Gavrilopoulos, Nea Makri High School, Athens, Greece.

J341. Let ABC be a triangle and let P be a point in its interior. Let $PA = x$, $PB = y$, $PC = z$. Prove that

$$\frac{(x+y+z)^9}{xyz} \geq 729a^2b^2c^2.$$

Proposed by Marcel Chiriţă, Bucharest, Romania

Solution by Polyahedra, Polk State College, USA

Let $\angle BPC = 2\alpha$, $\angle CPA = 2\beta$, and $\angle APB = 2\gamma$. Without loss of generality, assume that $\alpha \leq 90°$. Then

$$\begin{aligned}\cos 2\alpha + \cos 2\beta + \cos 2\gamma &= 2\cos^2\alpha - 1 + 2\cos(\beta+\gamma)\cos(\beta-\gamma)\\ &= 2\cos^2\alpha - 1 - 2\cos\alpha\cos(\beta-\gamma)\\ &= 2\left(\cos\alpha - \frac{1}{2}\right)^2 + 2\left[1-\cos(\beta-\gamma)\right]\cos\alpha - \frac{3}{2} \geq -\frac{3}{2}.\end{aligned}$$

Now, by the law of cosines,

$$a^2 = y^2 + z^2 - 2yz\cos 2\alpha = (y+z)^2 - 2yz(1+\cos 2\alpha).$$

Hence,

$$\begin{aligned}xa^2 + yb^2 + zc^2 &= x(y+z)^2 + y(z+x)^2 + z(x+y)^2 - 2xyz(3+\cos 2\alpha+\cos 2\beta+\cos 2\gamma)\\ &\leq x(y+z)^2 + y(z+x)^2 + z(x+y)^2 - 3xyz\\ &= (x+y+z)(xy+yz+zx) \leq \frac{1}{3}(x+y+z)^3.\end{aligned}$$

Finally, by the AM-GM inequality,

$$3^6xa^2yb^2zc^2 \leq 3^3(xa^2+yb^2+zc^2)^3 \leq (x+y+z)^9,$$

completing the proof.

Also solved by Daniel Lasaosa, Pamplona, Spain; Nermin Hodzic, University of Tuzla at Tuzla, Bosnia and Herzegovina; Albert Stadler, Herrliberg, Switzerland; Arber Avdullahu, Mehmet Akif College, Kosovo; Michael Tang, Edina High School, MN, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

J342. Solve in positive real numbers the system of equations

$$\frac{x^3}{4}+y^2+\frac{1}{z}=\frac{y^3}{4}+z^2+\frac{1}{x}=\frac{z^3}{4}+x^2+\frac{1}{y}=2.$$

Proposed by Titu Andreescu, University of Texas at Dallas

Solution by Alessandro Ventullo, Milan, Italy

Adding the three equations, we get

$$\left(\frac{x^3}{4}+x^2+\frac{1}{x}-2\right)+\left(\frac{y^3}{4}+y^2+\frac{1}{y}-2\right)+\left(\frac{z^3}{4}+z^2+\frac{1}{z}-2\right)=0.$$

Let $f(t)=\dfrac{t^3}{4}+t^2+\dfrac{1}{t}-2$. We have

$$f(t)=\frac{t^4+4t^3-8t+4}{4t}=\frac{(t^2+2t-2)^2}{4t}\geq 0$$

for all $t\in\mathbb{R}^+$ and $f(t)=0$ if and only if $t^2+2t-2=0$, i.e. $t=\sqrt{3}-1$. It follows that $f(x)+f(y)+f(z)=0$ if and only if $x=y=z=\sqrt{3}-1$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Kwon Il Kobe Ko, Cushing Academy, Ashburnham, MA, USA; Polyahedra, Polk State College, FL, USA; Phillip Ahn, Concord Academy, Concord, MA, USA; Thomas Choi, Phillips Academy Andover, Andover, MA, USA; Arkady Alt, San Jose, CA, USA; George Gavrilopoulos, Nea Makri High School, Athens, Greece; Haimoshri Das, South Point High School, India; Nermin Hodzic, University of Tuzla at Tuzla, Bosnia and Herzegovina; Nick Iliopoulos, Music Junior HS, Trikala, Greece; Nicusor Zlota ,Traian Vuia Technical College, Focsani, Romania; Paul Revenant, Lycée Champollion, Grenoble, France; Neculai Stanciu and Titu Zvonaru, Romania; Rade Krenkov, SOU Goce Delcev, Valandovo, Macedonia.

Senior problems

S337. Let a, b, c be positive real numbers such that $ab + bc + ca = 3$. Prove that

$$\left(a^3+1\right)\left(b^3+1\right)\left(c^3+1\right) \geq 8.$$

Proposed by Titu Andreescu, University of Texas at Dallas

First solution by Adnan Ali, Student in A.E.C.S-4, Mumbai, India
Notice that from Hölder's Inequality,

$$(a^3+1+a^3+1)(b^3+b^3+1+1)(1+c^3+c^3+1) \geq (ab+bc+ca+1)^3 = 64,$$

from which implies the result.

Second solution by Adnan Ali, Student in A.E.C.S-4, Mumbai, India
Notice that from Hölder's and Cauchy-Schwarz Inequality,

$$(a^3+1+a^3+a^3+1+1)(b^3+b^3+1+1+b^3+1)(1+c^3+1+c^3+1+c^3) \geq (ab+bc+ca+a+b+c)^3$$
$$\geq \left(ab+bc+ca+\sqrt{3(ab+bc+ca)}\right)^3 = 6^3,$$

Also solved by Russelle Guadalupe, Institute of Mathematics, College of Science, University of the Philippines-Diliman, Quezon City, Philippines; Daniel Lasaosa, Pamplona, Spain; Kwon Il Kobe Ko, Cushing Academy, Ashburnham, MA, USA; Phillip Ahn, Concord Academy, Concord, MA, USA; Thomas Choi, Phillips Academy Andover, Andover, MA, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Sonny An; S.Viswanathan, Center for Scientific Learning, Nagpur, Maharashtra, India; Albert Stadler, Herrliberg, Switzerland; Arber Avdullahu, Mehmet Akif College, Kosovo; Arkady Alt, San Jose, CA, USA; George Gavrilopoulos, Nea Makri High School, Athens, Greece; Nermin Hodzic, University of Tuzla at Tuzla, Bosnia and Herzegovina; Nicusor Zlota ,Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti , Università degli studi di Tor Vergata Roma, Roma, Italy; Paul Revenant, Lycée Champollion, Grenoble, France; Neculai Stanciu and Titu Zvonaru, Romania; Li Zhou, Polk State College, Winter Haven, FL, USA.

S338. Let ABC be a triangle with $\angle A$ being the largest angle. Let D, E, F be points on sides BC, CA, AB, such that AD is the altitude from A, DE is the internal angle bisector of $\angle ADC$, and DF is the internal angle bisector of $\angle ADB$. If $AE = AF$, prove that ABC is isosceles or right.

Proposed by Titu Zvonaru and Neculai Stanciu, Romania

Solution by Li Zhou, Polk State College, Winter Haven, FL, USA

Since DE and DF are angle bisectors of $\angle ADC$ and $\angle ADB$, we have $AE = \frac{AC \cdot AD}{AD+DC}$ and $AF = \frac{AB \cdot AD}{AD+BD}$. Hence the condition $AE = AF$ yields $AB(AD + DC) = AC(AD + BD)$. Squaring both sides and applying the Pythagorean theorem, we get

$$AB^2(AC^2 + 2AD \cdot DC) = AC^2(AB^2 + 2AD \cdot BD).$$

Thus $(AD^2 + BD^2)DC = (AD^2 + DC^2)BD$, which is equivalent to $(BD - DC)(BD \cdot DC - AD^2) = 0$. If $BD - DC = 0$, then $AB = AC$; if $BD \cdot DC - AD^2 = 0$, then $\angle A = 90°$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Kwon Il Kobe Ko, Cushing Academy, Ashburnham, MA, USA; Phillip Ahn, Concord Academy, Concord, MA, USA; Thomas Choi, Phillips Academy Andover, Andover, MA, USA; Michael Tang, Edina High School, MN, USA; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Andrea Fanchini, Cantù, Italy; Corneliu Mănescu-Avram, Technological Transportation High School, Ploieşti, Romania; Haimoshri Das, South Point High School, India; Nermin Hodzic, University of Tuzla at Tuzla, Bosnia and Herzegovina; Evgenidis Nikolaos, M. N. Raptou High School, Larissa, Greece; Paul Revenant, Lycée Champollion, Grenoble, France; Prithwijit De, HBCSE, Mumbai, India; Nick Iliopoulos, Music Junior HS, Trikala, Greece; Rade Krenkov, SOU Goce Delcev, Valandovo, Macedonia.

S339. Let p be a prime congruent to 2 mod 7. Solve in nonnegative integers the system of equations

$$7(x+y+z)(xy+yz+zx) = p\left(2p^2-1\right),$$

$$70xyz + 21(x-y)(y-z)(z-x) = 2p\left(p^2-4\right).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Note first that

$$(2x-y)(2y-z)(2z-x) = 2xy^2+2zx^2+2yz^2-4y^2z-4x^2y-4z^2x+7xyz =$$

$$= 10xyz + 3\left(xy^2+zx^2+yz^2-y^2z-x^2y-z^2x\right) -$$

$$-\left(xy^2+yz^2+zx^2+x^2y+y^2z+z^2x+3xyz\right) =$$

$$= 10xyz+3(x-y)(y-z)(z-x)-(x+y+z)(xy+yz+zx) =$$

$$= \frac{2p\left(p^2-4\right)-p\left(2p^2-1\right)}{7} = -p,$$

and since p is prime, it follows that one of $2x-y, 2y-z, 2z-x$ is either p or $-p$, and the other two are either 1 or -1. Now, note that the first equation is invariant under exchange of any two variables, and the second equation is invariant under cyclic permutation of the variables, or we may choose wlog $|2x-y|=p$ and $|2y-z|=|2z-x|=1$. Now, note that $(2x-y)+(2y-z)+(2z-x)=x+y+z$, or if $2x-y=-p$, then $2y-z=2z-x=\pm1$, and $x+y+z=\pm2-p\leq 0$, and since x,y,z are non-negative, we would have $x=y=z=0$, absurd. It follows that $2x-y=p$, and $2y-z=-(2z-x)=\pm1$, yielding two possible cases.

Case 1: $2y-z=1$ and consequently $2z-x=-1$, or $z=\frac{x-1}{2}$ and $y=\frac{x+1}{4}$. It follows that $x\equiv 3 \pmod 4$ for y,z to be integers, or some integer k exists such that $x=4k+3$, $y=k+1$, $z=2k+1$. But then $p=x+y+z=7k+5\equiv 5 \pmod 7$, in contradiction with the problem statement. No solution exists is this case.

Case 2: $2y-z=-1$ and consequently $2z-x=1$, or $z=\frac{x+1}{2}$ and $y=\frac{x-1}{4}$. It follows that $x\equiv 1 \pmod 4$ for y,z to be integers,or some integer k exists such that $x=4k+1$, $y=k$, $z=2k+1$, yielding indeed $p=x+y+z=7k+2\equiv 2 \pmod 7$, as desired. Direct substitution shows that this form for x,y,z is indeed a solution for any non-negative integer k, including $k=0$ which corresponds to $p=2$.

Restoring generality, we conclude that

$$(x,y,z) = (4k+1,k,2k+1),(k,2k+1,4k+1),(2k+1,4k+1,k),$$

where k is the non-negative integer integer such that $p=7k+2$, are all solutions, and there can be no others when p is a prime of this form.

Also solved by Adnan Ali, Student in A.E.C.S-4, Mumbai, India; David E. Manes, Oneonta, NY, USA; Nermin Hodzic, University of Tuzla at Tuzla, Bosnia and Herzegovina; Paul Revenant, Lycée Champollion, Grenoble, France.

S340. Prove that in a triangle with sides a, b, c and semiperimeter s,

$$\sum_{\text{cyc}} \frac{a}{\sqrt{(s-b)(s-c)}} + \frac{5}{4} \sum_{\text{cyc}} \frac{(a-b)^2}{(s-a)(s-b)} \geq \frac{3R}{r}.$$

Proposed by Mircea Lascu and Titu Zvonaru, Romania

Solution by Adnan Ali, Student in A.E.C.S-4, Mumbai, India
Using the formulae,

$$R = \frac{abc}{4\sqrt{(s(s-a)(s-b)(s-c)}}, \qquad r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

we have

$$\frac{3R}{r} = \frac{3abc}{4(s-a)(s-b)(s-c)}.$$

The idea is to get rid of denominators by making them equal on both the sides:

$$\sum_{cyc} \frac{a}{\sqrt{(s-b)(s-c)}} + \frac{5}{4} \sum_{cyc} \frac{(a-b)^2}{(s-a)(s-b)} = \sum_{cyc} \frac{4a(s-a)\sqrt{(s-b)(s-c)}}{4(s-a)(s-b)(s-c)} + \sum_{cyc} \frac{5(a-b)^2(s-c)}{4(s-a)(s-b)(s-c)}.$$

Thus we need to show that

$$\sum_{cyc} 4a(s-a)\sqrt{(s-b)(s-c)} + \sum_{cyc} 5(a-b)^2(s-c) \geq 3abc.$$

Using Ravi's Substitution, $a = x+y, b = y+z, c = z+x$, for positive real numbers x, y, z we get an equivalent inequality to prove that

$$\sum_{cyc} 4z(x+y)\sqrt{xy} + \sum_{cyc} 5y(z-x)^2 \geq 3(x+y)(y+z)(z+x).$$

From the AM-GM Inequality, we get

$$\sum_{cyc} 4z(x+y)\sqrt{xy} \geq \sum_{cyc} 4z \cdot 2\sqrt{xy} \cdot \sqrt{xy} = \sum_{cyc} 8xyz = 24xyz.$$

Also,

$$\sum_{cyc} 5y(z-x)^2 = 5\left(\sum_{cyc} xy(x+y)\right) - 30xyz.$$

Thus we need to prove that

$$5\left(\sum_{cyc} xy(x+y)\right) - 6xyz \geq 3\left(\sum_{cyc} xy(x+y)\right) + 6xyz \Leftrightarrow \sum_{cyc} xy(x+y) \geq 6xyz,$$

which follows easily from the AM-GM Inequality. Equality holds iff $a = b = c$.

Also solved by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Daniel Lasaosa, Pamplona, Spain; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Li Zhou, Polk State College, Winter Haven, FL, USA; Arkady Alt, San Jose, CA, USA; Nermin Hodzic, University of Tuzla at Tuzla, Bosnia and Herzegovina; Nicusor Zlota ,Traian Vuia Technical College, Focsani, Romania; Rao Yiyi, Wuhan, China.

S341. Let a, b, c, d be nonnegative real numbers such that $a+b+c+d=6$. Find the maximum value of $4a+2ab+2abc+abcd$.

Proposed by Marius Stănean, Zalău, Romania

Solution by Paolo Perfetti , Università degli studi di Tor Vergata Roma, Roma, Italy
Lagrange multipliers. For $(a,b,c,d)\neq(0,0,0,0)$ let's define the function

$$F(a,b,c,d,\lambda)=f(a,b,c,d)-\lambda(a+b+c+d-6)=4a+2ab+2abc+abcd-\lambda(a+b+c+d-6)$$

$$\begin{aligned}
F_a=0 &\iff 4+2b+2bc+bcd=\lambda \iff 4a+2ab+2abc+abcd=\lambda a\\
F_b=0 &\iff 2a+2ac+acd=\lambda \iff 2ab+2abc+abcd=\lambda b\\
F_c=0 &\iff 2ab+abd=\lambda \iff 2abc+abcd=\lambda c\\
F_d=0 &\iff abc=\lambda
\end{aligned}$$

which is

$$\begin{aligned}
F_a=0 &\iff 4a+2ab+2\lambda+\lambda d=\lambda a\\
F_b=0 &\iff 2ab+2\lambda+\lambda d=\lambda b\\
F_c=0 &\iff 2\lambda+\lambda d=\lambda c\\
F_d=0 &\iff abc=\lambda
\end{aligned}$$

By summing the first three we get

$$4a+4ab+6\lambda+3\lambda d=6\lambda-\lambda d \iff 4a+4ab=-5\lambda d$$

which implies $\lambda<0$ but this conflicts with $abc=\lambda$.

This means that in the open set $(0,6)^4$ there is no constrained critical point of $f(a,b,c,d)$ and then the maximum of $f(a,b,c,d)$ as well as its minimum of the set $[0,6]^4$, belongs to the boundary.

If $a=0$ we have $f(0,b,c,d)\equiv 0$.

If $b=0$ we have $f(a,0,c,d)=4a\leq 24$.
If $c=0$ we have $f(a,b,0,d)=4a+2ab$ and $a+b\leq 6$. It follows $4a+2ab\leq 4a+2a(6-a)$, $0\leq a\leq 6$. Easy computations show that the maximum occurs for $a=6$ and the value is 24.

If $d=0$ we have $f(a,b,c,0)=4a+2ab+2abc$, $a+b+c=6$. Clearly

$$f(a,b,c,0)\doteq g(a,b)\le 4a+2ab+2ab(6-a-b)=4a+14ab-2a^2b-2ab^2, \qquad 0\le a+b\le 6$$

To find the maximum of $g(a,b)$ we compute the gradient

$$g_a=0=4+14b-4ab-2b^2=0, \qquad g_b=a(14-2a-4b)=0$$

The second yields $a=7-2b$ which inserted into the first equation gives $6b^2-14b+4=0$ and then

$$(a,b)=(\frac{19}{3},\frac{1}{3})\doteq P_1, (a,b)=(3,2)\doteq P_2.$$

The determinant

$$g_{a,a}g_{b,b}-(g_{a,b})^2$$

is positive at P_1 and negative at P_2. Since $g_{a,a}(3,2)=-8$, it follows that P_1 is a point of local maximum while P_2 is a saddle.

To prove that P_1 is a point of absolute maximum in the set $0\le a+b\le 6$, we need to compute the maximum of $g(a,b)$ on the boundary. The boundary is made of three intervals. The first one is $a=0$, $0\le b\le 6$. In such a case $g(0,b)\equiv 0$.

The second one is $0\le a\le 6$ and $b=0$. In such a case $g(a,0)=4a\le 24$. Finally if $a+b=6$, we have $g(a,6-a)=16a-a^2\le 32=g(4,2)$.

It follows that the maximum is

$$f(3,2,1,0)=36.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Paul Revenant, Lycée Champollion, Grenoble, France; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; S.Viswanathan, Center for Scientific Learning, Nagpur, Maharashtra, India; Li Zhou, Polk State College, Winter Haven, FL, USA; Rao Yiyi, Wuhan,China; Adnan Ali, Student in A.E.C.S-4, Mumbai, India.

S342. Let $a \geq b \geq c$ be positive real numbers. Prove that for all $t \in \left[0, \frac{\pi}{4}\right]$,

$$\frac{a-b}{a\sin t + b\cos t} + \frac{b-c}{b\sin t + c\cos t} + \frac{c-a}{c\sin t + a\cos t} \geq 0.$$

Proposed by Titu Andreescu, University of Texas at Dallas

Solution by Daniel Lasaosa, Pamplona, Spain

Multiplying throughout by the product of the (clearly positive) denominators and rearranging terms, the proposed inequality is equivalent to

$$(a^2b + b^2c + c^2a - 3abc)\sin t(\cos t - \sin t) +$$

$$+(ab^2 + bc^2 + ca^2 - 3abc)\cos t(\cos t - \sin t) +$$

$$+(a-b)(b-c)(a-c)\sin t \cos t \geq 0.$$

Since $t \in \left[0, \frac{\pi}{4}\right]$, we have $\cos t - \sin t \geq 0$, with equality iff $t = \frac{\pi}{4}$, whereas $\sin t \geq 0$ with equality iff $t = 0$, and $\cos t > 0$. Moreover, by the AM-GM inequality we have $a^2b+b^2c+c^2a-3abc \geq 0$ and $ab^2+bc^2+ca^2-3abc \geq 0$, with equality in either case iff $a = b = c$. Clearly $a-b, b-c, a-c$ are non-negative, their product being zero iff at least two out of a, b, c are equal. The conclusion follows. Equality holds for all $t \in \left[0, \frac{\pi}{4}\right]$ iff $a = b = c$, or whenever a, b, c are not all equal, iff $t = \frac{\pi}{4}$ (otherwise the second term cannot be zero) and simultaneously at least two out of a, b, c are equal (otherwise the third term cannot be simultaneously zero).

Also solved by Ioan Viorel Codreanu, Satulung, Maramures, Romania; Li Zhou, Polk State College, Winter Haven, FL, USA; Rao Yiyi, Wuhan,China; Arkady Alt, San Jose, CA, USA; Nermin Hodzic, University of Tuzla at Tuzla, Bosnia and Herzegovina; Nicusor Zlota ,Traian Vuia Technical College, Focsani, Romania; Neculai Stanciu and Titu Zvonaru, Romania.

Undergraduate problems

U337. Let n be a positive integer. Find all real polynomials f and g such that

$$(x^2+x+1)^n f(x^2-x+1) = (x^2-x+1)^n g(x^2+x+1),$$

for all real numbers x.

Proposed by Marcel Chiriţă, Bucharest, Romania

Solution by Alessandro Ventullo, Milan, Italy

Observe that $x^2-x+1=(1-x)^2-(1-x)+1$. Hence, by substitution $x\mapsto 1-x$, we get

$$(x^2-3x+3)^n f(x^2-x+1) = (x^2-x+1)^n g(x^2-3x+3)$$

and dividing by the original equation, we have

$$\frac{g(x^2-3x+3)}{g(x^2+x+1)} = \left(\frac{x^2-3x+3}{x^2+x+1}\right)^n. \tag{1}$$

Moreover, since $x^2+x+1=(-1-x)^2-(-1-x)+1$, by substitution $x\mapsto -1-x$, we get

$$(x^2+x+1)^n f(x^2+3x+3) = (x^2+3x+3)^n g(x^2+x+1)$$

and dividing by the original equation, we have

$$\frac{f(x^2+3x+3)}{f(x^2-x+1)} = \left(\frac{x^2+3x+3}{x^2-x+1}\right)^n.$$

By substitution $x\mapsto -x$, we obtain

$$\frac{f(x^2-3x+3)}{f(x^2+x+1)} = \left(\frac{x^2-3x+3}{x^2+x+1}\right)^n. \tag{2}$$

Comparing (1) and (2), we obtain

$$\frac{f(x^2-3x+3)}{g(x^2-3x+3)} = \frac{f(x^2+x+1)}{g(x^2+x+1)}$$

for all $x\in\mathbb{R}$. It follows that $\dfrac{f(x^2-3x+3)}{g(x^2-3x+3)}$ and $\dfrac{f(x^2+x+1)}{g(x^2+x+1)}$ have the same derivative for all $x\in\mathbb{R}$. So,

$$\begin{aligned}(2x-3)\frac{f'(x^2-3x+3)g(x^2-3x+3)-f(x^2-3x+3)g'(x^2-3x+3)}{(g(x^2-3x+3))^2} &= \\ = \quad (2x+1)\frac{f'(x^2+x+1)g(x^2+x+1)-f(x^2+x+1)g'(x^2+x+1)}{(g(x^2+x+1))^2}&.\end{aligned}$$

for all $x\in\mathbb{R}$. Since $(1-x)^2+(1-x)+1=x^2-3x+3$, by substition $x\mapsto 1-x$ into the right-hand side, we have

$$(2x-3)\frac{f'(x^2-3x+3)g(x^2-3x+3)-f(x^2-3x+3)g'(x^2-3x+3)}{g^2(x^2-3x+3)} = 0$$

for all $x\in\mathbb{R}$. It follows that $\dfrac{f(x^2-3x+3)}{g(x^2-3x+3)}$ is constant on $\mathbb{R}$, i.e. $f(x)=\lambda g(x)$ for some $\lambda\in\mathbb{R}$. Substituting into the original equation, we get $\lambda=1$. Now, from equation (2), we get

$$\frac{f(x^2-3x+3)}{(x^2-3x+3)^n} = \frac{f(x^2+x+1)}{(x^2+x+1)^n}$$

for all $x \in \mathbb{R}$. It follows that $\dfrac{f(x^2-3x+3)}{(x^2-3x+3)^n}$ and $\dfrac{f(x^2+x+1)}{(x^2+x+1)^n}$ have the same derivative for all $x \in \mathbb{R}$. So,

$$\begin{aligned} &(2x-3)\frac{f'(x^2-3x+3)(x^2-3x+3)-nf(x^2-3x+3)}{(x^2-3x+3)^{n+1}} = \\ = \quad &(2x+1)\frac{f'(x^2+x+1)(x^2+x+1)-nf(x^2+x+1)}{(x^2+x+1)^{n+1}}. \end{aligned}$$

for all $x \in \mathbb{R}$. By substitution $x \mapsto 1-x$ into the right-hand side, we get once again that

$$\frac{d}{dx}\frac{f(x^2-3x+3)}{(x^2-3x+3)^n} = 0$$

for all $x \in \mathbb{R}$, so $f(x) = cx^n$ for some $c \in \mathbb{R}$. It follows that $f(x) = g(x) = cx^n$, where $c \in \mathbb{R}$.

Also solved by Daniel Lasaosa, Pamplona, Spain; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Moubinool Omarjee, Lycée Henri IV, Paris, France.

U338. Determine the number of pairs (a, b) such that the equation $ax = b$ is solvable in the ring $(\mathbb{Z}_{2015}, +, \cdot)$.

Proposed by Dorin Andrica, Cluj-Napoca, Romania

Solution by Joel Schlosberg Bayside, NY, USA

For $a \in \mathbb{Z}_{2015}$, let $d_a = \gcd(a, 2015)$ and $d'_a = \frac{2015}{d_a}$. Since 2015 is squarefree, d_a and d'_a are relatively prime. Suppose some prime p divides both a/d_a and d'_a. Then $pd_a|a$ and $pd_a|d'_a d_a = 2015$, making pd_a a common divisor of a and 2015; yet $pd_a > d_a = \gcd(a, 2015)$, a contradiction. Thus, a/d_a and d'_a are relatively prime.

If $ax = b$ is solvable in $\mathbb{Z}_{2015}$, then $b \equiv ax \equiv 0 \pmod{d_a}$. Conversely, if $d_a|b$, let $(a/d_a)^{-1}$ be the multiplicative inverse of a/d_a modulo d'_a (which exists by relative primality) and $x = (a/d_a)^{-1}(b/d_a)$. Then

$$(a/d_a)x \equiv (b/d_a) \pmod{d'_a}$$

$$ax \equiv b \pmod{d'_a d_a = 2015},$$

solving $ax = b$ in $\mathbb{Z}_{2015}$.

Therefore, for a given a, the number of b such that $ax = b$ is solvable in $\mathbb{Z}_{2015}$ is the number of b in $\mathbb{Z}_{2015}$ divisible by d_a, which is $2015/d_a$.

For $d|2015$, $a \in \mathbb{Z}_{2015}$ has $d_a = d$ iff $d|a$ and a/d is relatively prime to $2015/d$. Such a are of the form $a = da'$ where a' is invertible in $Z_{2015/d}$. By the definition of Euler's totient function ϕ, there are $\phi(2015/d)$ such elements a' in $Z_{2015/d}$, and since each distinct a' corresponds to a distinct a, exactly as many $a \in \mathbb{Z}_{2015}$ with $d_a = d$.

Therefore, the total number of pairs (a, b) is

$$\sum_{d|2015} \phi(2015/d)\frac{2015}{d} = \sum_{d'|2015} \phi(d')d'.$$

Since the prime factorization of 2015 is $5 \cdot 13 \cdot 31$, and since ϕ is multiplicative, this becomes

$$1\phi(1) + 5\phi(5) + 13\phi(13) + 31\phi(31) + 5 \cdot 13\phi(5 \cdot 13) + 5 \cdot 31\phi(5 \cdot 31)$$

$$+13 \cdot 31\phi(13 \cdot 31) + 5 \cdot 13 \cdot 31\phi(5 \cdot 13 \cdot 31)$$

$$= (1 + 5\phi(5))(1 + 13\phi(13))(1 + 31\phi(31))$$

$$= (1 + 5(5 - 1))(1 + 13(13 - 1))(1 + 31(31 - 1))$$

$$= 3069507.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Alessandro Ventullo, Milan, Italy.

U339. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that (i) f is continuous on $\mathbb{Q}$ and (ii) $f(x) < f\left(x + \frac{1}{n}\right)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}^+$. Prove that f is increasing over $\mathbb{R}$.

Proposed by Mihai Piticari, Câmpulung Moldovenesc, Romania

Solution by Joel Schlosberg Bayside, NY, USA

Consider any $x, y \in \mathbb{R}$ with $x < y$. For $m \in \mathbb{Z}^+$,

$$f(x) < f\left(x + \tfrac{1}{n}\right) < f\left(x + \tfrac{2}{n}\right) < \cdots < f\left(x + \tfrac{m}{n}\right),$$

so $f(x+q) \in (f(x), \infty)$ for all $q \in \mathbb{Q}^+$.

Since f is continuous, the preimage of the open set $(f(x), \infty)$ under f is open. Since the preimage contains $\{x+q | q \in \mathbb{Q}^+\}$, which is a dense subset of (x, ∞), the preimage contains $(x, \infty) \ni y$, so $f(y) > f(x)$. Therefore, f is increasing over $\mathbb{R}$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy.

U340. Define

$$A_n = \frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \cdots + \frac{n}{n^2+n^2}.$$

Find

$$\lim_{n\to\infty} n\left(n\left(\frac{\pi}{4} - A_n\right) - \frac{1}{4}\right).$$

Proposed by Yong Xi Wang, China

Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA

Let f be three-times differentiable on the interval $[0,1]$, and let $[a,b] \subset [0,1]$. By Taylor's Theorem,

$$f(x) = f(b) + f'(b)(x-b) + \frac{1}{2}f''(b)(x-b)^2 + \frac{1}{6}f'''(\xi)(x-b)^3,$$

for some ξ between x and b. It follows that

$$\int_a^b f(x)\,dx = f(b)(b-a) - \frac{1}{2}f'(b)(b-a)^2 + \frac{1}{6}f''(b)(b-a)^3 + O((b-a)^4).$$

Now, let n be a positive integer. Then, for $k = 0, 1, 2, \ldots, n-1$,

$$\int_{(k-1)/n}^{k/n} f(x)\,dx = \frac{1}{n}f\left(\frac{k}{n}\right) - \frac{1}{2n^2}f'\left(\frac{k}{n}\right) + \frac{1}{6n^3}f''\left(\frac{k}{n}\right) + O\left(\frac{1}{n^4}\right).$$

Summing over k then yields

$$\int_0^1 f(x)\,dx = \frac{1}{n}\sum_{k=1}^n f\left(\frac{k}{n}\right) - \frac{1}{2n^2}\sum_{k=1}^n f'\left(\frac{k}{n}\right) + \frac{1}{6n^3}\sum_{k=1}^n f''\left(\frac{k}{n}\right) + O\left(\frac{1}{n^3}\right). \quad (3)$$

Similarly

$$f(1) - f(0) = \int_0^1 f'(x)\,dx = \frac{1}{n}\sum_{k=1}^n f'\left(\frac{k}{n}\right) - \frac{1}{2n^2}\sum_{k=1}^n f''\left(\frac{k}{n}\right) + O\left(\frac{1}{n^2}\right) \quad (4)$$

and

$$f'(1) - f'(0) = \int_0^1 f''(x)\,dx = \frac{1}{n}\sum_{k=1}^n f''\left(\frac{k}{n}\right) + O\left(\frac{1}{n}\right). \quad (5)$$

Combining (1), (2), and (3) then yields

$$\int_0^1 f(x)\,dx = \frac{1}{n}\sum_{k=1}^n f\left(\frac{k}{n}\right) - \frac{1}{2n}[f(1)-f(0)] - \frac{1}{12n^2}[f'(1)-f'(0)] + O\left(\frac{1}{n^3}\right).$$

Now, let

$$f(x) = \frac{1}{1+x^2}.$$

Then

$$\begin{aligned}
\int_0^1 f(x)\,dx &= \frac{\pi}{4};\\
\frac{1}{n}\sum_{k=1}^n f\left(\frac{k}{n}\right) &= \frac{1}{n}\sum_{k=1}^n \frac{1}{1+(k/n)^2} = \sum_{k=1}^n \frac{n}{n^2+k^2} = A_n;\\
f(1) - f(0) &= \frac{1}{2} - 1 = -\frac{1}{2};\\
f'(1) - f'(0) &= -\frac{1}{2} - 0 = -\frac{1}{2};
\end{aligned}$$

so that

$$\frac{\pi}{4} = A_n + \frac{1}{4n} + \frac{1}{24n^2} + O\left(\frac{1}{n^3}\right).$$

Finally,

$$\lim_{n\to\infty} n\left(n\left(\frac{\pi}{4} - A_n\right) - \frac{1}{4}\right) = \frac{1}{24}.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Alessandro Ventullo, Milan, Italy; Moubinool Omarjee, Lycée Henri IV, Paris, France; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Li Zhou, Polk State College, Winter Haven, FL, USA.

U341. Let $f:[0,1]\to\mathbb{R}$ be a non-decreasing function such that the sequence $\{I_n\}_{n\geq 1}$ defined by

$$I_n=\int_0^1 f(x)e^{nx}dx$$

is bounded. Prove that $f(x)=0$ for all $x\in(0,1)$.

Proposed by Mihai Piticari, Câmpulung Moldovenesc, Romania

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

$$\begin{aligned}&I_n=\int_0^1 f(x)e^{nx}dx\geq f(0)\cdot\int_0^1 e^{nx}dx=f(0)\cdot\frac{e^n-1}{n}\text{ and } \ I_n \text{ is bounded}\\&\quad\Rightarrow\ f(0)=0\ \Rightarrow\ f(x)\geq 0,\ \forall x\in[0,1].\end{aligned}$$

Assume by contradiction there exists $\exists x_0\in(0,1):f(x_0)>0$. Then we have

$$\begin{aligned}I_n&=\int_0^{x_0} f(x)e^{nx}dx+\int_{x_0}^1 f(x)e^{nx}dx\\&\geq\int_{x_0}^1 f(x)e^{nx}dx\geq f(x_0)\int_{x_0}^1 e^{nx}dx=f(x_0)\cdot\frac{e^n-e^{nx_0}}{n}\to\infty.\end{aligned}$$

hence we have a contradiction.

Also solved by Daniel Lasaosa, Pamplona, Spain; Paolo Perfetti , Università degli studi di Tor Vergata Roma, Roma, Italy.

U342. Let A be an $m \times n$ matrix with real entries, and let $w_1, \ldots, w_n$ be a nonnegative real numbers associated with the columns of A. For every $j = 1, \ldots, n$ we say that w_j represents the *weight* of column A_j. Describe a polynomial time algorithm for finding a subset of linearly independent columns of A whose sum of weights is maximum.

Proposed by Cosmin Pohoata, Columbia University, USA

Solution by the author

The greedy approach works (and gives polynomial time).

Relabel the weights $w_1, \ldots, w_n$ such that $w_1 \geq \ldots \geq w_n$ - this step requires a sorting of the nonnegative reals $w_1, \ldots, w_n$ and so it can be easily done in polynomial time.

Let the set of columns of A be $\mathcal{C}$. Initialize the subset of columns $S = \emptyset$. At iteration i of the algorithm, choose a column C from $\mathcal{C} - S$ of maximal weight that is linearly independent of the columns from S and update S to $S \cup \{C\}$. When an iteration finds no such column, stop and output S.

It is easy to check that the algorithm works and that it is of polynomial time (provided the fact that checking linear independence can be done in polynomial time).

O337. Does there exist an irreducible polynomial $P(x)$ with integer coefficients such that $P(n)$ is a perfect power greater than 1 for all $n \in \mathbb{N}$?

Proposed by Oleksiy Klurman, Université de Montréal, Canada

Solution by the author

The answer is "No". First observe that since $P(x)$ is irreducible then $gcd(P(n), P'(n)) \leq M$ for some $M > 0$ and all n.

This implies that there exist infinitely many primes p, such that $p|P(n)$ and p does not divide $P'(n)$ for some $n \in \mathbb{Z}$. The result then follows from the following lemma.

Lemma. Suppose $p^k \mid\mid P(n_0)$ and $p \nmid P'(n_0)$. Then, there exist n_p such that $p \mid\mid P(n_p)$.

Proof. We proceed in the spirit of Hensel's lemma. For $t \in \mathbb{N}$ consider the Taylor expansion

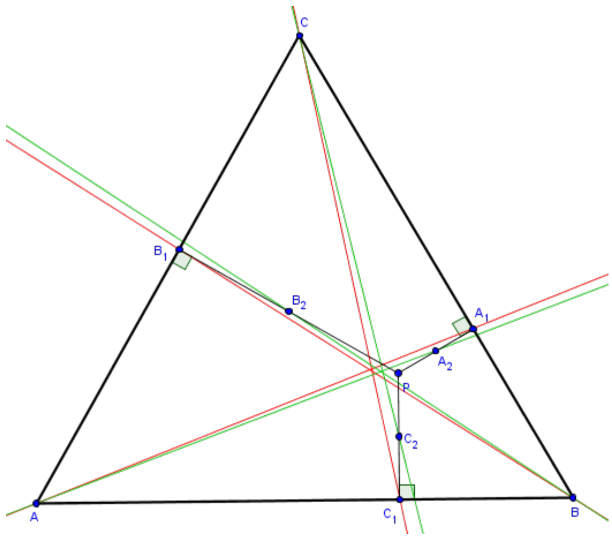
$$P(n_0 + tp) = P(n_0) + tpP'(n_0) + p^2 G(n_0, t),$$

for some $G(n_0, t) \in \mathbb{Z}$. Now if $k = 1$, we may take $t = 0$ and $n_0 = n_p$. Alternatively, if $k \geq 2$ take $t = 1$ and set $n_p = n_0 + p$.

O338. Let P be a point inside equilateral triangle ABC. Denote by A_1, B_1, C_1 the projections of P onto triangle's sides and denote by A_2, B_2, C_2 the midpoints of PA_1, PA_2, PA_3, respectively. Prove that AA_1, BB_1, CC_1 are concurrent if an only if AA_2, BB_2, CC_2 are concurrent.

Proposed by Nairi Sedrakyan, Armenia

Solution by Andrea Fanchini, Cantù, Italy



Using barycentric coordinates we denote $P(u : v : w)$ and using Conway's notation being the triangle equilateral we have $a = b = c$ and $S_A = S_B = S_C = \frac{a^2}{2}$.
Furthermore the infinite perpendicular points of the sides are

$$BC_{\infty\perp}(-a^2 : S_C : S_B) = (-2 : 1 : 1), \quad CA_{\infty\perp}(S_C : -b^2 : S_A) = (1 : -2 : 1), \quad AB_{\infty\perp}(S_B : S_A : -c^2) = (1 : 1 : -2)$$

then line $PBC_{\infty\perp}$ have equation

$$\begin{vmatrix} -2 & 1 & 1 \\ u & v & w \\ x & y & z \end{vmatrix} = 0 \quad \Rightarrow \quad PBC_{\infty\perp} \equiv (w - v)x + (2w + u)y - (2v + u)z = 0$$

so $A_1 = PBC_{\infty\perp} \cap BC = (0 : 2v + u : 2w + u), \Rightarrow \left(0, \frac{2v+u}{2(u+v+w)}, \frac{2w+u}{2(u+v+w)}\right)$.
Now line AA_1 have equation

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2v + u & 2w + u \\ x & y & z \end{vmatrix} = 0 \quad \Rightarrow \quad AA_1 \equiv -(2w + u)y + (2v + u)z = 0$$

Furtermore the midpoint A_2 will have coordinates $A_2 = (2u : 4v + u : 4w + u)$.
Then the line AA_2 have equation

$$\begin{vmatrix} 1 & 0 & 0 \\ 2u & 4v + u & 4w + u \\ x & y & z \end{vmatrix} = 0 \quad \Rightarrow \quad AA_2 \equiv -(4w + u)y + (4v + u)z = 0$$

Ciclically we find that the lines BB_1, CC_1 and BB_2, CC_2 have equation

$$BB_1 \equiv (2w + v)x - (2u + v)z = 0, \quad CC_1 \equiv -(2v + w)x + (2u + w)y = 0$$

$$BB_2 \equiv (4w + v)x - (4u + v)z = 0, \quad CC_2 \equiv -(4v + w)x + (4u + w)y = 0$$

Now three lines $p_i x + q_i y + r_i z = 0$, with $i = 1, 2, 3$, are concurrent if and only if

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0$$

so AA_1, BB_1, CC_1 are concurrent if and only if

$$\begin{vmatrix} 0 & -(2w+u) & 2v+u \\ 2w+v & 0 & -(2u+v) \\ -(2v+w) & 2u+w & 0 \end{vmatrix} = 0$$

that is if $u^2(w-v) + v^2(u-w) + w^2(v-u) = 0$.
Now AA_2, BB_2, CC_2 are concurrent if and only if

$$\begin{vmatrix} 0 & -(4w+u) & 4v+u \\ 4w+v & 0 & -(4u+v) \\ -(4v+w) & 4u+w & 0 \end{vmatrix} = 0$$

that is if $u^2(w-v) + v^2(u-w) + w^2(v-u) = 0$, so we prove that AA_1, BB_1, CC_1 are concurrent if and only if AA_2, BB_2, CC_2 are concurrent, q.e.d.

Also solved by Sardor Bozorboyev, S.H.Sirojjidinov lyceum, Tashkent, Uzbekistan; Daniel Lasaosa, Pamplona, Spain.

O339. Let a, b, c be positive real numbers satisfying

$$\frac{1}{a^3+b^3+1}+\frac{1}{b^3+c^3+1}+\frac{1}{c^3+a^3+1}\geq 1.$$

Prove that

$$(a+b)(b+c)(c+a)\leq 6+\frac{2}{3}\left(a^3+b^3+c^3\right).$$

Proposed by Nguyen Viet Hung, Hanoi, Vietnam

Solution by Li Zhou, Polk State College, Winter Haven, FL, USA

By clearing the denominators, we see that the given condition becomes

$$2(1+a^3+b^3+c^3)\geq (a^3+b^3)(b^3+c^3)(c^3+a^3).$$

Since $a^3+b^3\geq 2(\frac{a+b}{2})^3$, we further get that $2(1+a^3+b^3+c^3)\geq 8x^3$, where $x=\frac{1}{8}(a+b)(b+c)(c+a)$. Therefore

$$2(a^3+b^3+c^3)+18-24x\geq 8x^3-24x+16=8(x-1)^2(x+2)\geq 0,$$

which is equivalent to the claimed inequality.

Also solved by Sardor Bozorboyev, S.H.Sirojjidinov lyceum, Tashkent, Uzbekistan; Daniel Lasaosa, Pamplona, Spain; S.Viswanathan, Center for Scientific Learning, Nagpur, Maharashtra, India; George Gavrilopoulos, Nea Makri High School, Athens, Greece; Nicusor Zlota ,Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti , Università degli studi di Tor Vergata Roma, Roma, Italy.

O340. Let ABC be a triangle and let BB' and CC' be the altitudes from B and C, respectively. Ray $C'B'$ intersects the circumcircle of triangle ABC at B''. Let α_A be the measure of angle ABB''. Similarly, define angles α_B and α_C. Prove that

$$\sin\alpha_A \sin\alpha_B \sin\alpha_C \le \frac{3\sqrt{3}}{32}.$$

Proposed by Dorin Andrica, Cluj-Napoca, Romania

Solution by Daniel Lasaosa, Pamplona, Spain

Note first that $AB'C'$ is similar to ABC, with proportionality factor equal to $\cos A$ (it suffices to note that $BC'B'C$ is cyclic with circumcircle with diameter BC since $\angle BB'C = \angle BC'C = 90°$). Moreover, since $AB'C'$ can be obtained from ABC by reflecting on the internal bisector or angle A, and then performing a homothety with center A, it follows that the altitude from A in $AB'C'$ is the symmetric of AA' with respect to the internal bisector of angle A, hence the altitude from A in $AB'C'$ passes through the circumcenter O of ABC. Or the triangle whose vertices are A, and the intersections of $B'C'$ with the circumcircle of ABC, is isosceles at A, and $\angle AB''B' = \angle ABB'' = \alpha_A$. It follows that the length of the altitude from A onto $B'C'$ is $AB'' \sin\alpha_A = 2R\sin^2\alpha_A = 2R\sin B\sin C\cos A$, or $\sin\alpha_A = \sqrt{\sin B\sin C\cos A}$, and similarly for α_B and α_C, or

$$\sin\alpha_A \sin\alpha_B \sin\alpha_C = \sin A\sin B\sin C\sqrt{\cos A\cos B\cos C}.$$

Now, ABC is acute (if for example angle A is obtuse, $C'B'$ does not intersect the circumcircle of ABC and B'' is not defined), hence

$$\cos A\cos B\cos C = \frac{\cos C\left(\cos(A-B) - \cos C\right)}{2} \le \frac{\cos C - \cos^2 C}{2} =$$

$$= \frac{1-(1-2\cos C)^2}{8} \le \frac{1}{8},$$

with equality iff $C = 60°$ and simultaneously $A = B$, i.e. iff ABC is equilateral. On the other hand, and denoting by S the area of ABC, we have

$$\sin A\sin B\sin C = \frac{abc}{8R^3} = \frac{S}{2R^2} \le (\sin 60°)^3 = \frac{3\sqrt{3}}{8},$$

since it is well known that for fixed circumradius, the area of ABC is largest when ABC is equilateral. It follows that

$$\sin\alpha_A \sin\alpha_B \sin\alpha_C = \sin A\sin B\sin C\sqrt{\cos A\cos B\cos C} \le \frac{3\sqrt{3}}{8}\sqrt{\frac{1}{8}} = \frac{3\sqrt{6}}{32},$$

with equality iff ABC is equilateral.

Also solved by Sardor Bozorboyev, S.H.Sirojjidinov lyceum, Tashkent, Uzbekistan; Li Zhou, Polk State College, Winter Haven, FL, USA.

O341. Let a be a positive integer. Find all nonzero integer polynomials $P(X), Q(X)$ such that

$$P(X)^2 + aP(X)Q(X) + Q(X)^2 = P(X).$$

Proposed by Mircea Becheanu, Bucharest, Romania

Solution by Daniel Lasaosa, Pamplona, Spain

Assume that either $\deg(P)$ or $\deg(Q)$ is at least 1. Note that if $\deg(P) > \deg(Q)$, then the LHS has degree $2\deg(P)$, while the RHS has degree $\deg(P)$, absurd. If $\deg(Q) > \deg(P)$, then the LHS has degree $2\deg(Q)$, but $2\deg(Q) > 2\deg(P) \geq \deg(P)$, absurd again. We therefore need either $\deg(P) = \deg(Q) \geq 1$, or $P(x), Q(x)$ both constant.

When $P(x), Q(x)$ are not both constant, let $\deg(P) = \deg(Q) = u \geq 1$. Let A, B be the (nonzero by definition) leading coefficients of $P(x), Q(x)$, or the coefficients of x^{2u} in the LHS and the RHS are respectively $A^2 + aAB + B^2$ and 0. Note therefore that

$$\frac{A}{B} = \frac{-a \pm \sqrt{a^2-4}}{2},$$

where $\sqrt{a^2-4}$ must be at the same time rational (because A, B are nonzero integers) and the square root of an integer (because a is an integer). Hence $a^2 - 4$ must be a perfect square, or since the only two squares which differ by 4 are $0, 4$, it follows that $a^2 = 4$, and $a = 2$ because a is positive by definition. Therefore, $P(x) = (P(x) + Q(x))^2$ is the square of a polynomial $p(x) = P(x) + Q(x)$ with integer coefficients, or when $P(x), Q(x)$ are not constant, solutions exist only iff $a = 2$, in which case for any nonconstant polynomial $p(x)$ with integer coefficients,

$$P(x) = (p(x))^2, \qquad Q(x) = p(x) - (p(x))^2,$$

is a solution, and there can be no solutions of any other form, or for any other positive integer a.

Consider now the case where $P(x), Q(x)$ are both constant, or the problem reduces to finding all integer solutions to the equation $p^2 + apq + q^2 = p$ for positive integer a. When $a = 1$, the quadratic equation $q^2 + pq + p^2 - p = 0$ in unknown q has discriminant $4p - 3p^4$, always negative when either $p < 0$ or $p \geq 2$, hence the only nonzero solution can be $p = 1$, which in turn results in $q = -1$ for q to be nonzero. There can be no other solution than $(p, q) = (1, -1)$ when $a = 1$.

Consider now $a \geq 2$. Note now that quadratic equation $p^2 + (aq-1)p + q^2 = 0$ in unknown p has two roots with product q^2. If both roots are equal, then $|p| = |q|$, for $p^2(a+2) = |p|$, or $|p|(a+2) = 1$, absurd since $a + 2 \geq 3$. It follows that if (p, q) is a solution in nonzero integers, then so is $(p', q) = \left(\frac{q^2}{p}, q\right)$, where $\frac{q^2}{p} = 1 - aq - p$ is clearly a nonzero integer. Similarly, quadratic equation $q^2 + apq + p^2 - p = 0$ can only have two equal roots if $p^2 - p = p(p-1)$ is a perfect square, or since $p - 1, p$ are coprime, both must be perfect squares. But the only two consecutive integers which are perfect squares are $0, 1$, or both roots can only be equal for nonzero p when $p = 1$, yielding $q(q + a) = 0$, which clearly has two distinct roots, and only $q = -a$ is nonzero. In any other case, note that if (p, q) is a solution, then so is $(p, q') = \left(p, \frac{p^2-p}{q}\right)$, where $\frac{p^2-p}{q} = -ap - q$ is clearly a nonzero integer unless $p = 1$. Since $|p| = |q|$ has been ruled out, note that if $|p| > |q|$ then $|p'| < |q|$, whereas if $|q| > |p|$, then $q' < |p-1| < |p|$, or starting from any solution (p, q), we may generate a new solution with lower value of $|pq|$. Since $|pq|$ is an integer, the process cannot continue indefinitely, and by the previous results we will eventually end up in a solution with $p = 1$, and consequently $q = -a$ Note that from this solution, we may generate back the initial solution, since the steps are reversible. In other words, there are infinitely many solutions for any a, which can be generated from $(p_0, q_0) = (1, -a)$ by

$$(p_{2n+1}, q_{2n+1}) = \left(\frac{{q_{2n}}^2}{p_{2n}}, q_{2n}\right), \qquad (p_{2n+2}, q_{2n+2}) = \left(p_{2n+1}, \frac{{p_{2n+1}}^2 - p_{2n+1}}{q_{2n+1}}\right).$$

This is true for all $a \geq 2$, since it can easily be proved by induction that $p_{2n+2} = p_{2n+1} > p_{2n} > 0$ for all $n \geq 0$, and similarly $q_{2n+2} > q_{2n+1} = q_{2n}$ for all $n \geq 0$. In the case $a = 1$, which was treated separately earlier on, we find $(p_1, q_1) = (p_0, q_0) = (1, -1)$, and $q_2 = 0$, or no other solutions are generated by this recursive method. All other values of $a \geq 2$ have infinitely many solutions, and the previously describe recursion indeed generates solutions with increasing product $|p_n q_n|$.

Note that we can give a closed form for this sequence of solutions, by defining $r_0 = 1$, $r_1 = a$, and for all $n \geq 0$, $r_{n+2} = ar_{n+1} - r_n$, and then taking

$$p_{2n-1} = p_{2n} = {r_n}^2, \qquad q_{2n} = q_{2n+1} = -r_n r_{n+1}.$$

Indeed,

$${p_{2n}}^2 + ap_{2n}q_{2n} + {q_{2n}}^2 - p_{2n} = \left({r_{n-1}}^2 - r_n r_{n-2} - 1\right) {r_n}^2$$

and it can be trivially found by backward induction that

$${r_{n-1}}^2 - r_n r_{n-2} - 1 = {r_{n-2}}^2 - r_{n-1} r_{n-3} - 1 = {r_1}^2 - r_2 r_0 - 1 = a^2 - (a^2 - 1) - 1 = 0.$$

Solving the recursive relation by the usual methods yields, for $a \geq 3$,

$$r_n = \frac{1}{\sqrt{a^2-4}} \left(\left(\frac{a + \sqrt{a^2-4}}{2} \right)^{n+1} - \left(\frac{a - \sqrt{a^2-4}}{2} \right)^{n+1} \right),$$

$$p_{2n-1} = p_{2n} = \frac{1}{a^2-4} \left(\left(\frac{a + \sqrt{a^2-4}}{2} \right)^{2n+2} + \left(\frac{a - \sqrt{a^2-4}}{2} \right)^{2n+2} - 2 \right),$$

$$q_{2n} = q_{2n+1} = -\frac{1}{a^2-4} \left(\left(\frac{a + \sqrt{a^2-4}}{2} \right)^{2n+3} + \left(\frac{a - \sqrt{a^2-4}}{2} \right)^{2n+3} - a \right).$$

For $a = 2$, the characteristic equation of the recursive relation becomes $\rho^2 - 2\rho + 1 = (\rho - 1)^2 = 0$, or the solution is of the form $r_n = An + B$, hence for the initial conditions $r_0 = 1$ and $r_1 = a = 2$ to be satisfied we find $r_n = n + 1$, yielding

$$p_{2n-1} = p_{2n} = (n+1)^2, \qquad q_{2n} = q_{2n+1} = -(n+1)(n+2).$$

Also solved by Sardor Bozorboyev, S.H.Sirojjidinov lyceum, Tashkent, Uzbekistan.

O342. Let $n \geq 2015$ be a positive integer and let $A \subset \{1, \ldots, n\}$ such that $|A| \geq \lfloor \frac{n+1}{2} \rfloor$. Prove that A contains a three term arithmetic progression.

Proposed by Oleksiy Klurman, Université de Montréal, Canada

First solution by the author

We begin with the following simple lemma.

Lemma. If $B \subset \{1, 2, \ldots, 8\}$ and $|B| \geq 5$, then B contains a three term arithmetic progression.

Proof. Suppose that this is not the case and consider the sets $B_1 = \{1, 2, 3, 4\}$ and $B_2 = \{5, 6, 7, 8\}$. One of them must contain at least three elements. Without loss of generality, we may assume that $B_1 \cap A \geq 3$. 3:We now have two possibilities: $1, 2, 4 \in B$ or $1, 3, 4 \in B$ (in the other cases we readily have the arithmetic progression in B_1). In the first case, numbers 6 and 7 cannot belong to B_2, and thus $B_2 = \{5, 8\}$. This is also impossible since we then have a progression of the form $2, 5, 8$. The second case is totally analogous. This completes the proof of the lemma.

We now turn to the solution of the main problem. Let $n = 8k + r \geq 1000$ and n is odd. We may assume that $|A| = 4k + \left\lfloor \dfrac{r+1}{2} \right\rfloor$. For any $0 \leq k_0 \leq k$ consider partition of the set

$$\{1, 2, \ldots, n\} \setminus [8k_0 + 1, 8k_0 + r] = [1, 8] \cup \cdots \cup [8(k_0 - 1) + 1, 8k_0] \cup [8k_0 + r + 1, 8(k_0 + 1) + r] \cup \ldots$$

Each interval from this partition contains at most 4 elements A and therefore the interval $[8k_0 + 1, 8k_0 + r]$ must contain at least $\left\lfloor \dfrac{r+1}{2} \right\rfloor$ elements from A.

Consider the interval $[8t + 1, 8t + r]$ where t is an even number and assign to it the array $(x_1, x_2, \ldots, x_r)$ of 0 and 1 in the following way: $x_1 = 1$ iff $8t + i \in A$ and 0 otherwise. Observe, that for r elements we have at most $\binom{r}{r+1/2} \leq \binom{7}{4} = 35$ different arrays. On the other hand, the number of intervals $[8t + 1, 8t + r]$ with even t is $\left\lfloor \dfrac{k+2}{2} \right\rfloor = \left\lfloor \dfrac{n-r}{16} + 1 \right\rfloor \geq 50$. By the pigeonhole principle there exist intervals $[8m + 1, 8m + r]$ and $[8n + 1, 8n + r]$ that have identical corresponding arrays.

This implies that the interval $[4(m + n) + 1, 4(m + n) + r]$ cannot contain more then $r - \left\lfloor \dfrac{r+1}{2} \right\rfloor$ elements from A. This contradiction finishes the proof for odd $n \geq 1013$.

If $n = 2k$, then $[1, 2, \ldots, n] = [1, 2, \ldots, 2\lfloor k/2 \rfloor - 1] \cup [2\lfloor k/2 \rfloor, \ldots, 2k] = A_1 \cup A_2$ and each subset contains odd number of elements. Clearly, $|A_2| > |A_1| \geq 1011$ and we can use the same arguments as in the previous case.

Second solution by Daniel Lasaosa, Pamplona, Spain

We first note that given a set K of integers k, there exists a three-term arithmetic progression in K iff there exists a three-term arithmetic progression in $K-d$, iff there exists a three-term arithmetic progression in $d-K$ for any integer d. Denote now by $m(n)$ the maximum number m of elements of $\{1,2,\ldots,n\}$ such that no three of them are in arithmetic progression. The observation $m(n_1+n_2)\leq m(n_1)+m(n_2)$ is clearly true, since if we can find m elements in $\{1,2,\ldots,n_1+n_2\}$ such that no three of them are in arithmetic progression, then there are $m_1\leq m(n_1)$ of those elements in $\{1,2,\ldots,n_1\}$, and $m_2\leq m(n_2)$ of those elements in $\{n_1+1,n_1+2,\ldots,n_1+n_2\}$, with $m(n_1+n_2)=m_1+m_2$. As a trivial consequence, $m(n)$ is non-decreasing.

Claim 1: $m(8)\leq 4$.

Proof 1: Assume that a 5element subset K of $\{1,2,\ldots,8\}$ exists such that no three of them are in arithmetic progression. Considering $9-K$, wlog three of them are in $\{1,2,3,4\}$, or since $1,2,3$ and $2,3,4$ are arithmetic progressions, K must include either $1,2,4$ or $1,3,4$. Now $1,2,4$ rules out $6,7$ but $2,5,8$ is an arithmetic progression, while $1,3,4$ rules out $5,7$ but $4,6,8$ is an arithmetic progression, with contradiction in both cases. The Claim 1 follows.

Claim 2: $m(13)=7$ with equality for the subsets $\{1,2,4,5,10,11,13\}$, $\{1,2,4,5,10,12,13\}$, $\{1,2,4,8,10,11,13\}$, $\{1,2,4,9,10,12,13\}$, $\{1,3,4,6,10,12,13\}$, $\{1,3,4,9,10,12,13\}$.

Proof 2: Assume that a 7-element subset K of $\{1,2,\ldots,13\}$ exists such that 7 is not in K, or wlog 4 elements of K must be in $\{1,2,\ldots,6\}$. It is not hard to see that these 4 elements must be $\{1,2,4,5\}$, $\{1,2,5,6\}$, $\{1,3,4,6\}$ or $\{2,3,5,6\}$. These subsets rule out respectively $3,6,7,8,9$, and $3,4,7,8,9,10,11$, and $2,5,7,8,9,11$, and $1,4,7,8,9,10$. In the second case there are only 2 elements left to choose, while in the fourth case there are exactly 3 elements left to choose, $11,12,13$, which are already in arithmetic progression. In the third case we obtain $\{1,3,4,6,10,12,13\}$, and in the first case we must choose 3 elements not in arithmetic progression from $\{10,11,12,13\}$, which can be done in exactly two ways, for sets $\{1,2,4,5,10,11,13\}$, $\{1,2,4,5,10,12,13\}$. Once we add sets $14-K$ for each one of the K's found, the remaining sets must contain 7, and by our previous argument, must contain exactly 3 elements in $\{1,2,\ldots,6\}$, and exactly 3 elements in $\{8,9,\ldots,13\}$, where moreover $k,14-k$ cannot simultaneously appear. Therefore, if $5,7$ are in the set, $3,6,9$ are not in the set, or $11,8$ are in the set, contradiction since $5,8,11$ are an arithmetic progression. We reach a similar contradiction if $7,9$ are in the set. There are therefore no more sets for $m(13)=7$. The Claim 2 follows.

Claim 3: $m(17)\leq 8$.

Proof 3: Assume that a 9-element subset K of $\{1,2,\ldots,17\}$ can be found which contains no three-element arithmetic progression. All of the subsets in Claim 2 rule out all but at most one of the elements in $\{14,15,16,17\}$, or there are at most 6 elements in $\{1,2,\ldots,13\}$, hence at least 3 elements in $\{14,15,16,17\}$, and similarly at least 3 elements in $\{1,2,3,4\}$. Clearly all 4 element subsets of $\{1,2,3,4\}$ contain a three-element arithmetic progression, or either $\{1,2,4\}$ or $\{1,3,4\}$ must be in K. Similarly, either $\{14,16,17\}$ or $\{14,15,17\}$ must be in K. Now, $1,2,4$ rule out $6,7$ and $1,3,4$ rule out $5,7$, while $14,16,17$ rule out $11,12$ and $14,15,17$ rule out $11,13$. Note further that neither 7 nor 11 can be in K, while $1,17$ are both in K, or 9 cannot be in K. Note finally that since 4 is in K, $8,12$ cannot be simultaneously in K, and since 14 is in K, $6,10$ cannot be simultaneously in K. It follows that, wlog by the symmetry around 9, 5 must be in K, or $\{1,2,4\}$ must be in K, hence $6,8$ cannot be in K. This means that two out of $10,12,13$ must be in K, which rules out 14, contradiction. The Claim 3 follows.

Claim 4: $m(19)\leq 9$.

Proof 4: Assume that a 10-element subset K of $\{1,2,\ldots,19\}$ can be found which contains no three-element arithmetic progression. All of the subsets in Claim 2 rule out all but at most one element in $\{14,15,\ldots,19\}$, or there are at most 6 elements in $\{1,2,\ldots,13\}$, hence at least 4 elements in $\{14,15,\ldots,19\}$, and similarly there are at least 4 elements in $\{1,2,\ldots,6\}$. Note that every 5-element subset of $\{1,2,\ldots,6\}$ contains one of the two arithmetic progressions $1,2,3$ or $4,5,6$, or there are exactly four elements in $\{1,2,\ldots,6\}$, and four elements in $\{14,15,\ldots,19\}$. The 4-element subsets of $\{1,2,\ldots,6\}$ which do not contain a three-term arithmetic progression are $\{1,2,4,5\}$, $\{1,2,5,6\}$, $\{1,3,4,6\}$ and $\{2,3,5,6\}$. They respectively rule out $7,8,9$, and $7,8,9,10,11$, and $7,8,9,11$, and $7,8,9,10$. Similarly, $\{15,16,18,19\}$, $\{14,15,18,19\}$,

$\{14,16,17,19\}$ and $\{14,15,17,18\}$ respectively rule out $11,12,13$, and $9,10,11,12,13$, and $9,11,12,13$, and $10,11,12,13$. It follows that $7,8,9,11,12,13$ can never be in K, or 10 must be in K, which means that $20-k$ cannot be in K as long as k is in K. But if 1 is in K, 19 is not in K, or $14,15,17,18$ are in K, hence $2,3,5,6$ are not in K, absurd.

Claim 5: $m(21) \leq 10$.

Proof 5: Assume that an 11-element subset K of $\{1,2,\ldots,21\}$ can be found which contains no three-element arithmetic progression. All of the subsets in Claim 2 rule out all but at most one element in $\{14,15,\ldots,21\}$, or there are at most 6 elements in $\{1,2,\ldots,13\}$, hence at least 5 elements in $\{14,15,\ldots,21\}$, and similarly there are at least 5 elements in $\{1,2,\ldots,8\}$, absurd because of Claim 1. The Claim 5 follows.

It follows from Claims 3, 4 and 5 and by the fact that $m(n)$ is non-decreasing, that

$$m(33) \leq m(16)+m(17) \leq 16, \qquad m(34) \leq 2m(17) \leq 16,$$

$$m(35) \leq m(17)+m(18) \leq 17, \qquad m(36) \leq m(17)+m(19) \leq 17,$$

$$m(37) \leq m(18)+m(19) \leq 18, \qquad m(38) \leq 2m(19) \leq 18,$$

$$m(39) \leq m(19)+m(20) \leq 19, \qquad m(40) \leq m(19)+m(21) \leq 19.$$

Or $m(n) \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ for 8 consecutive values of n starting at 33, or since $m(8)=4$, this result is by trivial induction clearly true for all $n \geq 33$. Therefore, for any $n \geq 33$, taking $\left\lfloor \frac{n+1}{2} \right\rfloor$ elements out of $\{1,2,\ldots,n\}$, we guarantee that there will be at least one three-term arithmetic progression. The conclusion follows, the lower bound having been improved from 2015 to 33.

Note: Using numerical computation, the bound for $m(19)$ can be lowered to 8, the bound for $m(21)$ to 9, and so on, or the bound $\left\lfloor \frac{n+1}{2} \right\rfloor$ can also be significantly reduced. In fact, for $n \geq 70$ we have $m(n) < \frac{n}{3}$, and increasing the base case for the induction we can obtain still higher values of u such that $m(n) < \frac{n}{u}$ for $n \geq 2015$.

Third solution by Li Zhou, Polk State College, USA

This is a well-known topic started by Erdős and Turan in 1936. See, for example, Arun Sharma, Sequences of Integers Avoiding 3-term Arithmetic Progressions, *Electronic J. of Combinatorics* **19** (2012), which gives $r(27) = 11$, where $r(n)$ is the cardinality of the largest subsets of $\{1, \ldots, n\}$ avoiding 3-term arithmetic progression (It is easier to prove the weaker $r(27) \leq 13$ by writing $1, 2, \ldots, 27$ in a 3×9 array.). Hence, for $n \geq 2015$,

$$r(n) \leq \frac{13n}{27} + 26 < \frac{n-1}{2} < \left\lfloor \frac{n+1}{2} \right\rfloor,$$

completing the proof.