Junior problems

J343. Prove that the number $102400\ldots002401$, having a total of 2014 zeros, is composite.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J344. Find the maximum possible value of k for which

$$\frac{a^2+b^2+c^2}{3}-\left(\frac{a+b+c}{3}\right)^2 \geq k\cdot \max((a-b)^2,(b-c)^2,(c-a)^2)$$

for all $a,b,c\in \mathbf{R}$.

Proposed by Dominik Teiml, University of Oxford, United Kingdom

J345. Let a and b be positive real numbers such that

$$a^6-3b^5+5b^3-3a=b^6+3a^5-5a^3+3b.$$

Find $a-b$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J346. We are given a sequence of $12n$ numbers that are consecutive terms of an arithmetic progression. We randomly choose four numbers. What is the probability that among the chosen numbers there will be three in arithmetic progression?

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J347. Let x, y, z be positive real numbers. Prove that

$$\frac{x}{2x+y+z}+\frac{y}{x+2y+z}+\frac{z}{x+y+2z}\geq \frac{6xyz}{(x+y)(y+z)(z+x)}.$$

Proposed by Titu Zvonaru and Neculai Stanciu, Romania

J348. Let $ABCDEFG$ be a regular heptagon. Prove that

$$\frac{AD^3}{AC^3}=\frac{AC+2AD}{AB+AD}.$$

Proposed by Dragoljub Milosevic, Gornji Milanovac, Serbia

Senior problems

S343. Let a, b, c, d be positive real numbers. Prove that

$$\frac{a+b+c+d}{\sqrt[4]{abcd}}+\frac{16abcd}{(a+b)(b+c)(c+d)(d+a)}\geq 5.$$

Proposed by Titu Andreescu, USA and Alok Kumar, India

S344. Find all non-zero polynomials $P \in \mathbb{Z}[X]$ such that $a^2+b^2-c^2|P(a)+P(b)-P(c)$.

Proposed by Vlad Matei, University of Wisconsin, USA

S345. Solve in positive integers the system of equations

$$\begin{cases}(x-3)(yz+3)=6x+5y+6z\\(y-3)(zx+3)=2x+6y\\(z-3)(xy+3)=4x+y+6z.\end{cases}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S346. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}+\frac{15}{(a+b+c)^2}\geq\frac{6}{ab+bc+ca}.$$

Proposed by Marius Stănean, Zalău, Romania

S347. Prove that a convex quadrilateral $ABCD$ is cyclic if and only if the common tangent to the incircles of triangles ABD and ACD, different from AD, is parallel to BC.

Proposed by Nairi Sedrakyan, Armenia

S348. Find all functions $f:\mathbf{R}\times\mathbf{R}\to\mathbf{R}$ such that for all $a, b, c \in \mathbf{R}$

$$f(a^2, f(b,c)+1)=a^2(bc+1).$$

Proposed by Mehtaab Sawhney, USA

Undergraduate problems

U343. Evaluate

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+\pi(k)\log n},$$

where $\pi(k)$ denotes the number of primes not exceeding k.

Proposed by Albert Stadler, Herrliberg, Switzerland

U344. Evaluate the following sum

$$\sum_{n=0}^{\infty}\frac{3^n(2^{3^n-1}+1)}{4^{3^n}+2^{3^n}+1}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U345. Let R be a ring. We say that the pair $(a,b)\in R\times R$ satisfies property (P) if the unique solution of the equation $axa=bxb$ is $x=0$. Prove that if (a,b) has property (P) and $a-b$ is invertible, then the equation $axa-bxb=a+b$ has a unique solution in R.

Proposed by Dorin Andrica, "Babes-Bolyai" University, Cluj Napoca, Romania

U346. Let $f:[0,1]\to\mathbb{R}$ be a twice differentiable function for which $[f'(x)]^2+f(x)f''(x)\geq 1$. Prove that

$$\int_0^1 f^2(x)dx \geq f^2\left(\frac{1}{2}\right)+\frac{1}{12}.$$

Proposed by Marcel Chiriță, Bucharest, Romania

U347. Find all differentiable functions $f:[0,\infty)\to\mathbf{R}$ such that $f(0)=0$, f' is increasing and for all $x\geq 0$

$$x^2f'(x)=f^2(f(x)).$$

Proposed by Stanescu Florin, Gaesti, Romania

U348. Evaluate the linear integral

$$\oint_c \frac{(1+x^2-y^2)\ dx+2xy\ dy}{(1+x^2-y^2)^2+4x^2y^2}$$

where c is the square with vertices $(2,0)$, $(2,2)$, $(-2,2)$, and $(-2,0)$ traversed counterclockwise.

Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

Olympiad problems

O343. Let $a_1, \ldots, a_n$ be positive real numbers such that

$$\sqrt{a}_1 + \sqrt{a}_2 + ... + \sqrt{a}_n = a_1 + a_2 + ... + a_n.$$

Prove that

$$\sqrt{a_1^2+1} + \sqrt{a_2^2+1} + ... + \sqrt{a_n^2+1} \le n\sqrt{2}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O344. Consider the sequence $a_n = [(\sqrt[3]{65}-4)^{-n}]$, where $n \in \mathbb{N}^*$. Prove that $a_n \equiv 2, 4 \pmod{15}$.

Proposed by Vlad Matei, University of Wisconsin, USA

O345. Let A_1, B_1, C_1 be points on the sides BC, CA, AB of a triangle ABC. Let r_A, r_B, r_C, r_1 be the inradii of triangles $AB_1C_1, BC_1A_1, CA_1B_1$ and $A_1B_1C_1$ respectively. Prove that

$$R_1 r_1 \ge 2\min(r_A r_B, r_B r_C, r_C r_A),$$

where R_1 is the circumcircle of triangle $A_1B_1C_1$.

Proposed by Nairi Sedrakyan, Armenia

O346. Define the sequence $(a_n)_{n\ge 0}$ by $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, $a_3 = 6$ and

$$a_{n+4} = 2a_{n+3} + a_{n+2} - 2a_{n+1} - a_n, \quad n \ge 0.$$

Prove that n^2 divides a_n for infinitely many positive integers.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj Napoca, Romania

O347. Let $a, b, c, d \ge 0$ be real numbers such that $a+b+c+d = 1$. Prove that

$$\sqrt{a + \frac{2(b-c)^2}{9} + \frac{2(c-d)^2}{9} + \frac{2(d-b)^2}{9}} + \sqrt{b} + \sqrt{c} + \sqrt{d} \le 2.$$

Proposed by Marius Stanean, Zalau, Romania

O348. Let $ABCDE$ be a convex pentagon with area S, and let R_1, R_2, R_3, R_4, R_5 be the circumradii of triangles ABC, BCD, CDE, DEA, EAB, respectively. Prove that

$$R_1^4 + R_2^4 + R_3^4 + R_4^4 + R_5^4 \ge \frac{4}{5\sin^2 108^\circ} S^2.$$

Proposed by Nairi Sedrakyan, Armenia