A FORGOTTEN COAXALITY LEMMA

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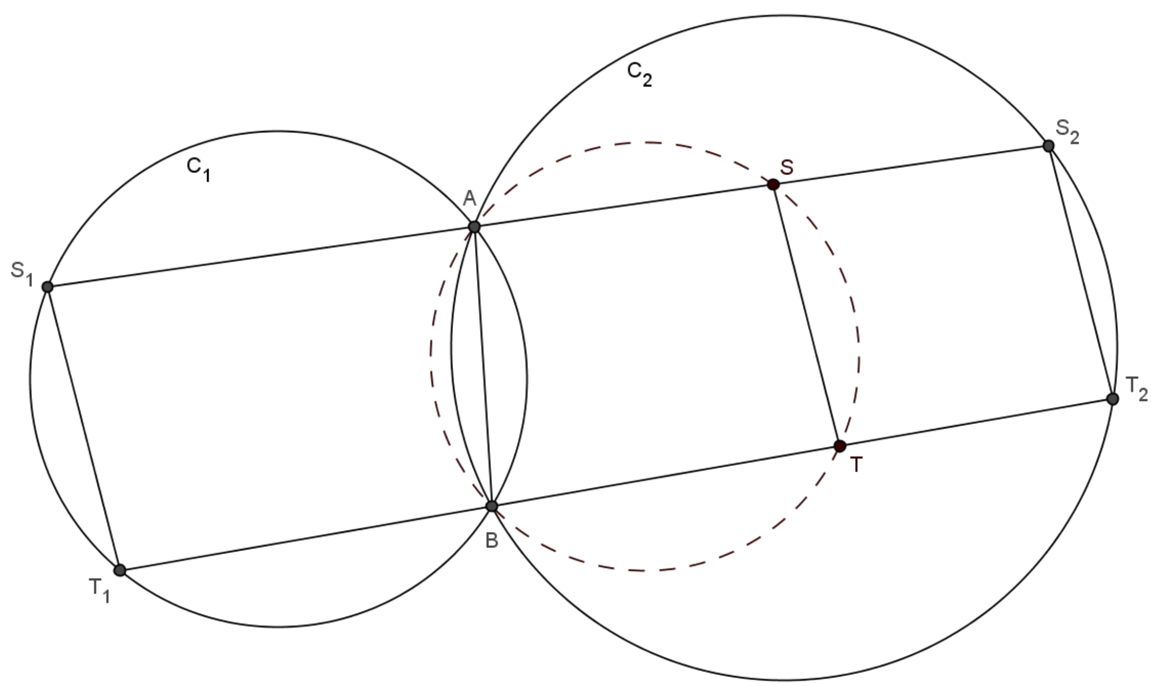
ABSTRACT. There are a lot of problems involving coaxality at olympiads. Sometimes, problems look pretty nasty and ask to prove that three circles are coaxal. In the following article, we will show how this kind of problems can be tackled using a very nice lemma.

We will start by presenting the lemma that will help us solving some difficult coaxality problems. Denote by $\omega(P,C)$ the power of point P with respect to circle C. Then,

Lemma: Let circles C_1 and C_2 intersect at points A and B. Now, let C_3 be a circle which passes through A and B. Then, C_3 is the locus of points P such that $\dfrac{\omega(P,C_1)}{\omega(P,C_2)} = k$, where k is a constant.



Proof: We will prove the lemma as follows. Take two points S and T such that $\frac{\omega(S,C_1)}{\omega(S,C_2)} = \frac{\omega(T,C_1)}{\omega(T,C_2)} = k$. Firstly, we will prove that A, B, S, T lie on a circle. Let SA intersect circles C_1 and C_2 at points S_1 and S_2, respectively. Define similarly points T_1 and T_2 for the line TB. Now, we know that

$$\frac{SS_2 \cdot SA}{SA \cdot SS_1} = \frac{TT_2 \cdot TB}{TB \cdot TT_1} \Leftrightarrow \frac{SS_2}{SS_1} = \frac{TT_2}{TT_1} \quad (1)$$

and also it is easy to see that S_2T_2 is parallel to the line S_1T_1, since $\angle T_1S_1A = \angle ABT_2 = 180 - \angle S_1S_2T_2$. Combining this with (1), we see that ST is also parallel to the lines S_2T_2 and S_1T_1 and this means that the quadrilater $ABST$ is cyclic. Now, for the second part, we suppose that $ABST$ is cyclic and prove that $\frac{\omega(S,C_1)}{\omega(S,C_2)} = \frac{\omega(T,C_1)}{\omega(T,C_2)}$. But this is easy since we know that $ST \parallel S_1T_1 \parallel S_2T_2$. Thus,

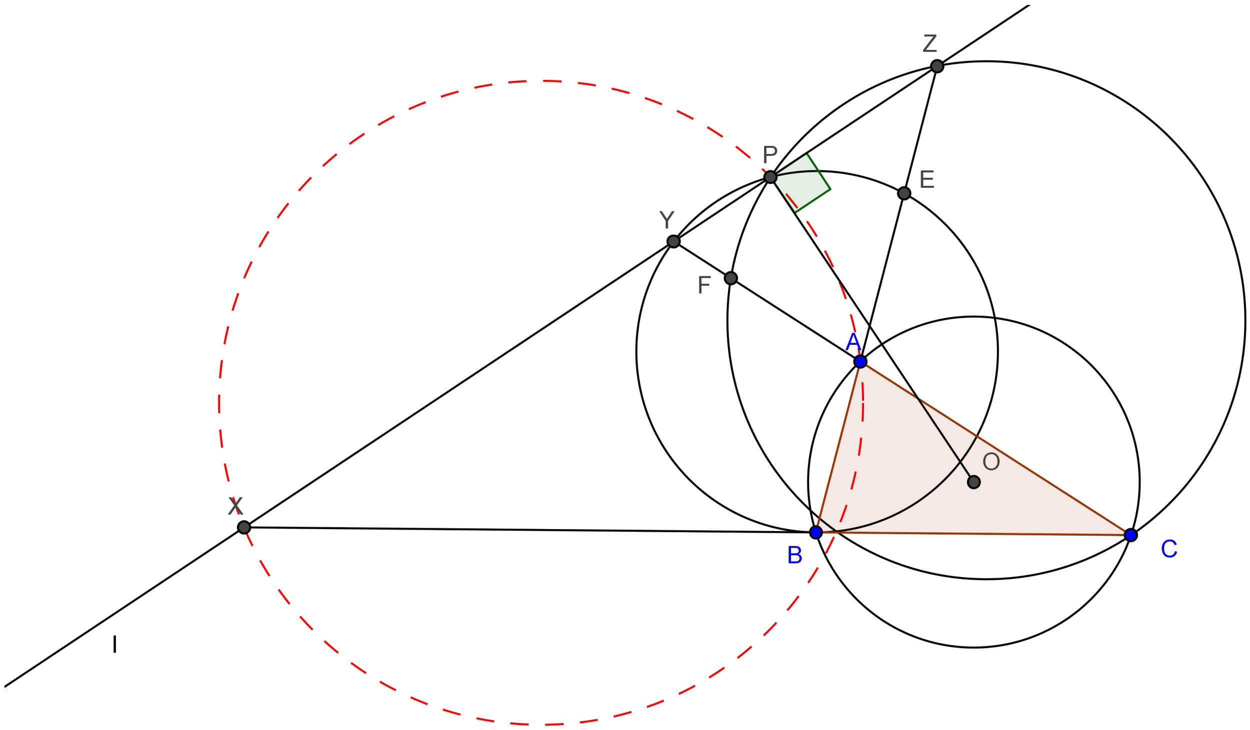
$$\frac{SS_2}{SS_1} = \frac{TT_2}{TT_1} \Leftrightarrow \frac{SS_2 \cdot SA}{SA \cdot SS_1} = \frac{TT_2 \cdot TB}{TB \cdot TT_1}$$

which is what we wanted to prove. In this way, our lemma is proven and we are ready to proceed.■

Moreover, the lemma is true for nonintersecting circles and the proof goes along the same lines. The only difference is that we will need to use 3D geometry in order to be able to visualize the radical axes.

Now, we will start with a very beautiful problem listed as $G8$ on the IMO Shortlist 2012 and proposed by the great mathematician, Cosmin Pohoata. The problem goes as follows:

Let ABC be a triangle with circumcircle ω and ℓ a line without common points with ω. Denote by P the foot of the perpendicular from the center of ω to ℓ. The side-lines BC, CA, AB intersect ℓ at the points X, Y, Z different from P. Prove that the circumcircles of the triangles AXP, BYP and CZP have a common point different from P or are mutually tangent at P.



Proof: As in our lemma, let C_1 be the circumcircle of triangle BYP and C_2 be the circumcircle of triangle CZP. Now, let $\{E\} = C_1 \cap AB$ and $\{F\} = C_2 \cap AC$. Using our lemma, we see that it is enough to prove that

$$\frac{\omega(X, C_1)}{\omega(X, C_2)} = \frac{\omega(A, C_1)}{\omega(A, C_2)} \Leftrightarrow \frac{AE}{AF} = \frac{CY}{BZ}.$$

Now, all we have to do is to figure out a way of doing that. We denote by R the radius of the circumcircle of triangle ABC. Then,

$$AZ{\cdot}ZB - AY{\cdot}YC = OY^2 - OZ^2 = YP^2 - ZP^2 = YP{\cdot}YZ - PZ{\cdot}YZ = YF{\cdot}YC - ZE{\cdot}ZB$$

and so, we got that

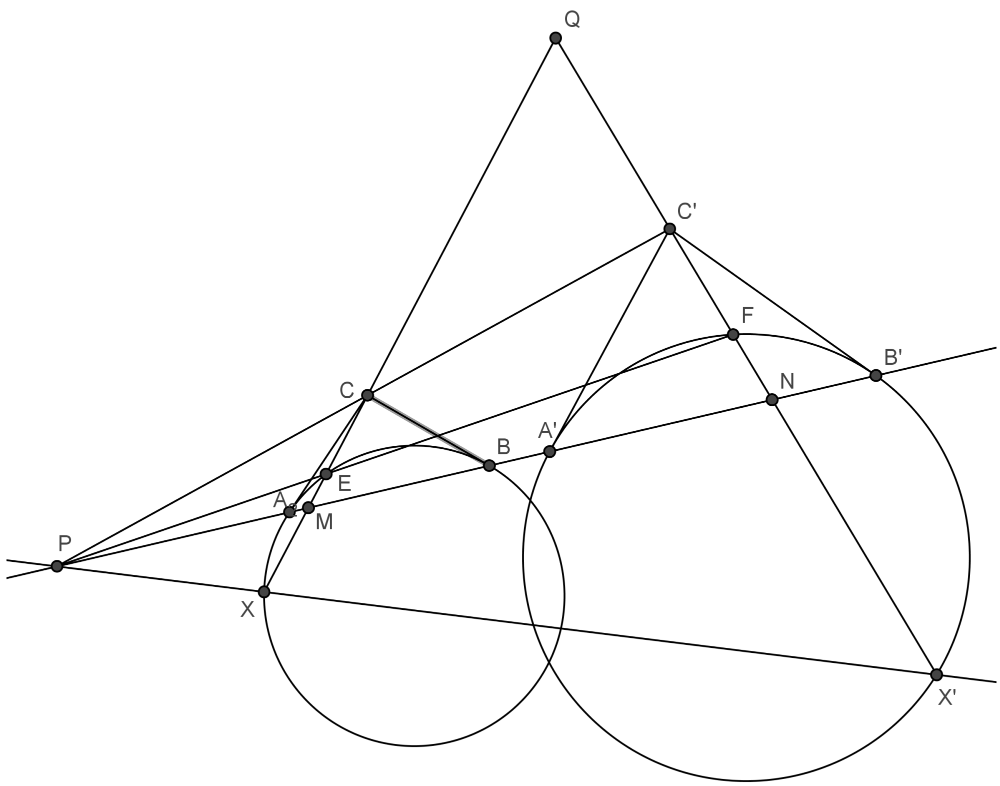
$$ZB(AZ + ZE) = CY(AY + YF) \Leftrightarrow ZB \cdot AE = CY \cdot AF \Leftrightarrow \frac{AE}{AF} = \frac{CY}{ZB}$$

which is what we were looking for.■

We will continue with a problem which was solved by no more than 2 students in the actual TST.

(Romanian TST 2010) Let ℓ be a line, and let γ and γ' be two circles. The line ℓ meets γ at points A and B, and γ' at points A' and B'. The tangents to γ at A

and B meet at point C, and the tangents to γ' at A' and B' meet at point C'. The lines ℓ and CC' meet at point P. Let λ be a variable line through P and let X be one of the points where λ meets γ, and X' be one of the points where λ meets γ'. Prove that the point of intersection of the lines CX and $C'X'$ lies on a fixed circle.



Proof: Let $Q \equiv CX \cap C'X'$. CX cuts ℓ at M and γ again at E. $C'X'$ cuts ℓ at N and γ' again at F. Since ℓ is polar of C and C' WRT γ and γ', then the pencils $P(C, M, E, X)$ and $P(C', N, F, X')$ are harmonic $\Longrightarrow P, E, F$ are collinear. Hence, by Menelaus' theorem for $\triangle QCC'$ cut by XX' and EF, we get that

$$\frac{QX}{QX'} = \frac{PC'}{PC} \cdot \frac{CX}{C'X'}, \quad \frac{QE}{QF} = \frac{PC'}{PC} \cdot \frac{CE}{C'F}$$

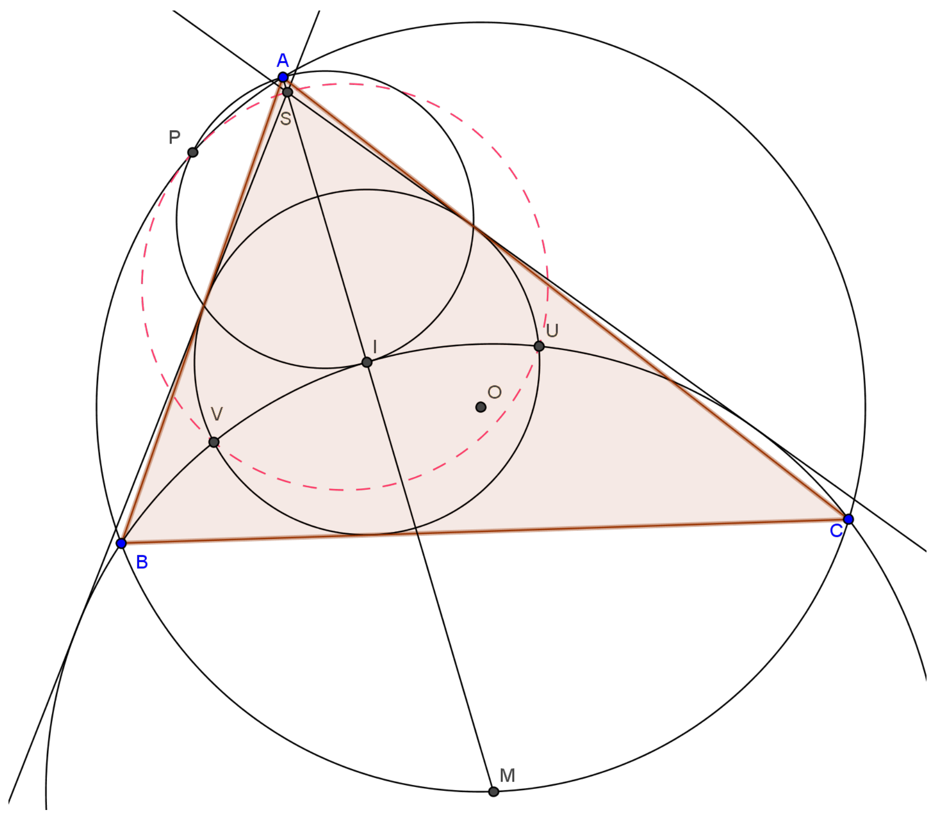
and so,

$$\frac{QX \cdot QE}{QX' \cdot QF} = \left(\frac{PC'}{PC}\right)^2 \cdot \frac{CX \cdot CE}{C'X' \cdot C'F} = \left(\frac{PC' \cdot CA}{PC \cdot C'A'}\right)^2 = \text{const.}$$

Thus, the ratio of the powers of Q with respect to γ and γ' is constant and using our lemma we get that the locus of Q is a circle ω coaxal with γ and γ'. This circle ω then passes through the intersections $CA \cap C'A'$, $CB \cap C'B'$, $CA \cap C'B'$ and $CB \cap C'A'$. ■

We conclude with the final example and what could be more spectacular than a Russian problem.

(Russia) Let $\triangle ABC$ be a triangle with $\omega(I, r)$ its incircle. The common tangents of circles ω and the circumcircle of $\triangle BIC$ intersect at point S. Also, these two circles intersect at points U and V. Prove that the circumcircle of $\triangle SUV$ is tangent to the circumcircle of $\triangle ABC$.



Proof: Consider λ to be the circle with AI as its diameter and let it cut the circumcircle of $\triangle ABC$ at a point P, different from A. Let M be the circumcenter of $\triangle BIC$. We will denote by C_{XYZ} the circumcircle of $\triangle XYZ$ and by R_{XYZ}, the radius of the circumcircle of $\triangle XYZ$. Now, we will prove that $PSUV$ is cyclic. For this, using our lemma, it is enough to prove that

$$\frac{\omega(P,\omega)}{\omega(P,C_{BIC})} = \frac{\omega(S,\omega)}{\omega(S,C_{BIC})}$$

or equivalently

$$\frac{PI^2 - r^2}{PM^2 - R_{BIC}^2} = \frac{r^2}{R_{BIC}^2} \Leftrightarrow \frac{PI}{PM} = \frac{r}{R_{BIC}} = \frac{r}{MI} \ (*)$$

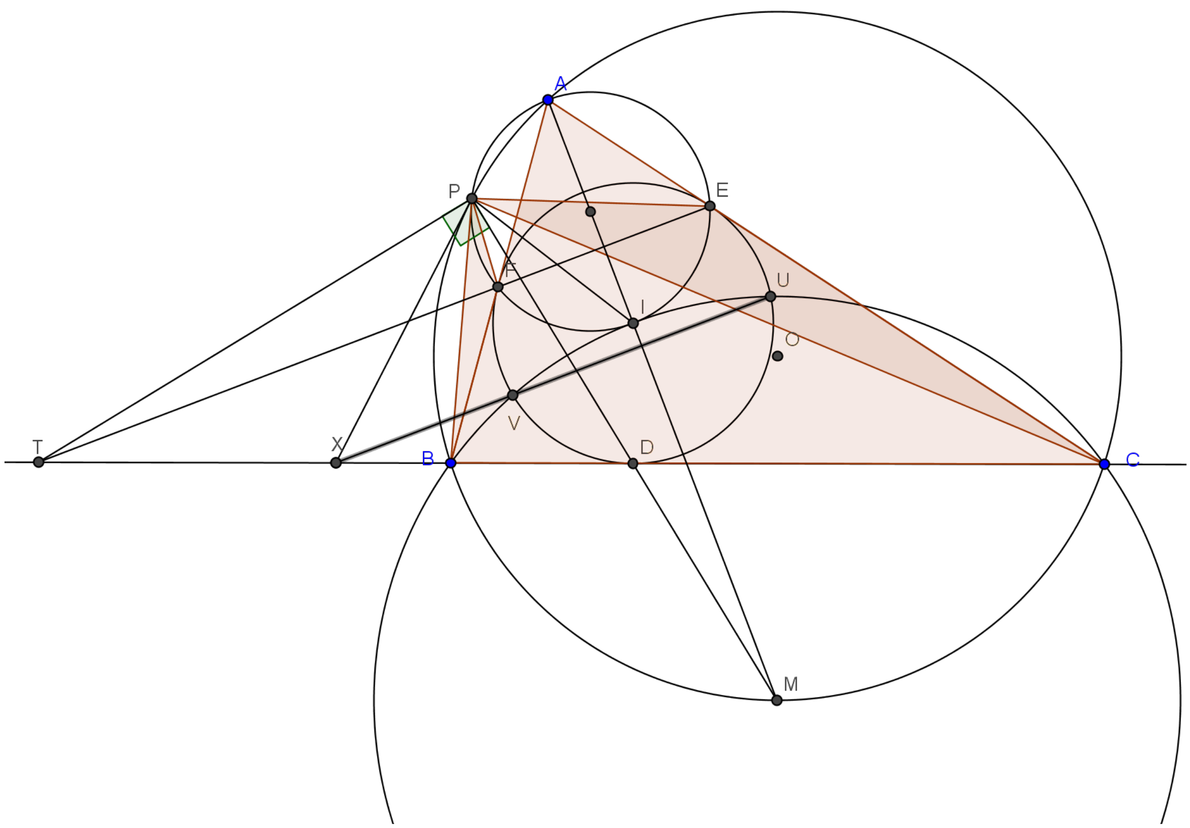
since it is well known that $MI = MB = MC$, using the sinus theorem, we get that

$$\frac{AI \cdot \sin PAI}{2R_{ABC} \cdot \sin PAM} = \frac{AI}{2R_{ABC}}.$$

Thus, letting $R = R_{ABC}$, $(*)$ is quivalent to

$$\frac{AI}{2R} = \frac{r}{MI} \Leftrightarrow AI \cdot IM = 2Rr \Leftrightarrow R^2 - OI^2 = 2Rr$$

which is true by Euler's formula.
Now, we will continue with the second part of the proof.



Let D, E, F be the contact points of ω with the the sides BC, AC, AB respectively, $\{X\} = UV \cap BC$ and $\{T\} = EF \cap BC$. Now, we first notice that X is the radical center of the circles C_{ABC}, ω, C_{BIC} and C_{SUV}. Thus, we get that $XD^2 = XB \cdot XC$ and since $(T, D; B, C)$ forms a harmonic division, we deduce that X is the midpoint of the segment TD. Since $PAEF$ is cyclic, we know that $\angle PFA = \angle PEA \Rightarrow \angle PFB = \angle PEC$. Moreover, $\angle PBF = \angle PCE$ since $PACB$ is cyclic. Therefore, we see that $\triangle PBF$ is similar to $\triangle PCE$, so

$$\frac{PB}{BF} = \frac{PC}{CE} \Leftrightarrow \frac{PB}{PC} = \frac{DB}{DC}$$

which implies that $P\!-\!D\!-\!M$ are collinear and since $P(T, D; B, C)$ forms a harmonic pencil, we get that $\angle TPD = 90$. Since X is the midpoint of TD, we get that $XP = XD$. But, $XP^2 = XD^2 = XB \cdot XC = XV \cdot XU$ so we got that XP is the common tangent at point P of circles C_{ABC} and C_{SPUV}, which means that these two circles are tangent Q.E.D.$\blacksquare$