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INEQUALITIES ON RATIOS OF RADII OF TANGENT CIRCLES

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Abstract. Some inequalities involving ratios of radii of internally tangent circles which intersect the given line in fixed points are studied. By considering special and degenerate cases of the construction, some surprising inequalities are obtained. Analogous results for externally tangent circles are also discussed.

Introduction. If *A*, *B*, *C*, and *D* are fixed points on a line in the given order then the locus of points *M* not on line *AD* for which $\angle AMB = \angle CMD$ is a circle whose diameter lies on the line *AD* (Figure 0). This circle, which is sometimes called *Apollonius circle*, is also interesting for the other reason. Note that the circumscribed circles of the triangles *AMD* and *CMB* are tangent if *M* lies on the Apollonius circle. If we fix the circumscribed circle of *AMD* and move point *M* along this circle then the ratio $\angle CMB / \angle AMD$ decreases. So in a certain sense Apollonius circle is the locus of points M for which the ratio $\angle CMB / \angle AMD$ is maximal. On the other hand it would be interesting to find maximal and minimal values of the ratio $\angle CMB / \angle AMD$ if *M* is on the Apollonius circle. We have not succeeded in solving the last problem completely. For more information on the history of the question see [1]. But our investigations led to some interesting inequalities which we collected in the present paper.

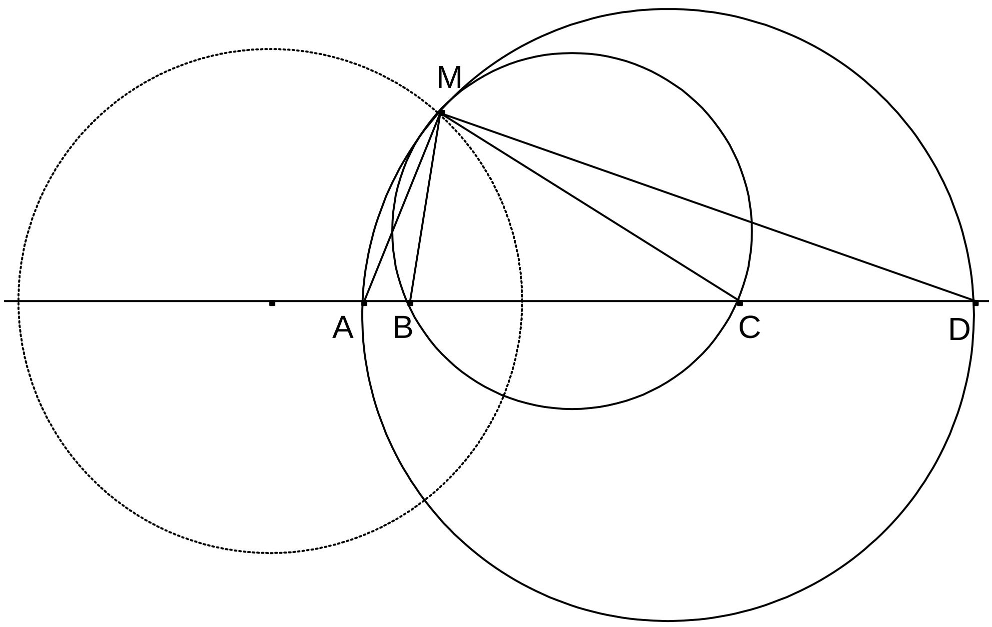

Figure 0.

1. Theorem. Let *A*, *B*, *C*, and *D* be points on line *k* in this order, and *M* be a point not on *k* such that $\angle AMB = \angle CMD$. Then

$$\frac{\sin \angle BMC}{\sin \angle AMD} > \frac{|BC|}{|AD|}.$$

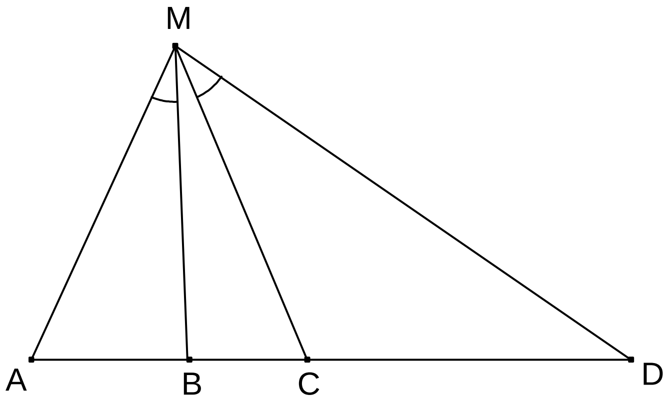

Figure 1.

Proof. We shall first prove that circumscribed circles of triangles *AMD* and *BMC* with radii *R* and *r* respectively, are tangent at point *M*. Let O_1 and O_2 be circumcenters of triangles *AMD* and *BMC*. Drop perpendicular *MH* to line *AD*. It is easy to show that $\angle AMH = \angle CMO_1$ and $\angle BMH = \angle CMO_2$. By subtracting we obtain $\angle AMB = \angle CMD + \angle O_1MO_2$. Note that $\angle AMB = \angle CMD$. It follows that points *M*, O_1 and O_2 are collinear.

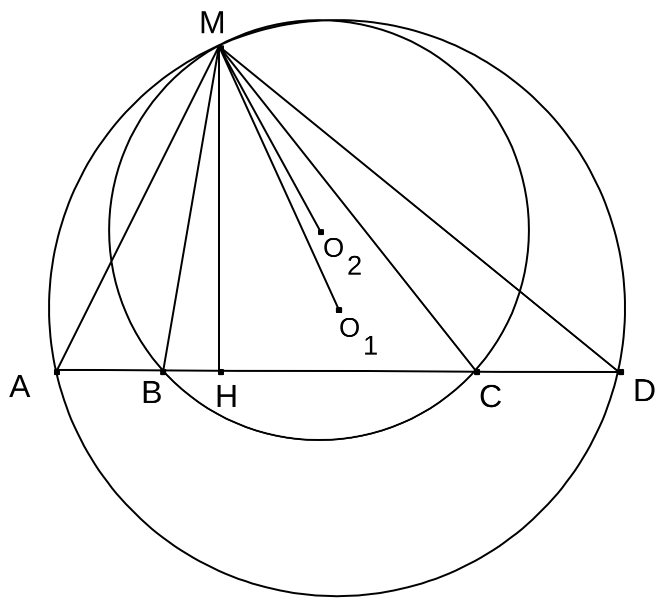

Figure 2.

Therefore these circles are tangent at point *M*. We see that *R*>*r*. By sine theorem

$$\frac{\sin\angle BMC}{\sin\angle AMD}=\frac{R}{r}\cdot\frac{BC}{AD}>\frac{|BC|}{|AD|}.$$

2. Theorem. Let *A*, *B*, *C*, and *D* be points on line *k* in this order, and *M* be a point not on *k* such that $\angle AMB=\angle CMD$. Then

$$\frac{|BC|\cdot|AD|}{\left(\sqrt{|AC|\cdot|BD|}-\sqrt{|AB|\cdot|CD|}\right)^2}\geq\frac{\sin\angle BMC}{\sin\angle AMD}\geq\frac{|BC|\cdot|AD|}{\left(\sqrt{|AC|\cdot|BD|}+\sqrt{|AB|\cdot|CD|}\right)^2}.$$

Proof. Suppose first that $|AB|<|CD|$. Then common tangent line of circles at point M intersect line *AD* at point *E*. Denote |*AB*|=*a*, |*CD*|=*b*, |*BC*|=*c*, |*EA*|=*x* and |*EM*|=*y*. It follows that $y^2=x(x+a+b+c)$ and $y^2=(x+a)(x+a+c)$.

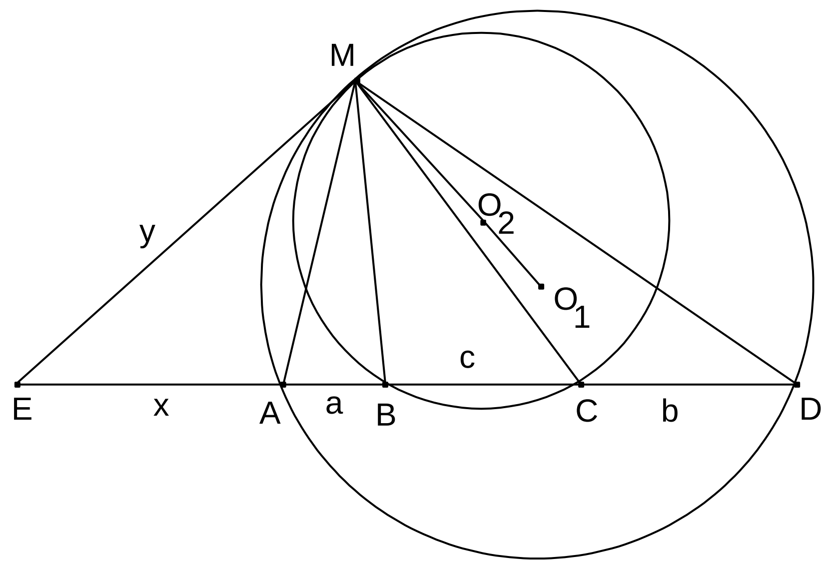

Figure 3.

By subtracting or dividing we obtain $x=\frac{a(a+c)}{(b-a)}$. Putting this in one of the previous equalities gives $y=\frac{\sqrt{ab(a+c)(b+c)}}{b-a}$. Denote $\angle AMB=\mu$. Drop the perpendiculars O_1K and O_2L to line *AD*. We obtain $EH=y\cos\mu$,

$AK = KD = \frac{a+b+c}{2}$, $BL = LC = \frac{c}{2}$, $HL = x + a + \frac{c}{2} - y\cos\mu$, $HK = x + \frac{a+b+c}{2} - y\cos\mu$.

Consequently,

$$\frac{R}{r} = \frac{x + \frac{a+b+c}{2} - y\cos\mu}{x + a + \frac{c}{2} - y\cos\mu}.$$

Since $a < b$, fraction *R/r* decreases as angle *μ* increases from 0 to *π*. Therefore

$$\frac{x + \frac{a+b+c}{2} + y}{x + a + \frac{c}{2} + y} \le \frac{R}{r} \le \frac{x + \frac{a+b+c}{2} - y}{x + a + \frac{c}{2} - y}.$$

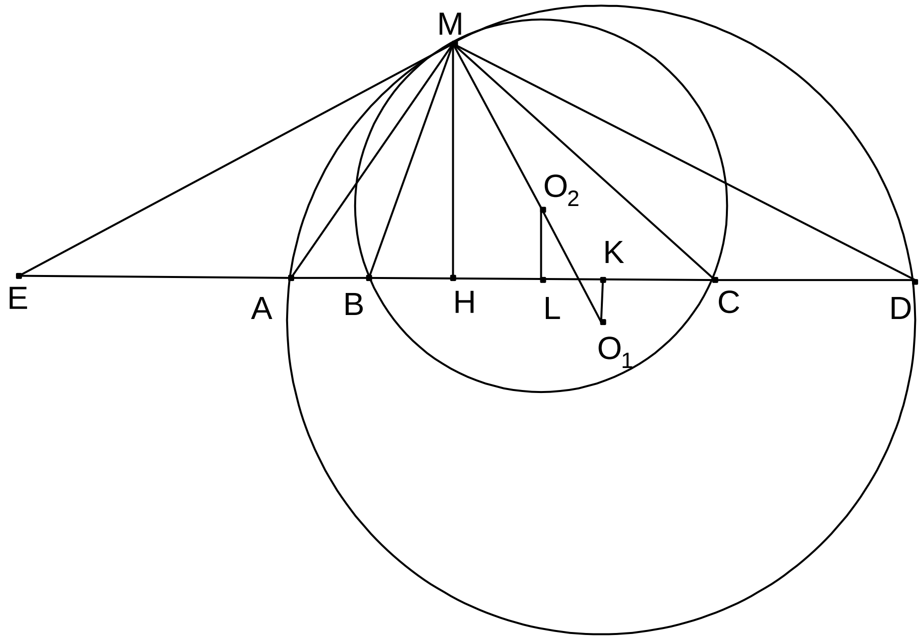

Figure 4.

It remains only to simplify the expressions on both sides of this double inequality.

$$\frac{x+\frac{a+b+c}{2}\pm y}{x+a+\frac{c}{2}\pm y}=\frac{\frac{a(a+c)}{(b-a)}+\frac{a+b+c}{2}\pm\frac{\sqrt{ab(a+c)(b+c)}}{b-a}}{\frac{a(a+c)}{(b-a)}+a+\frac{c}{2}\pm\frac{\sqrt{ab(a+c)(b+c)}}{b-a}}=$$

$$=\frac{2a(a+c)+(b-a)(a+b+c)\pm 2\sqrt{ab(a+c)(b+c)}}{2a(a+c)+(b-a)(2a+c)\pm 2\sqrt{ab(a+c)(b+c)}}=$$

$$=\frac{a(a+c)+b(b+c)\pm 2\sqrt{ab(a+c)(b+c)}}{a(b+c)+b(a+c)\pm 2\sqrt{ab(a+c)(b+c)}}=$$

$$=\frac{\left(\sqrt{a(a+c)}\pm\sqrt{b(b+c)}\right)^2}{\left(\sqrt{a(b+c)}\pm\sqrt{b(a+c)}\right)^2}.$$

By multiplying the numerator with its conjugate and then dividing with the same conjugate we obtain

$$\frac{\left(\sqrt{a(a+c)}\pm\sqrt{b(b+c)}\right)^2}{\left(\sqrt{a(b+c)}\pm\sqrt{b(a+c)}\right)^2}=$$

$$=\frac{\left(\sqrt{a(a+c)}\pm\sqrt{b(b+c)}\right)^2}{\left(\sqrt{a(b+c)}\pm\sqrt{b(a+c)}\right)^2}\cdot\frac{\left(\sqrt{a(a+c)}\mp\sqrt{b(b+c)}\right)^2}{\left(\sqrt{a(a+c)}\mp\sqrt{b(b+c)}\right)^2}=$$

$$=\left(\frac{a(a+c)-b(b+c)}{a\sqrt{(a+c)(b+c)}-b\sqrt{(a+c)(b+c)}\pm(a+c)\sqrt{ab}\mp(b+c)\sqrt{ab}}\right)^2=$$

$$=\left(\frac{(a+b+c)(a-b)}{\left(\sqrt{(a+c)(b+c)}\pm\sqrt{ab}\right)(a-b)}\right)^2=\left(\frac{a+b+c}{\sqrt{(a+c)(b+c)}\pm\sqrt{ab}}\right)^2.$$

Finally,

$$\frac{c(a+b+c)}{\left(\sqrt{(a+c)(b+c)}+\sqrt{ab}\right)^2}\le\frac{\sin\angle BMC}{\sin\angle AMD}=\frac{R}{r}\cdot\frac{c}{a+b+c}\le\frac{c(a+b+c)}{\left(\sqrt{(a+c)(b+c)}-\sqrt{ab}\right)^2}.$$

The case $|AB|>|CD|$ is analogous. For the case $|AB|=|CD|$ ($b=a$) one must pass to the limit in the last double inequality by tending $b \to a$.

Note. The following problem is open: Prove that $\frac{c(a+b+c)}{\left(\sqrt{(a+c)(b+c)}+\sqrt{ab}\right)^2} \le \frac{\angle BMC}{\angle AMD}$.

By proving this inequality the following chain of inequalities will be completed [1]:

$$\frac{c}{(a+b+c)} \le \frac{c(a+b+c)}{\left(\sqrt{(a+c)(b+c)}+\sqrt{ab}\right)^2} \le \frac{\angle BMC}{\angle AMD} \le \frac{\sin\angle BMC}{\sin\angle AMD} \le \frac{c(a+b+c)}{\left(\sqrt{(a+c)(b+c)}-\sqrt{ab}\right)^2}.$$

3. Corollary. Let *R* and *r*<*R* be radii of two circles which are tangent at point *M*. Chord *AD* of greater circle intersects the other circle at points *B* and *C*. Then

$$\left(\frac{|AD|}{\sqrt{|AC|\cdot|BD|}+\sqrt{|AB|\cdot|CD|}}\right)^2 \le \frac{R}{r} \le \left(\frac{|AD|}{\sqrt{|AC|\cdot|BD|}-\sqrt{|AB|\cdot|CD|}}\right)^2.$$

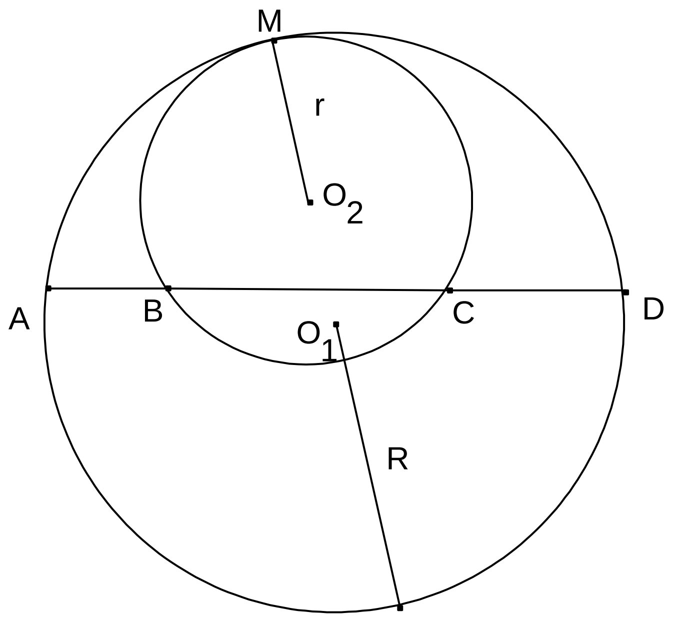

Figure 5.

Note. In the notations of previous problems this inequality can also be written as

$$\frac{a+b+c}{\sqrt{(a+c)(b+c)}+\sqrt{ab}} \le \sqrt{\frac{R}{r}} \le \frac{a+b+c}{\sqrt{(a+c)(b+c)}-\sqrt{ab}}.$$

4. Corollary. [3] Let *R* and *r*<*R* be radii of two circles which are tangent at point *M*. Chord *AC* of greater circle is tangent to the other circle at point *B*. Then

$$\frac{|AC|}{2\sqrt{|AB|\cdot|BC|}} \le \sqrt{\frac{R}{r}}.$$

Note. This follows from previous problem (simply put |*BC*|=0). It is interesting that the above inequality gives upper bound $\sqrt{\frac{R}{r}}$ as a ratio of arithmetic mean $\frac{|AB|+|BC|}{2}$ and geometric mean $\sqrt{|AB|\cdot|BC|}$ of two segments *AB* and *BC*. Lower bound is of course 1.

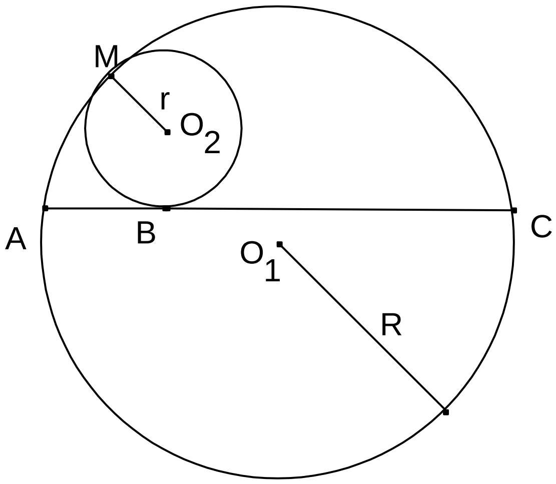

Figure 6.

5. Corollary. Chord *AC* of circle w_1 passes through midpoint *B* of chord *DE* of the same circle w_1. Circle w_2 is tangent to line *AC* at point *B* and circle w_1 at point *M*. Line *MB* intersects w_1 at point *F*. Then

$$\frac{|AC|}{|DE|} \le \sqrt{\frac{|MF|}{|MB|}}.$$

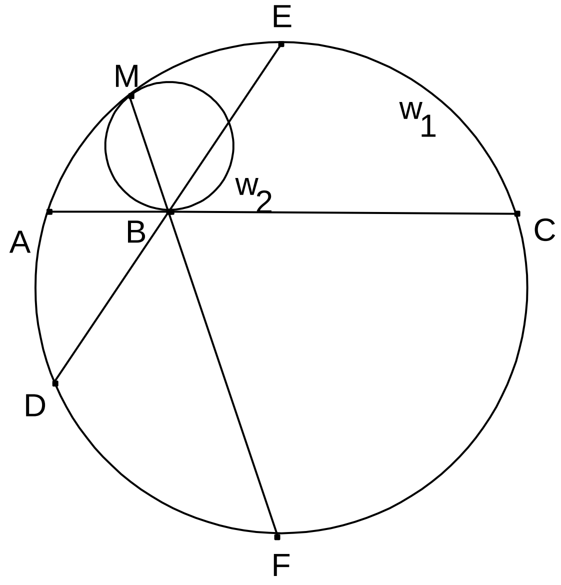

Figure 7.

6. Corollary. [2] Two circles w_1 and w_2 intersecting at points *B* and *C* are tangent to circle *w* internally at points *M* and *N*, respectively. Line *BC* intersects circle *w* at points *A* and *D*. Let r_1 and r_2 be radii of circles w_1 and w_2, respectively. Then

$$\frac{\sqrt{|AC|\cdot|BD|}-\sqrt{|AB|\cdot|CD|}}{\sqrt{|AC|\cdot|BD|}+\sqrt{|AB|\cdot|CD|}} \le \sqrt{\frac{r_1}{r_2}}.$$

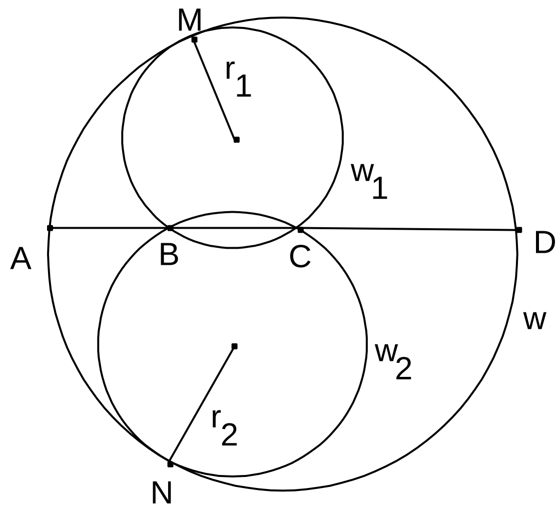

Figure 8.

Proof. Let radius of circle *w* be *R*. By the inequality in Exercise 3,

$$\left(\frac{\sqrt{|AC|\cdot|BD|}-\sqrt{|AB|\cdot|CD|}}{|AD|}\right)^2 \le \frac{r_1}{R},$$

$$\left(\frac{|AD|}{\sqrt{|AC|\cdot|BD|}+\sqrt{|AB|\cdot|CD|}}\right)^2 \le \frac{R}{r_2}.$$

Multiplying we obtain the required inequality.

Note. By replacing r_1 with r_2 and vice versa we can also obtain the upper bound for $\sqrt{\frac{r_1}{r_2}}$:

$$\sqrt{\frac{r_1}{r_2}} \le \frac{\sqrt{|AC|\cdot|BD|}+\sqrt{|AB|\cdot|CD|}}{\sqrt{|AC|\cdot|BD|}-\sqrt{|AB|\cdot|CD|}}.$$

So in fact the following double inequality holds true

$$\frac{\sqrt{|AC|\cdot|BD|}-\sqrt{|AB|\cdot|CD|}}{\sqrt{|AC|\cdot|BD|}+\sqrt{|AB|\cdot|CD|}} \le \sqrt{\frac{r_1}{r_2}} \le \frac{\sqrt{|AC|\cdot|BD|}+\sqrt{|AB|\cdot|CD|}}{\sqrt{|AC|\cdot|BD|}-\sqrt{|AB|\cdot|CD|}}.$$

The following theorem and its consequence can be proved using the same method.

7. Theorem. A circle *w* passing through the points *A* and *B* is externally tangent to a circle w_1. Line *AB* intersects the circle w_1 at points *C* and *D*. Let r_1 and *R* be radii of circles w_1 and *w*, respectively. Then

$$\frac{|AB|}{\sqrt{|AC|\cdot|BD|}+\sqrt{|BC|\cdot|AD|}} \le \sqrt{\frac{R}{r_1}} \le \frac{|AD|}{\sqrt{|AC|\cdot|BD|}-\sqrt{|BC|\cdot|AD|}}.$$

If w_2 is another circle passing through the points *C* and *D*, and externally tangent to the circle *w* then

$$\frac{\sqrt{|AC|\cdot|BD|}-\sqrt{|BC|\cdot|AD|}}{\sqrt{|AC|\cdot|BD|}+\sqrt{|BC|\cdot|AD|}} \le \sqrt{\frac{r_1}{r_2}} \le \frac{\sqrt{|AC|\cdot|BD|}+\sqrt{|BC|\cdot|AD|}}{\sqrt{|AC|\cdot|BD|}-\sqrt{|BC|\cdot|AD|}},$$

where r_2 is the radius of circle w_2.

Corollary. Let circles w and w_1 of radii R and r be externally tangent. Let the extension of chord AB of the circle w be tangent to the circle w_1 at the point C. Let CD be tangent to the circle w. Then

$$\frac{|AB|}{2\cdot|CD|}\le\sqrt{\frac{R}{r}}.$$

Theorem. Let circles w and w_1 of radii R and r be externally tangent. A line through the center of circle w_1 is tangent to the circle w at the point A. Let AB be tangent to circle w_1 at the point B. Similarly, a line through the center of circle w is tangent to the circle w_1 at the point C. Let CD be tangent to circle w at the point D. Then

$$|AB|=|CD|\ge\sqrt{Rr}.$$

Problems for further explorations. Do the last equality and the inequality hold true if the circles w and w_1 are

1) nonintersecting

2) intersecting?

References.

1. Y.N. Aliyev, Use of dynamic geometry software in teaching and research, 5th International Conference on AICT, 12-14 October 2011, Qafqaz University, Baku, 212-214.
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3. Y.N. Aliyev, Aufgabe 1318, Elem. Math. 68 (2013) 128; Solution: 69 (2014) 164-165.

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