Junior problems

J343. Prove that the number $102400\ldots002401$, having a total of 2014 zeros, is composite.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Polyahedra, Polk State College, USA
By $4a^4+b^4=(2a^2-2ab+b^2)(2a^2+2ab+b^2)$, the given number equals

$$\begin{aligned} 1024\cdot 10^{2016}+2401 &= 4\left(4\cdot 10^{504}\right)^4+7^4 \\ &= \left(32\cdot 10^{1008}-56\cdot 10^{504}+49\right)\left(32\cdot 10^{1008}+56\cdot 10^{504}+49\right). \end{aligned}$$

Second solution by Polyahedra, Polk State College, USA
Because $10^4\equiv -1 \pmod{137}$, the given number equals

$$1024\left(10^4\right)^{504}+2401\equiv 1024(-1)^{504}+2401\equiv 3425\equiv 0 \pmod{137}.$$

Also solved by Adnan Ali, A.E.C.S-4, Mumbai, India; Alok Kumar, Delhi, India; Arkady Alt, San Jose, CA, USA; Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; David Stoner, Harvard University, Cambridge, MA, USA; Devesh Rajpal, Raipur, Chhattisgarh, India; Ercole Suppa, Teramo, Italy; Haimoshri Das, South Point High School, India; José Hernández Santiago, México; Paul Revenant, Lycée Champollion, Grenoble, France; Robert Bosch, Archimedean Academy, Florida, USA and Jorge Erick López, IMPA, Brazil; Albert Stadler, Herrliberg, Switzerland; Hyun Jin Kim, Stuyvesant High School, New York, NY, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Théo Lenoir, Institut Saint-Lô, Agneaux, France; Michael Tang, Edina High School, MN, USA; Ioan Viorel Codreanu, Satulung, Maramures, Romania.

J344. Find the maximum possible value of k for which

$$\frac{a^2+b^2+c^2}{3}-\left(\frac{a+b+c}{3}\right)^2 \geq k\cdot\max\left((a-b)^2,(b-c)^2,(c-a)^2\right)$$

for all $a, b, c \in \mathbb{R}$.

Proposed by Dominik Teiml, University of Oxford, United Kingdom

Solution by Daniel Lasaosa, Pamplona, Spain

Note that the problem is invariant under exchange of any two of a, b, c, or we may assume wlog $a \geq b \geq c$, and denote $u = a - c$, $v = a - b$ with $u \geq v \geq 0$, and

$$\frac{a^2+b^2+c^2}{3}-\left(\frac{a+b+c}{3}\right)^2 = \frac{(a-b)^2+(b-c)^2+(c-a)^2}{9} = \frac{v^2+(u-v)^2+u^2}{9} =$$

$$= \frac{3u^2+(u-2v)^2}{18} \geq \frac{u^2}{6} = \frac{1}{6}\max\left((a-b)^2,(b-c)^2,(c-a)^2\right),$$

or the maximum value is $k = \frac{1}{6}$, with equality iff $u = 2v$, ie iff a, b, c are in arithmetic progression.

Also solved by Polyahedra, Polk State College, USA; Hyun Jin Kim, Stuyvesant High School, New York, NY, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Théo Lenoir, Institut Saint-Lô, Agneaux, France; Alok Kumar, Delhi, India; Arkady Alt, San Jose, CA, USA; Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; Ercole Suppa, Teramo, Italy; Joel Schlosberg, Bayside, NY, USA; Nick Iliopoulos, Music Junior HS, Trikala, Greece; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Paul Revenant, Lycée Champollion, Grenoble, France; Robert Bosch, Archimedean Academy, Florida, USA and Jorge Erick López, IMPA, Brazil; Albert Stadler, Herrliberg, Switzerland.

J345. Let a and b be positive real numbers such that

$$a^6 - 3b^5 + 5b^3 - 3a = b^6 + 3a^5 - 5a^3 + 3b.$$

Find $a - b$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Robert Bosch, Archimedean Academy, Florida, USA and Jorge Erick López, IMPA, Brazil

We have

$$\begin{aligned} a^6 - 3a^5 + 5a^3 - 3a &= b^6 + 3b^5 - 5b^3 + 3b, \\ (a^2 - a - 1)^3 + 1 &= (b^2 + b - 1)^3 + 1, \\ (a^2 - a - 1)^3 &= (b^2 + b - 1)^3, \end{aligned}$$

but $x^3 = y^3$ for real x, y if and only if $x = y$. Obtaining $a^2 - a - 1 = b^2 + b - 1$, it follows that $(a - b - 1)(a + b) = 0$ and finally $a - b = 1$ since $a + b > 0$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Polyahedra, Polk State College, USA; Hyun Jin Kim, Stuyvesant High School, New York, NY, USA; Michael Tang, Edina High School, MN, USA; Adnan Ali, A.E.C.S-4, Mumbai, India; Arkady Alt, San Jose, CA, USA; Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; Ercole Suppa, Teramo, Italy; Joseph Lee, Loomis Chaffee School, Windsor, CT, USA; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paul Revenant, Lycée Champollion, Grenoble, France; Rade Krenkov, SOUUD, Dimitar Vlahov, Strumica, Macedonia.

J346. We are given a sequence of $12n$ numbers that are consecutive terms of an arithmetic progression. We randomly choose four numbers. What is the probability that among the chosen numbers there will be three in arithmetic progression?

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Polyahedra, Polk State College, USA

Without loss of generality, assume the given sequence is $S = \{1, 2, \ldots, 12n\}$.
For $d \geq 1$, let $A_d = \big\{(a, a+d, a+2d, a+3d) \in S^4\big\}$ and $B_d = \big\{(a, a+d, a+2d, b) \in S^4 : b - a \neq 0, d, 2d, \frac{d}{2}, \frac{3d}{2}\big\}$. Then $|A_d| = 12n - 3d$, $|B_d| = (12n-2d)(12n-3)$ for odd d, and $|B_d| = (12n-2d)(12n-5)$ for even d. By PIE (the principle of inclusion-exclusion), the total number of favorable outcomes is $\sum_{d=1}^{6n} |B_d| - \sum_{d=1}^{4n} |A_d|$, which equals

$$\begin{aligned} & (12n-3)\sum_{i=1}^{3n}(12n-4i+2) + (12n-5)\sum_{i=1}^{3n}(12n-4i) - \sum_{i=1}^{4n}(12n-3i) \\ = \; & (12n-3)(3n)(6n) + (12n-5)(3n)(6n-2) - 2n(12n-3) = 12n(36n^2-20n+3). \end{aligned}$$

Hence, the probability we are seeking is

$$\frac{12n(36n^2-20n+3)}{\binom{12n}{4}} = \frac{4(36n^2-20n+3)}{(12n-1)(6n-1)(4n-1)}.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Robert Bosch, Archimedean Academy, Florida, USA and Jorge Erick López, IMPA, Brazil.

J347. Let x, y, z be positive real numbers. Prove that

$$\frac{x}{2x+y+z}+\frac{y}{x+2y+z}+\frac{z}{x+y+2z}\geq\frac{6xyz}{(x+y)(y+z)(z+x)}.$$

Proposed by Titu Zvonaru and Neculai Stanciu, Romania

Solution by Adnan Ali, A.E.C.S-4, Mumbai, India

Observe that the inequality is equivalent to

$$\sum\nolimits_{cyc}\frac{1}{yz(2x+y+z)}\geq\frac{6}{(x+y)(y+z)(z+x)}.$$

But by the Cauchy-Schwartz Inequality,

$$\sum\nolimits_{cyc}\frac{1}{yz(2x+y+z)}\geq\frac{9}{\sum\limits_{cyc}yz(2x+y+z)}=\frac{9}{(x+y)(y+z)(z+x)+4xyz}.$$

Hence, it suffices to prove that

$$\frac{9}{(x+y)(y+z)(z+x)+4xyz}\geq\frac{6}{(x+y)(y+z)(z+x)}\Leftrightarrow(x+y)(y+z)(z+x)\geq 8xyz,$$

which follows by the AM-GM Inequality. Equality holds iff $x=y=z$.

Also solved by Rade Krenkov, SOUUD, Dimitar Vlahov, Strumica, Macedonia; Daniel Lasaosa, Pamplona, Spain; Polyahedra, Polk State College, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Ioan Viorel Codreanu, Satulung, Maramures, Romania; George Gavrilopoulos, Nea Makri High School, Athens, Greece; Alok Kumar, Delhi, India; Andrea Fanchini, Cantù, Italy; Arkady Alt, San Jose, CA, USA; David Stoner, Harvard University, Cambridge, MA, USA; Ercole Suppa, Teramo, Italy; Haimoshri Das, South Point High School, India; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Paul Revenant, Lycée Champollion, Grenoble, France; Robert Bosch, Archimedean Academy, Florida, USA and Jorge Erick López, IMPA, Brazil; Brian Bradie, Christopher Newport University, Newport News, VA, USA.

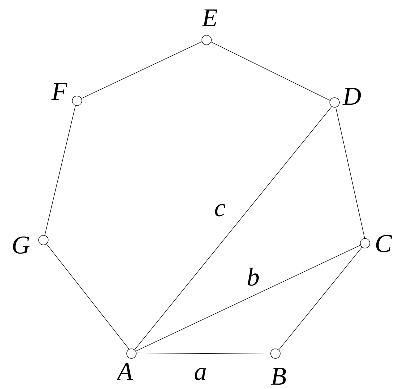
J348. Let $ABCDEFG$ be a regular heptagon. Prove that

$$\frac{AD^3}{AC^3} = \frac{AC + 2AD}{AB + AD}.$$

Proposed by Dragoljub Milosevic, Gornji Milanovac, Serbia

Solution by Ercole Suppa, Teramo, Italy

Let $AB = a$, $AC = b$, $AD = c$ as shown in the following figure.



Ptolemy's theorem applied to quadrilaterals $ABCD$, $ABCE$,$ABDF$ yields

$$a^2 + ac = b^2 \quad (1)$$
$$ab + ac = bc \quad (2)$$
$$b^2 + ab = c^2 \quad (3)$$

Now, by using (1),(2),(3) we get

$$\begin{aligned}\frac{AC + 2AD}{AB + AD} &= \frac{b + 2c}{a + c} = \frac{a(b + 2c)}{a^2 + ac} \overset{(1)}{=} \frac{a(b + 2c)}{b^2} = \\ &= \frac{ab(b + 2c)}{b^3} \overset{(2)}{=} \frac{(bc - ac)(b + 2c)}{b^3} = \\ &= \frac{c\left(b^2 + 2bc - ab - 2ac\right)}{b^3} = \frac{c\left[b^2 - ab + 2(bc - ac)\right]}{b^3} \overset{(2)}{=} \\ &= \frac{c\left(b^2 - ab + 2ab\right)}{b^3} = \frac{c\left(b^2 + ab\right)}{b^3} \overset{(3)}{=} \\ &= \frac{c^3}{b^3} = \frac{AD^3}{AC^3}\end{aligned}$$

as desired. The proof is complete.

Also solved by Rade Krenkov, SOUUD, Dimitar Vlahov, Strumica, Macedonia; Daniel Lasaosa, Pamplona, Spain; Polyahedra, Polk State College, USA; Michael Tang, Edina High School, MN, USA; Adnan Ali, A.E.C.S-4, Mumbai, India; Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Robert Bosch, Archimedean Academy, Florida, USA and Jorge Erick López, IMPA, Brazil; SooYoung Choi, Mount Michael Benedictine, Elkhorn, NE, USA.

Senior problems

S343. Let a, b, c, d be positive real numbers. Prove that

$$\frac{a+b+c+d}{\sqrt[4]{abcd}} + \frac{16abcd}{(a+b)(b+c)(c+d)(d+a)} \geq 5.$$

Proposed by Titu Andreescu, USA and Alok Kumar, India

Solution by the authors

$$\frac{a+b}{2\sqrt[4]{abcd}} + \frac{b+c}{2\sqrt[4]{abcd}} + \frac{c+d}{2\sqrt[4]{abcd}} + \frac{d+a}{2\sqrt[4]{abcd}} + \frac{16}{(a+b)(b+c)(c+d)(d+a)} \geq 5$$

by AM-GM.

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA, USA; Daniel Lasaosa, Pamplona, Spain; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Michael Tang, Edina High School, MN, USA; Ioan Viorel Codreanu, Satulung, Maramures, Romania; SooYoung Choi, Mount Michael Benedictine, Elkhorn, NE, USA; Adnan Ali, A.E.C.S-4, Mumbai, India; David Stoner, Harvard University, Cambridge, MA, USA; Ercole Suppa, Teramo, Italy; Francesco De Sclavis, Università degli Studi di Roma Tor Vergata, Rome, Italy; George Gavrilopoulos, Nea Makri High School, Athens, Greece; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Paul Revenant, Lycée Champollion, Grenoble, France; Rade Krenkov, SOUUD, Dimitar Vlahov, Strumica, Macedonia; Robert Bosch, Archimedean Academy, Florida, USA and Jorge Erick López, IMPA, Brazil; Anant Mudgal, Delhi Public School, Faridabad, Haryana, India; Li Zhou, Polk State College, Winter Haven, FL, USA.

S344. Find all non-zero polynomials $P \in \mathbb{Z}[X]$ such that $a^2+b^2-c^2 | P(a)+P(b)-P(c)$.

Proposed by Vlad Matei, University of Wisconsin, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Taking $b=c$, we find that for all $a \in \mathbb{Z}$, we have $a^2 | P(a)$. If the constant or linear term coefficients in $P(x)$ are non-zero, we may choose $a=p$, where p is a prime that does not divide the constant or the linear term coefficients, reaching a contradiction, or $P(x)=x^2Q(x)$ for some polynomial $Q(x)$ with integer coefficients.

Let now u, v be any two distinct positive integers with $u>v$, and let $a=u^2-v^2$, $b=2uv$ and $c=u^2+v^2$, or $a^2+b^2=c^2$, or by the condition given in the problem statement we need $P(c)=P(a)+P(b)$. Now, if the degree of $Q(x)$ is at least 1, then the absolute value of $Q(x)$ is increasing for all sufficiently large x, and taking a, b, c sufficiently large, we would have $Q(c)>Q(a), Q(b)$, for

$$P(c)=c^2Q(c)=a^2Q(c)+b^2Q(c)>a^2Q(a)+b^2Q(b)=P(a)+P(b),$$

reaching a contradiction, or $Q(x)$ must be constant, ie an integer. It follows that necessarily $P(x)=kx^2$ for some nonzero integer k, and clearly $P(a)+P(b)-P(c)=k\left(a^2+b^2-c^2\right)$ is a multiple of $a^2+b^2-c^2$, or $P(x)$ is a solution iff it is a polynomial of this form.

Also solved by Robert Bosch, Archimedean Academy, Florida, USA and Jorge Erick López, IMPA, Brazi; Anant Mudgal, Delhi Public School, Faridabad, Haryana, India.

S345. Solve in positive integers the system of equations

$$\begin{cases} (x-3)(yz+3) = 6x+5y+6z, \\ (y-3)(zx+3) = 2x+6y, \\ (z-3)(xy+3) = 4x+y+6z. \end{cases}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by the author

Summing the three equations we obtain

$$3xyz - 3(xy+yz+zx) = 9(x+y+z)+27,$$

and so

$$4xyz = (x+3)(y+3)(z+3),$$

This is equivalent to

$$\left(1+\frac{3}{x}\right)\left(1+\frac{3}{y}\right)\left(1+\frac{3}{z}\right) = 4.$$

From the given equations it follows that $x, y, z \geq 4$ and that x, y, z are not all equal. If $\min(x,y,z) > 4$, then

$$\left(1+\frac{3}{x}\right)\left(1+\frac{3}{y}\right)\left(1+\frac{3}{z}\right) < \left(1+\frac{3}{5}\right)\left(1+\frac{3}{5}\right)\left(1+\frac{3}{6}\right) = \frac{96}{25} < 4,$$

a contradiction. Hence $\min(x,y,z) = 4$ and let $\{a,b\} = \{x,y,z\} - \{4\}$. Then

$$\left(\frac{7}{4}\right)\left(1+\frac{3}{a}\right)\left(1+\frac{3}{b}\right) = 4,$$

which reduces to $(3a-7)(3b-7) = 7 \times 16$.

It follows that 7 divides a or b, and since a and b are greater than 3, the only possibility is $\{a,b\} = \{5,7\}$. So $\{x,y,z\} = \{4,5,7\}$ and, looking back at the original equations, we see that the only solution is

$$(x,y,z) = (7,4,5).$$

Second solution by Robert Bosch, Archimedean Academy, Florida, USA and Jorge Erick López, IMPA, Brazil

The only solution is $(x,y,z) = (7,4,5)$. The right side on each equation is positive, so $x,y,z \geq 4$. Now by the second equation we obtain

$$4 \leq x \Rightarrow 2x+24 \leq 8x \Rightarrow 2x+24 \leq 2zx \Rightarrow y = \frac{2x+3zx+9}{zx-3} \leq 5.$$

Hence, $y=4$ or $y=5$. When $y=5$ we have equality in the previous inequality, therefore $x=z=4$, but $(4,5,4)$ is not solution of the system. For $y=4$ in the second equation we get $zx+3 = 2x+24$, or equivalently $x(z-2) = 21$, and finally $x=7$ and $z=5$, that satisfies the other two equations.

Also solved by Daniel Lasaosa, Pamplona, Spain; Antoine Faisant; David E. Manes, Oneonta, NY, USA; Ercole Suppa, Teramo, Italy; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paul Revenant, Lycée Champollion, Grenoble, France; Rade Krenkov, SOUUD, Dimitar Vlahov, Strumica, Macedonia.

S346. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}+\frac{15}{(a+b+c)^2}\geq\frac{6}{ab+bc+ca}.$$

Proposed by Marius Stănean, Zalău, Romania

Solution by Adnan Ali, Student in A.E.C.S-4, Mumbai, India
W.L.O.G assume that $a \geq b \geq c$. Let

$$f(a,b,c)=\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}+\frac{15}{(a+b+c)^2}-\frac{6}{ab+bc+ca}.$$

Then

$$\begin{aligned}f(a,b,c)-f(a,b+c,0)&=\frac{1}{(a+b)^2}-\frac{1}{(a+b+c)^2}+\frac{1}{(c+a)^2}-\frac{1}{a^2}-\frac{6}{ab+bc+ca}+\frac{6}{a(b+c)}\\&\geq\frac{6bc}{a(b+c)(ab+bc+ca)}-\frac{c(2a+c)}{a^2(a+c)^2}.\end{aligned}$$

Now notice that since $a \geq b \geq c$, we have

$$6ab\geq(2a+c)(b+c),\qquad (a+c)^2\geq(ab+bc+ca)$$

proving that $f(a,b,c) \geq f(a,b+c,0)$. So, now it remains to show that $f(a,b+c,0) \geq 0$. Let $b+c=d$, then by the AM-GM Inequality,

$$ad\left(\frac{1}{a^2}+\frac{1}{d^2}+\frac{16}{(a+d)^2}\right)=-2+\frac{(a+d)^2}{ad}+\frac{16ad}{(a+d)^2}\geq-2+2\sqrt{16}=6,$$

and the conclusion follows. Equality holds when $a=b, c=0$ and its respective permutations.

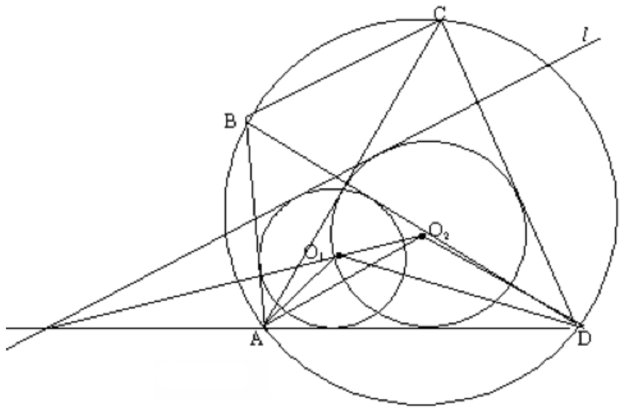
Also solved by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Arkady Alt, San Jose, CA, USA; David Stoner, Harvard University, Cambridge, MA, USA; Ercole Suppa, Teramo, Italy; George Gavrilopoulos, Nea Makri High School, Athens, Greece; Nick Iliopoulos, Music Junior HS, Trikala, Greece; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Robert Bosch, Archimedean Academy, Florida, USA and Jorge Erick López, IMPA, Brazil; Subhadeep Dey, Shyamnagar, West Bengal, India.

S347. Prove that a convex quadrilateral $ABCD$ is cyclic if and only if the common tangent to the incircles of triangles ABD and ACD, different from AD, is parallel to BC.

Proposed by Nairi Sedrakyan, Armenia

Solution by the author

Denote the centers of the circles inscribed in triangles ABD and ACD as O_1 and O_2 respectively, and the common tabgent, different from AD, as l.



Notince that l and AD are symmetric to each other with respect to O_1O_2, thus l and BC are parallel if and only if $\angle CBD - \angle ADB = 2(\angle O_2O_1D - \angle O_1DA)$, i.e.

$$\angle CBD = 2\angle O_2O_1D \tag{1}$$

If quadrilateral $ABCD$ can be inscribed into a circle, then $\angle ABD = \angle ACD$ and $\angle AO_1D = 90^\circ + \frac{1}{2}\angle ABD = 90^\circ + \frac{1}{2}\angle ACD = \angle AO_2D \Rightarrow$ points A, O_1, O_2, D are laying on the same circle. Hence, $\angle CBD = \angle CAD = 2\angle O_2AD = 2\angle O_2O_1D$, i.e. $lPBC$.

Now let us prove that quadrilateral $ABCD$ is circumscribable. Let $\angle CAD = \alpha, \angle CAB = \alpha_1, \angle ABD = \beta, \angle DBC = \beta_1, \angle ACB = \gamma, \angle ACD = \gamma_1, \angle CDB = \delta, \angle BDA = \delta_1$.

According to the Laq of Sines

$$1 = \frac{AB}{BC} \cdot \frac{BC}{CD} \cdot \frac{CD}{DA} \cdot \frac{DA}{AB} = \frac{\sin\gamma}{\sin\alpha_1} \cdot \frac{\sin\delta}{\sin\beta_1} \cdot \frac{\sin\alpha}{\sin\gamma_1} \cdot \frac{\sin\beta}{\sin\delta_1}$$

thus

$$\sin\alpha \sin\ beta \sin\gamma \sin\delta = \sin\alpha_1 \sin\ beta_1 \sin\gamma_1 \sin\delta_1 \tag{2}$$

Therefore, according to (1)

$$\angle O_2O_1D = \frac{\beta_1}{2} \text{ and } \angle AO_2O_1 = \frac{\gamma}{2}$$

On the other side $\angle O_2AD = \frac{\alpha}{2}, \angle O_2AO_1 = \frac{\alpha_1}{2}, \angle AO_1D = 90^\circ + \frac{\beta}{2}, \angle AO_2D = 90^\circ + \frac{\gamma_1}{2}, \angle O_1DA = \frac{\delta_1}{2}, \angle O_2DO_1 = \frac{\delta}{2}$.

Writing down expresison (2) for AO_1O_2D, get

$$\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2}\sin\frac{\delta}{2} = \sin\frac{\alpha_1}{2}\sin\frac{\beta_1}{2}\cos\frac{\gamma_1}{2}\sin\frac{\delta_1}{2} \tag{3}$$

Divide (2) by (3) and get

$$\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2}\cos\frac{\delta}{2} = \cos\frac{\alpha_1}{2}\cos\frac{\beta_1}{2}\sin\frac{\gamma_1}{2}\cos\frac{\delta_1}{2} \tag{4}$$

Let us prove that $\beta = \gamma_1$. Indeed, assume $\beta \neq \gamma_1$, then WLOG we can assume $\beta < \gamma_1$. In this case point C will be located inside the circle circumscribed around triangle ABD. Therefore, $\alpha > \beta_1$ and $\gamma > \delta_1 \Rightarrow$ from (3) we get

$$\cos\frac{\beta}{2}\sin\frac{\delta}{2} < \sin\frac{\alpha_1}{2}\cos\frac{\gamma_1}{2}$$

and from (4) we get

$$\cos\frac{\alpha_1}{2}\sin\frac{\gamma_1}{2} < \cos\frac{\delta}{2}\sin\frac{\beta}{2}$$

Adding up last two expressions we get $\sin\dfrac{\delta-\beta}{2} < \sin\dfrac{\alpha_1-\gamma_1}{2}$, which is impossible since $\delta - \beta = \alpha_1 - \gamma_1$.

Hence $\beta = \gamma_1$ and the conclusion follows.

S348. Find all functions $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ such that for all $a, b, c \in \mathbf{R}$

$$f\left(a^2, f(b,c)+1\right) = a^2(bc+1).$$

Proposed by Mehtaab Sawhney, USA

Solution by Daniel Lasaosa, Pamplona, Spain

For any real x, take $a = b = 1$, $c = x - 1$, or $f(a^2, f(b,c)+1) = x$, and f is surjective. Assume now that $f(b,c) = f(u,v)$ for some $(b,c) \neq (u,v)$. Then, for any nonzero a, we have

$$a^2(bc - uv) = f\left(a^2, f(b,c)+1\right) - f\left(a^2, f(u,v)+1\right) = 0,$$

or if $f(b,c) = f(u,v)$, we have $bc = uv$. Reciprocally, assume that $bc = uv$, or for any nonzero a we have

$$f(a^2, f(b,c)+1) = a^2(bc+1) = a^2(uv+1) = f(a^2, f(u,v)+1),$$

hence by the previous result we have

$$a^2\left(f(b,c)+1\right) = a^2\left(f(u,v)+1\right), \qquad a^2\left(f(b,c) - f(u,v)\right) = 0,$$

and we conclude that $f(b,c) = f(u,v)$ iff $bc = uv$. It then suffices to find $f(1,x) = f(x,1)$ for all real x, since for any $(y,z) \in \mathbb{R}$, $f(y,z) = f(1,yz) = f(yz,1)$. Now, since f is surjective, clearly b, c exist such that $f(b,c) = 0$, and choosing any such pair we have $f(a^2, 1) = a^2$ for any real a, hence $f(1,x) = f(x,1) = x$ for all non-negative real x (this is consistent for $x = 0$ with taking $a = 0$. Now, taking $a = b = 1$, we obtain

$$f\left(1, f(1,c)+1\right) = c+1.$$

Note that this is zero when $c = -1$, and since $f(1,0) = f(0,1) = 0$, by previous results we obtain that $f(b,c) = 0$ iff $bc = 0$. This produces $f(1,-1)+1 = 0$, or $f(1,-1) = -1$. Finally, choose b, c such that $f(b,c) = -2$, which is always possible since f is surjective, or for any real a, we have

$$f\left(a^2, -1\right) = a^2(bc+1).$$

In particular, taking $a = 1$ we have $-1 = f(1,-1) = bc+1$, or $bc = -2$, ie $f(a^2,-1) = -a^2$ for all real a, and $f(-1,x) = -x$ for all non-negative real x, hence $f(x) = x$ for all negative x.

Or $f(b,c) = f(1,bc) = bc$ for all $(b,c) \in \mathbb{R} \times \mathbb{R}$ is the only function that may satisfy the proposed equation, and direct substitution easily yields that it is indeed a solution. The conclusion follows.

Also solved by Théo Lenoir, Institut Saint-Lô, Agneaux, France; Joel Schlosberg, Bayside, NY, USA; Robert Bosch, Archimedean Academy, Florida, USA and Jorge Erick López, IMPA, Brazil.

U343. Evaluate

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+\pi(k)\log n}$$

where $\pi(k)$ denotes the number of prime not exceeding k.

Proposed by Albert Stadler, Herrliberg, Switzerland

Solution by Robert Bosch, Archimedean Academy, Florida, USA and Jorge Erick López, IMPA, Brazil

Let $l_m := \inf_{k\geq m}\pi(k)\frac{\log k}{k}$ and $u_m := \sup_{k\geq m}\pi(k)\frac{\log k}{k}$, then $l_m \longrightarrow 1$ and $u_m \longrightarrow 1$ by the Prime Number Theorem. For $n\geq k\geq m>1$ we have

$$\begin{aligned} n+\pi(k)\log(n) &\geq n+l_m k\frac{\log n}{\log k}\geq n+l_m k,\\ n+\pi(k)\log(n) &\leq n+u_m k+u_m k\frac{\log(n/k)}{\log k}\leq n+u_m k+u_m\frac{e^{-1}n}{\log m}, \end{aligned}$$

using $\frac{k}{n}\log\frac{n}{k}\leq e^{-1}$. Then

$$\begin{aligned} \limsup_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+\pi(k)\log(n)} &= \limsup_{n\to\infty}\sum_{k=m}^{n}\frac{1}{n+\pi(k)\log(n)},\\ &\leq \limsup_{n\to\infty}\sum_{k=m}^{n}\frac{1}{n+l_m k},\\ &= \limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+l_m k/n},\\ &= \int_0^1\frac{dx}{1+l_m x}. \end{aligned}$$

In a similar way we arrive to

$$\liminf_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+\pi(k)\log(n)}\geq\int_0^1\frac{dx}{1+\frac{u_m e^{-1}}{\log m}+u_m x}.$$

Taking $m\to\infty$, the limit is $\int_0^1\frac{dx}{1+x}=\log(1+x)|_0^1=\log 2.$

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy

U344. Evaluate the following sum

$$\sum_{n=0}^{\infty} \frac{3^n \left(2^{3^n-1}+1\right)}{4^{3n}+2^{3n}+1}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Joel Schlosberg, Bayside, NY, USA

For n a nonnegative integer,

$$\frac{3^n(2^{3^n}+2)}{4^{3^n}+2^{3^n}+1} - \frac{3^n}{2^{3^n}-1} = 3^n \cdot \frac{(2^{3^n}+2)(2^{3^n}-1) - 4^{3^n} - 2^{3^n} - 1}{8^{3^n}-1}$$

$$= -\frac{3^{n+1}}{2^{3^{n+1}}-1},$$

so for N a nonnegative integer, by a telescoping sum,

$$\sum_{n=0}^{N} \frac{3^n(2^{3^n-1}+1)}{4^{3^n}+2^{3^n}+1} = \sum_{n=0}^{N} \frac{1}{2}\left(\frac{3^n}{2^{3^n}-1} - \frac{3^{n+1}}{2^{3^{n+1}}-1}\right)$$

$$= \frac{1}{2}\left(1 - \frac{3^{N+1}}{2^{3^{N+1}}-1}\right).$$

Since $x = o(2^x - 1)$,

$$\sum_{n=0}^{\infty} \frac{3^n(2^{3^n-1}+1)}{4^{3^n}+2^{3^n}+1} = \lim_{N\to\infty} \frac{1}{2}\left(1 - \frac{3^{N+1}}{2^{3^{N+1}}-1}\right) = \frac{1}{2}.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Hyun Jin Kim, Stuyvesant High School, New York, NY, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Li Zhou, Polk State College, Winter Haven, FL, USA; Adnan Ali, A.E.C.S-4, Mumbai, India; Arkady Alt, San Jose, CA, USA; G. C. Greubel, Newport News, VA, USA; Moubinool Omarjee, Lycée Henri IV, Paris, France; Robert Bosch, Archimedean Academy, Florida, USA and Jorge Erick López, IMPA, Brazil; Albert Stadler, Herrliberg, Switzerland.

U345. Let R be a ring. We say that the pair $(a,b) \in R \times R$ satisfies property (P) if the unique solution of the equation $axa = bxb$ is $x = 0$. Prove that if (a,b) has property (P) and $a-b$ is invertible, then the equation $axa - bxb = a + b$ has a unique solution in R.

Proposed by Dorin Andrica, "Babes-Bolyai" University, Cluj Napoca, Romania

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

Let (a,b) has property (P) and $(a-b)$ is invertible.

$$\exists c \in \mathbb{R} : c(a-b) = 1 \Leftrightarrow ca - cb = 1$$
$$(a-b)c = 1 \Leftrightarrow ac - bc = 1$$

Since prove that c is solution of $axa - bxb = a + b$.

$$aca - bcb = (1 + bc)a - bcb = a + bca - bcb = a + bc(a-b) = a + b.$$

Now prove that this solution is unique. Assume by contradiction there exists 2 solution.

$$ax_1a - bx_1b = ax_2a - bx_2b \Rightarrow a(x_1 - x_2)a = b(x_1 - x_2)b$$

(a,b) has property (P), hence $x_1 - x_2 = 0$. Hence we have contradiction. Thus

$$axa - bxb = a + b$$

equation has unique solution.

Also solved by Daniel Lasaosa, Pamplona, Spain; Shohruh Ibragimov, National University of Uzbekistan, Uzbekistan; Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; David Stoner, Harvard University, Cambridge, MA, USA; Joel Schlosberg, Bayside, NY, USA; Robert Bosch, Archimedean Academy, Florida, USA and Jorge Erick López, IMPA, Brazil.

U346. Let $f : [0,1] \to \mathbb{R}$ be a twice differentiable function for which $[f'(x)]^2 + f(x)f''(x) \geq 1$. Prove that

$$\int_0^1 f^2(x)dx \geq f^2\left(\frac{1}{2}\right) + \frac{1}{12}.$$

Proposed by Marcel Chirita, Bucharest, Romania

Solution by Li Zhou, Polk State College, USA

Let $g(x) = f^2(x)$. Then $g''(x) = 2\left([f'(x)]^2 + f(x)f''(x)\right) \geq 2$ for all $x \in (0,1)$. By Taylor's theorem, for each $x \in [0,1]$, there exists $c(x)$ between x and $\frac{1}{2}$ such that

$$g(x) = g\left(\frac{1}{2}\right) + g'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) + \frac{1}{2}g''(c(x))\left(x - \frac{1}{2}\right)^2.$$

Hence,

$$\begin{aligned}\int_0^1 f^2(x)dx &= \int_0^1 g(x)dx = g\left(\frac{1}{2}\right) + g'\left(\frac{1}{2}\right)\int_0^1\left(x - \frac{1}{2}\right)dx + \frac{1}{2}\int_0^1 g''(c(x))\left(x - \frac{1}{2}\right)^2 dx \\ &\geq g\left(\frac{1}{2}\right) + 0 + \int_0^1\left(x - \frac{1}{2}\right)^2 dx = f^2\left(\frac{1}{2}\right) + \frac{1}{12}.\end{aligned}$$

Also solved by Daniel Lasaosa, Pamplona, Spain; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Brian Bradie, Christopher Newport University, Newport News, VA, USA; George Gavrilopoulos, Nea Makri High School, Athens, Greece; Joel Schlosberg, Bayside, NY, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Robert Bosch, Archimedean Academy, Florida, USA and Jorge Erick López, IMPA, Brazil; Shohruh Ibragimov, National University of Uzbekistan, Uzbekistan; Albert Stadler, Herrliberg, Switzerland; Stanescu Florin, Serban Cioculescu School, Gaesti, Dambovita, Romania.

U347. Find all differentiable functions $f:[0,\infty)\to\mathbf{R}$ such that $f(0)=0, f'$ is increasing and for all $x\geq 0$

$$x^2f'(x)=f^2(f(x))$$

Proposed by Stanescu Florin, Gaesti, Romania

Solution by Robert Bosch, Archimedean Academy, Florida, USA and Jorge Erick López, IMPA, Brazil

$f(x)=0$ and $f(x)=x$ are the only solutions. By the given equation $f'(x)\geq 0$, but $f(0)=0$, then $f(x)$ is a positive and increasing function, thus the same remains valid for the function $f\circ f(x)$. If $(f\circ f)(x)=0$, by the given equation $f'(x)=0$ with $f(0)=0$, then $f(x)=0$. If $f\circ f\not\equiv 0$, from $f(f(0))=0$ and $(f\circ f)(x)$ increasing, $a=\sup\{x, f(f(x))=0\}$ is well-defined and finite with $f(f(a))=0$ and $f(f(x))>0$ for all $x>a$. For $x>a$

$$\left(\frac{1}{f(f(x))}-\frac{1}{x}\right)'=\frac{1}{x^2}-\frac{f'(x)}{(f(f(x)))^2}=0,$$

therefore $f(f(x))=\dfrac{x}{1+cx}$ (for some constant c) for all $x\geq a$, hence $c\geq 0$ and $a=0$, since $f(f(a))=0$. Substituting in the original equation we have $f'(x)=\dfrac{1}{(1+cx)^2}$ for all $x\geq 0$, but $f'(x)$ is an increasing function, so $c=0$ and $f'(x)=1$ with $f(0)=0$, and finally $f(x)=x$ for all $x\geq 0$.

U348. Evaluate the linear integral

$$\oint_c \frac{(1+x^2-y^2)\,dx + 2xy\,dy}{(1+x^2-y^2)^2+4x^2y^2}$$

where c is the square [sic] with vertices $(2,0)$, $(2,2)$, $(-2,2)$, and $(-2,0)$ traversed counterclockwise.

Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

Solution by José Hernández Santiago, México

The integral under consideration is equal to the real part of the following complex integral

$$\oint_c \frac{1}{z^2+1}\,dz. \tag{1}$$

Indeed, if we write z in the form $x+iy$, where $x, y \in \mathbb{R}$, then

$$\begin{aligned} \mathrm{Re}\left(\oint_c \frac{1}{z^2+1}\,dz\right) &= \mathrm{Re}\left(\oint_c \frac{1}{(1+x^2-y^2)+2ixy}\,(dx+idy)\right) \\ &= \mathrm{Re}\left(\oint_c \frac{(1+x^2-y^2)-2ixy}{(1+x^2-y^2)^2+4x^2y^2}\,(dx+idy)\right) \\ &= \oint_c \frac{(1+x^2-y^2)\,dx+2xy\,dy}{(1+x^2-y^2)^2+4x^2y^2}. \end{aligned}$$

Thus, in order to solve the problem it suffices to evaluate the integral in (1). This can be done by means of the Cauchy Integral Formula. Let U be any region of the complex plane containing c. If $-i \in \mathbb{C}\setminus U$, then the function $f\colon U \to \mathbb{C}$ determined by the assignation $z \to \frac{1}{z+i}$ is holomorphic on U; the Cauchy Integral Formula gives us in this case that

$$\oint_c \frac{1}{z^2+1}\,dz = \oint_c \frac{\frac{1}{z+i}}{z-i}\,dz = 2\pi i f(i) = \pi.$$

Hence,

$$\oint_c \frac{(1+x^2-y^2)\,dx+2xy\,dy}{(1+x^2-y^2)^2+4x^2y^2} = \mathrm{Re}\left(\oint_c \frac{1}{z^2+1}\,dz\right) = \pi$$

and we are done.

Also solved by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Li Zhou, Polk State College, Winter Haven, FL, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Robert Bosch, Archimedean Academy, Florida, USA and Jorge Erick López, IMPA, Brazil.

O343. Let $a_1, \ldots, a_n$ be positive real numbers such that

$$\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n} = a_1 + a_2 + \cdots + a_n.$$

Prove that

$$\sqrt{{a_1}^2 + 1} + \sqrt{{a_2}^2 + 1} + \cdots + \sqrt{{a_n}^2 + 1} \leq n\sqrt{2}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Li Zhou, Polk State College, USA

Let $A = \sqrt{a_1} + \cdots + \sqrt{a_n} = a_1 + \cdots + a_n$. By the Cauchy-Schwarz inequality,

$$\begin{aligned} & \sqrt{a_1^2 + 1} + \cdots + \sqrt{a_n^2 + 1} \\ = & \sqrt{(a_1 + \sqrt{2a_1} + 1)(a_1 - \sqrt{2a_1} + 1)} + \cdots + \sqrt{(a_n + \sqrt{2a_n} + 1)(a_n - \sqrt{2a_n} + 1)} \\ \leq & \sqrt{(A + \sqrt{2}A + n)(A - \sqrt{2}A + n)} = \sqrt{2n^2 - (n - A)^2} \leq n\sqrt{2}. \end{aligned}$$

Also solved by Rade Krenkov, SOUUD, Dimitar Vlahov, Strumica, Macedonia; Daniel Lasaosa, Pamplona, Spain; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Adnan Ali, A.E.C.S-4, Mumbai, India; David Stoner, Harvard University, Cambridge, MA, USA; Ercole Suppa, Teramo, Italy; George Gavrilopoulos, Nea Makri High School, Athens, Greece; Joel Schlosberg, Bayside, NY, USA; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Robert Bosch, Archimedean Academy, Florida, USA and Jorge Erick López, IMPA, Brazil.

O344. Consider the sequence $a_n = \left[\left(\sqrt[3]{65} - 4 \right)^{-n} \right]$, where $n \in \mathbb{N}^*$. Prove that $a_n \equiv 2, 3 \pmod{15}$.

Proposed by Vlad Matei, University of Wisconsin, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Denote $r = \sqrt[3]{65}$ and $u = \left(\sqrt[3]{65} - 4 \right)^{-1} = \sqrt[3]{65}^2 + 4\sqrt{65} + 16 = r^2 + 4r + 16$. Note that

$$u^2 = 48r^2 + 193r + 776, \qquad u^3 = 2316r^2 + 9312r + 37441,$$

or u is one of the roots of equation

$$x^3 - 48x^2 - 12x - 1 = 0,$$

and denoting by v, w the other two, we have $v + w = 48 - u = 32 - 4r - r^2 = -12(r-4) - (r-4)^2$ and $vw = \frac{1}{u} = r - 4$. Note first that

$$r^3 = 4^3 + 1 = 4^3 + \frac{3 \cdot 4^2}{48} < \left(4 + \frac{1}{48} \right)^3,$$

or $0 < r - 4 < \frac{1}{48}$, hence $0 > v + w > -1$ and $1 > vw > 0$, or $v, w < 0$, and for every positive integer n,

$$0 < |v^n + w^n| = |v^n| + |w^n| \leq |v| + |w| < 1,$$

or $1 > v^n + w^n > 0$ iff n is even and $0 > v^n + w^n > -1$ iff n is odd. Note next that

$$v^2 + w^2 = (v+w)^2 - 2vw = 1552 - 193r - 48r^2,$$

$$v^3 + w^3 = (v+w)^3 - 3vw(v+w) = 74882 - 9312r - 2316r^2,$$

for

$$u + v + w = 48 \equiv 3 \pmod{15}, \qquad u^2 + v^2 + w^2 = 2328 \equiv 3 \pmod{15},$$

$$u^3 + v^3 + w^3 = 112323 \equiv 3 \pmod{15}.$$

Now, sequence $(b_n)_{n \geq 1}$ defined by $b_{n+3} = 48b_{n+2} + 12b_{n+1} + b_n$, with initial conditions $b_1 = 48$, $b_2 = 2328$, $b_3 = 112323$, has characteristic equation with roots u, v, w, or $b_n = u^n + v^n + w^n$ for $n \in \mathbb{N}^*$. Clearly all b_n are integers, and $b_n \equiv 3 \pmod{15}$ for all $n \in \mathbb{N}^*$, since $b_1 \equiv b_2 \equiv b_3 \equiv 3 \pmod{15}$, and by $b_{n+3} \equiv 3\left(b_{n+2} - b_{n+1}\right) + b_n \pmod{15}$. We therefore conclude that, for all even $n \in \mathbb{N}^*$, we have

$$a_n = [u^n] = u^n + v^n + w^n - 1 \equiv 2 \pmod{15},$$

while for all odd $n \in \mathbb{N}^*$, we have

$$a_n = [u^n] = u^n + v^n + w^n \equiv 3 \pmod{15}.$$

Therefore, we conclude that $a_n \equiv 2, 3 \pmod{15}$.

Also solved by Robert Bosch, Archimedean Academy, Florida, USA and Jorge Erick López, IMPA, Brazil; Li Zhou, Polk State College, Winter Haven, FL, USA.

O345. Let A_1, B_1, C_1 be points on the sides BC, CA, AB of a triangle ABC. Let r_A, r_B, r_C, r_1 be the inradii of triangles $AB_1C_1, BC_1A_1, CA_1B_1$ and $A_1B_1C_1$ respectively. Prove that

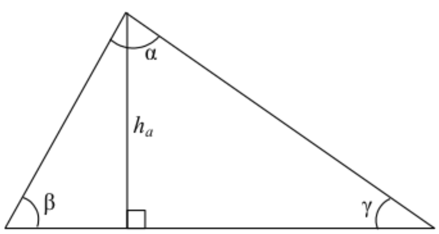
$$R_1 r_1 \geq 2\min(r_A r_B, r_B r_C, r_C r_A),$$

where R_1 is the circumcircle of triangle $A_1B_1C_1$.

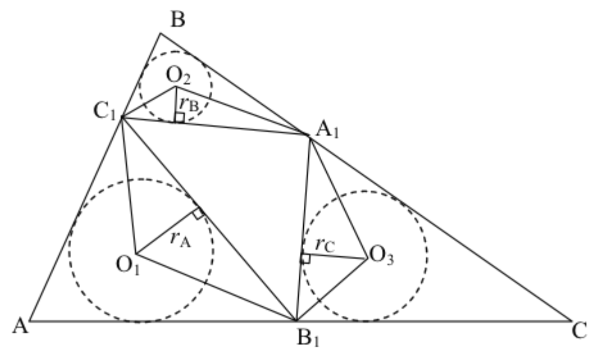
Proposed by Nairi Sedrakyan, Armenia

Solution by the author

First we prove that for any triangle $h_a \leq \frac{a}{2}\cot\frac{\alpha}{2}$.



(a)



(b)

Indeed,

$$\frac{h_a}{a} = \frac{\sin\beta\sin\gamma}{\sin\alpha} = \frac{\cos(\beta-\gamma)-\cos(\beta+\gamma)}{\sin\alpha} \leq \frac{1+\cos\alpha}{\sin\alpha} = \frac{1}{2}\cot\frac{\alpha}{2}$$

$$\Rightarrow \frac{h_a}{a} \leq \frac{1}{2}\cot\frac{\alpha}{2} \qquad (1)$$

Denote the centers of circumscribed triangles $AB_1C_1, BC_1A_1, CA_1B_1$ as O_1, O_2, O_3 respectively. Let $u = \frac{1}{2}\angle B_1O_1C_1, v = \frac{1}{2}\angle C_1O_2A_1, w = \frac{1}{2}\angle A_1O_3B_1$. Notice that $u = \frac{\pi}{4} + \frac{\angle A}{4}, v = \frac{\pi}{4} + \frac{\angle B}{4}, w = \frac{\pi}{4} + \frac{\angle C}{4}$, therefore $u + v + w = \pi$ and

$$\cot u \cdot \cot v + \cot v \cdot \cot w + \cot u \cdot \cot w = 1 \qquad (2)$$

Apply (1) and (2) for triangles $B_1O_1C_1, C_1O_2A_1, A_1O_3B_1$ and get

$$\frac{2r_A}{B_1C_1}\cdot\frac{2r_B}{A_1C_1} + \frac{2r_B}{A_1C_1}\cdot\frac{2r_C}{A_1B_1} + \frac{2r_C}{A_1B_1}\cdot\frac{2r_A}{B_1C_1} \leq 1.$$

Therefore,

$$2\min(r_Ar_B, r_Br_C, r_Cr_A) \leq \frac{A_1B_1\cdot B_1C_1\cdot A_1C_1}{2(A_1B_1+B_1C_1+A_1C_1)} = \frac{4R_1S_{A_1B_1C_1}}{2\cdot\frac{2S_{A_1B_1C_1}}{r_1}} = R_1r_1$$

and the conclusion folllows.

Also solved by Robert Bosch, Archimedean Academy, Florida, USA and Jorge Erick López, IMPA, Brazil.

O346. Define the sequence $(a_n)_{n\geq 0}$ by $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, $a_3 = 6$ and

$$a_{n+4} = 2a_{n+3} + a_{n+2} - 2a_{n+1} - a_n, \quad n \geq 0.$$

Prove that n^2 divides a_n for infinitely many positive integers.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj Napoca, Romania

Solution by Li Zhou, Polk State College, USA

Let (F_n) be the Fibonacci sequence, that is, $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Notice that $a_n = nF_n$ for $n = 0, 1, 2, 3$. As an induction hypothesis, assume that $k \geq 3$ and $a_n = nF_n$ for all $n = 0, 1, \dots, k$. Then

$$\begin{aligned} a_{k+1} &= 2kF_k + (k-1)F_{k-1} - 2(k-2)F_{k-2} - (k-3)F_{k-3} \\ &= 2kF_k + 2F_{k-1} - (k-1)F_{k-2} = (k+1)F_k + (k+1)F_{k-1} = (k+1)F_{k+1}, \end{aligned}$$

completing the induction step. Hence $a_n = nF_n$ for all $n \geq 0$. Thus, it suffices to show that $n|F_n$ for infinitely many n.

Indeed, in the solutions to U316, Math. Reflections, **5** (2014), it is shown that $2^{m+2}|F_{3\cdot 2^m}$ for all $m \geq 1$. Also, for $m \geq 2$, $144 = F_{(12,3\cdot 2^m)} = (F_{12}, F_{3\cdot 2^m})$, so $3|F_{3\cdot 2^m}$. Therefore $n|F_n$ for all $n = 3 \cdot 2^m$ with $m \geq 2$.

Also solved by Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Antoine Faisant; Arkady Alt, San Jose, CA, USA; Ercole Suppa, Teramo, Italy; G. C. Greubel, Newport News, VA, USA; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Paul Revenant, Lycée Champollion, Grenoble, France; Robert Bosch, Archimedean Academy, Florida, USA and Jorge Erick López, IMPA, Brazil.

O347. Let $a,b,c,d \ge 0$ be real numbers such that $a+b+c+d=1$. Prove that

$$\sqrt{a+\frac{2(b-c)^2}{9}+\frac{2(c-d)^2}{9}+\frac{2(d-b)^2}{9}}+\sqrt{b}+\sqrt{c}+\sqrt{d}\le 2$$

Proposed by Marius Stanean, Zalau, Romania

Solution by the author

First we prove the following Lemma.

Lemma: Let x,y,z be non-negative real numbers. Prove that

$$2\sum_{cyc}(y^2-z^2)^2 \le 3(x^2+y^2+z^2)\sum_{cyc}(y-z)^2.$$

Proof: Since the inequality is symmetric and homogeneous, WLOG we assume that $x+y+z=3$ and for an easy computing, denote $q=xy+yz+zx$. Rewrite the inequality to the following forms

$$2(x^4+y^4+z^4-x^2y^2-y^2z^2-z^2x^2)\le 3(x^2+y^2+z^2)(x^2+y^2+z^2-xy-yz-zx) \iff$$
$$2[(x^2+y^2+z^2)^2-3(x^2y^2+y^2z^2+z^2x^2)]\le 3(9-2q)(9-3q) \iff$$
$$2[(9-2q)^2-3(q^2-6xyz)]\le 3(9-2q)(9-3q) \iff$$
$$36xyz \le 81-63q+16q^2.$$

Now, since $3q\le (x+y+z)^2=9 \iff q\le 3$, there are a real number $t\in[0,1]$ such that $q=3(1-t^2)$. Therefore, the inequality becomes

$$4xyz \le 16t^4-11t^2+4. \tag{1}$$

Now we have $xyz\le(1-t^2)(1+2t)=1-3t^2+2t^3$ Returning back to (1) we only need to prove that

$$4-12t^2+8t^3\le 16t^4-11t^2+4 \iff$$
$$t^2(16t^2-8t+1)\ge 0 \iff t^2(4t-1)^2\ge 0$$

which is obviously true. Equality holds for $t=0$ which means $x=y=z$ or for $t=\frac{1}{4}$ which means $2x=2y=z$.

Let us now return to out inequality. According to Lemma we have

$$2\sum_{cyc}(b-c)^2\le 3(b+c+d)\sum_{cyc}(\sqrt{b}-\sqrt{c})^2$$

but $b+c+d\le 1$, so

$$\frac{2}{9}\sum_{cyc}(b-c)^2\le\frac{1}{3}\sum_{cyc}(\sqrt{b}-\sqrt{c})^2=b+c+d-\frac{(\sqrt{b}+\sqrt{c}+\sqrt{d})^2}{3}. \tag{2}$$

Applying the Cauchy-Schwarz inequality we have

$$\left(\sqrt{a+\frac{2(b-c)^2}{9}+\frac{2(c-d)^2}{9}+\frac{2(d-b)^2}{9}}+\sqrt{b}+\sqrt{c}+\sqrt{d}\right)^2\le$$
$$\left(a+\frac{2(b-c)^2}{9}+\frac{2(c-d)^2}{9}+\frac{2(d-b)^2}{9}+\frac{(\sqrt{b}+\sqrt{c}+\sqrt{d})^2}{3}\right)(1+3)\overset{(2)}{\le}$$
$$4(a=b=c=d=\frac{1}{4})$$

Equality holds when $a=b=c=d=\frac{1}{4}$

Also solved by Robert Bosch, Archimedean Academy, Florida, USA and Jorge Erick López, IMPA, Brazil

O348. Let $ABCDE$ be a convex pentagon with area S, and let R_1, R_2, R_3, R_4, R_5 be the circumradii of triangles ABC, BCD, CDE, DEA, EAB, respectively. Prove that

$$R_1^4 + R_2^4 + R_3^4 + R_4^4 + R_5^4 \geq \frac{4}{5\sin^2 108^\circ} S^2.$$

Proposed by Nairi Sedrakyan, Armenia

Solution by Robert Bosch, Archimedean Academy, Florida, USA and Jorge Erick López, IMPA, Brazil

First, let us only consider the triangle ABC. Let O_1 be its circumcenter, and M_1, M_2 the midpoints of AB and BC respectively. Denote the area of the quadrilateral $O_1M_1BM_2$ by S_1. We have the following inequality

$$S_1 \leq \frac{BC \cdot R}{4}.$$

The proof is as follows,

$$\begin{aligned}
\frac{1}{4}R^2(\sin(2A) + \sin(2C)) &\leq \frac{AC \cdot R}{4}, \\
\sin(2A) + \sin(2C) &\leq \frac{AC}{R}, \\
\sin(2A) + \sin(2C) &\leq 2\sin(B), \\
2\sin(A + C)\cos(A - C) &\leq 2\sin(B), \\
\cos(A - C) &\leq 1.
\end{aligned}$$

Clearly, $S \leq S_1 + S_2 + S_3 + S_4 + S_5$, suppose without loss of generality that $\min\{R_1, R_2, R_3, R_4, R_5\} = R_1$ and move clockwise. Thus

$$2S \leq R_5^2 \sin A + R_1^2 \sin B + R_2^2 \sin C + R_3^2 \sin D + R_4^2 \sin E.$$

Now, by Cauchy-Schwarz inequality it follows

$$4S^2 \leq \left(\sum_{i=1}^{5} R_i^4\right)\left(\sin^2 A + \sin^2 B + \sin^2 C + \sin^2 D + \sin^2 E\right).$$

To finish let us prove that

$$\sin^2 A + \sin^2 B + \sin^2 C + \sin^2 D + \sin^2 E \leq 5\sin^2(108^\circ).$$

We know that $A+B+C+D+E = 540^\circ$. Assuming $0 < A \leq B \leq C \leq D \leq E < 180^\circ$ we obtain if $A = 108^\circ$ then $B = C = D = E = 108^\circ$. $A < 108^\circ$ since if $A > 108^\circ$ then $A+B+C+D+E > 540^\circ$ a contradiction. In a similar way $E > 108^\circ$. Notice that $A+E < 270^\circ$ since if $A+E \geq 270^\circ$ then $B+C+D \leq 270^\circ$, so $B \leq 90^\circ$ and hence $A \leq 90^\circ$, therefore $E \geq 180^\circ$, a contradiction. By the identity $\sin^2 x - \sin^2 y = \sin(x-y)\sin(x+y)$ we obtain

$$\sin^2 A + \sin^2 E < \sin^2(108^\circ) + \sin^2(A + E - 108^\circ).$$

Let us see the proof. The difference $\sin^2(108^\circ) + \sin^2(A + E - 108^\circ) - \sin^2 A - \sin^2 E$ is positive due to

$$\begin{aligned}
& \sin^2(108^\circ) - \sin^2 A + \sin^2(A + E - 108^\circ) - \sin^2 E, \\
= \; & \sin(108^\circ - A)\sin(108^\circ + A) + \sin(A - 108^\circ)\sin(A + 2E - 108^\circ), \\
= \; & \sin(A - 108^\circ)\left(\sin(A + 2E - 108^\circ) - \sin(108^\circ + A)\right), \\
= \; & 2\cos(A + E)\sin(A - 108^\circ)\sin(E - 108^\circ) > 0,
\end{aligned}$$

since $90^\circ < A + E < 270^\circ$. This means that by the replacement of A with 108° and of E with $A + E - 108^\circ$ we increase the sum of the squared sinuses. Repeating this operation, we will make all the angles equal to 108°, and the inequality is proved.