Junior problems

J349. Prove that for each positive integer n, $6^{n+1} + 8^{n+1} + 27^n - 1$ has at least 11 proper positive divisors.

Proposed by Titu Andreescu, The University of Texas at Dallas, USA

J350. Let a, b, c be positive real numbers such that $ab + bc + ca = 1$. Prove that

$$\sqrt{a^4 + b^2} + \sqrt{b^4 + c^2} + \sqrt{c^4 + a^2} \geq 2.$$

Proposed by Titu Zvonaru, Comănești, România

J351. Find the sum of all six-digit positive integers such that if a and b are adjacent digits of such an integer, then $|a - b| \geq 2$.

Proposed by Neelabh Deka, India

J352. Let ABC be a triangle and let D be a point on side AC such that $\frac{1}{3}\angle BCA = \frac{1}{4}\angle ABD = \angle DBC$, and $AC = BD$. Find the angles of triangle ABC.

Proposed by Marius Stănean, Zalău, România

J353. Let a, b, c be nonnegative real numbers and let

$$\begin{aligned} A &= \frac{1}{4a+1} + \frac{1}{4b+1} + \frac{1}{4c+1}, \\ B &= \frac{1}{3a+b+1} + \frac{1}{3b+c+1} + \frac{1}{3c+a+1}, \\ C &= \frac{1}{2a+b+c+1} + \frac{1}{2b+c+a+1} + \frac{1}{2c+a+b+c}. \end{aligned}$$

Prove that $A \geq B \geq C$.

Proposed by Nguyen Viet Hung, Hanoi, Vietnam

J354. Evaluate

$$\sum_{n\geq 1} \frac{3n+1}{2n+1} \binom{2n}{n}^{-1}.$$

Proposed by Cody Johnson, Carnegie Mellon University, USA

Senior problems

S349. Each face of eight unit cubes is colored in one of the k colors, where $k \in \{2, 3, 4, 5, 6, 8, 12, 24\}$, so that there are $\frac{48}{k}$ faces of each color. Prove that from these unit cubes, we can assemble a 2×2 cube that has on its surface equal amount of squares of each color.

Proposed by Nairi Sedrakyan, Armenia

S350. Let a_1, a_2, ..., a_{15} be positive integers such that

$$(a_1 + 1)(a_2 + 1) \cdots (a_{15} + 1) = 2015 a_1 a_2 \cdots a_{15}.$$

Prove that there are at least six and at most ten numbers among a_1, a_2, ..., a_{15} that are equal to 1.

Proposed by Titu Zvonaru, Comănești and Neculai Stanciu, Buzău, România

S351. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$a + b + c \geq \frac{1}{a(b+1)} + \frac{1}{b(c+1)} + \frac{1}{c(a+1)} + \frac{3}{2}.$$

Proposed by Nguyen Viet Hung, Hanoi, Vietnam

S352. In the triangle ABC, let ω denote its Brocard angle, and let φ satisfy the identity

$$\tan \varphi = \tan A + \tan B + \tan C.$$

Prove that

$$\frac{\cos 2A + \cos 2B + \cos 2C}{\sin 2A + \sin 2B + \sin 2C} = -\frac{1}{4} \left(\cot \omega + 3 \cot \varphi \right).$$

Proposed by Oleg Faynshteyn, Leipzig, Germany

S353. Let a, b, c, x, y be positive real numbers such that $xy \geq 1$. Prove that

$$\frac{ab}{xa + yb + 2c} + \frac{bc}{xb + yc + 2a} + \frac{ca}{xc + ya + 2b} \leq \frac{a + b + c}{x + y + 2}$$

Proposed by Yong Xi Wang

S354. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x, y,

$$(f(x + y))^2 = (f(x))^2 + 2f(xy) + (f(y))^2.$$

Proposed by Oleksiy Klurman, Université de Montréal, Canada

Undergraduate problems

U349. Let $0 < x,\ y,\ z < 1$. Prove that

$$\frac{1}{1-x^4}+\frac{1}{1-y^4}+\frac{1}{1-z^4}+\frac{1}{1-x^2yz}+\frac{1}{1-y^2zx}+\frac{1}{1-z^2xy}\geq$$
$$\frac{1}{1-x^3y}+\frac{1}{1-xy^3}+\frac{1}{1-y^3z}+\frac{1}{1-yz^3}+\frac{1}{1-x^3z}+\frac{1}{1-xz^3}.$$

Proposed by Mehtaab Sawhney, Commack High School, New York, USA

U350. Let a and b be real numbers such that $a \geq 1$ and $b > a^2 - a + 1$. Prove that the equation $x^5 - ax^3 + a^2x - b = 0$ has a unique real solution x_0, and $2b - a^3 < x_0^6 < b^2 + a - a^3$.

Proposed by Corneliu Mănescu-Avram, Ploești, România

U351. Let $a \geq 0$. Evaluate

$$\lim_{n\to\infty}\frac{1}{2^n n^a}\sum_{k=0}^{n}\binom{n}{k}k^a.$$

Proposed by Albert Stadler, Herrliberg, Switzerland

U352. Evaluate

$$\sum_{n=1}^{\infty}\frac{n-1}{\binom{2n}{n}}.$$

Proposed by Cody Johnson, Carnegie Mellon, University, USA

U353. Let $a \in \mathbb{R}$ and let $f : (-1,\ 1) \to \mathbb{R}$ be a function differentiable at 0. Evaluate

$$\lim_{n\to\infty}\left[an-\sum_{k=1}^{n}f\left(\frac{k}{n^2}\right)\right].$$

Proposed by Dorin Andrica, Babeș-Bolyai University, România

U354. Let $f, g : [-1,\ 1] \to \mathbb{R}$ be increasing functions. Prove that if $f(-x) = -f(x)$, for all $x \in [-1,\ 1]$, then

$$\int_{-1}^{1} f(x)g(x)dx \geq 0.$$

Proposed by Marcel Chiriță, Bucharest, România

Olympiad problems

O349. Find all positive integers n such that

$$\sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor$$

is an even integer.

Proposed by Dorin Andrica, Babeș-Bolyai University, România

O350. Find all triples (x, y, z) of integers satisfying the equation $x^3 + 3xy + y^3 = 2^z + 1$.

Proposed by Titu Andreescu, The University of Texas at Dallas, USA

O351. Let ABC be a triangle with $\angle ABC = 60^\circ$ and $\angle BCA = 70^\circ$, and let point D lie on side BC. Prove that $\angle BAD = 20^\circ$ if and only if $AB + BD = AD + DC$.

Proposed by Mircea Lascu and Titu Zvonaru, România

O352. Solve in positive integers the system of equations

$$\begin{cases} \dfrac{1}{x} + \dfrac{2}{y} + \dfrac{3}{z} = 1 \\ x + 2y + 3z = \dfrac{50yz}{8 + yz}. \end{cases}$$

Proposed by Titu Andreescu, The University of Texas at Dallas, USA

O353. Let a, b, c, d be nonnegative real nubers such that $a \geq b \geq 1 \geq c \geq d$ and $a + b + c + d = 4$. Prove that $4(a^2 + b^2 + c^2 + d^2) \geq 12 + a^3 + b^3 + c^3 + d^3$.

Proposed by Marius Stănean, Zalău, România

O354. Find all primes p such that

$$\frac{p^2}{1 + \frac{1}{2} + \cdots + \frac{1}{p-1}}$$

is an integer.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA