Solving Some Problems Using the Mean Value Theorem

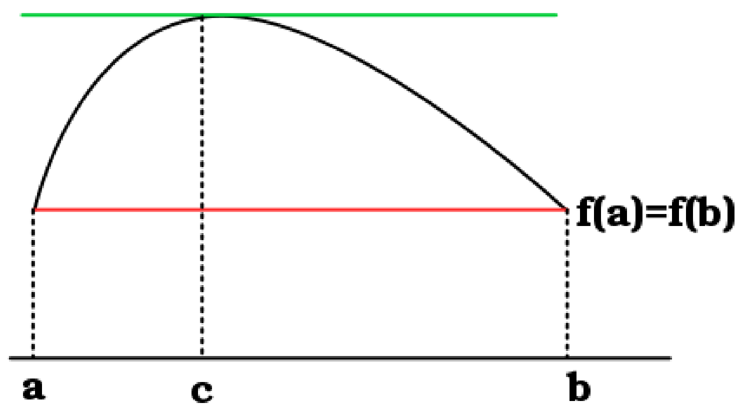
Phu Cuong Le Van-Senior
College of Education
Hue University, Vietnam

1 Introduction

Mean value theorems play an important role in analysis, being a useful tool in solving numerous problems. Before we approach problems, we will recall some important theorems that we will use in this paper.

Theorem 1.1. *(Rolle's theorem) Let* $f : [a,b] \to \mathbb{R}$ *be a continuous function on* $[a,b]$*, differentiable on* (a,b) *and such that* $f(a) = f(b)$*. Then, there is a point* $c \in (a,b)$ *such that* $f'(c) = 0$*.*

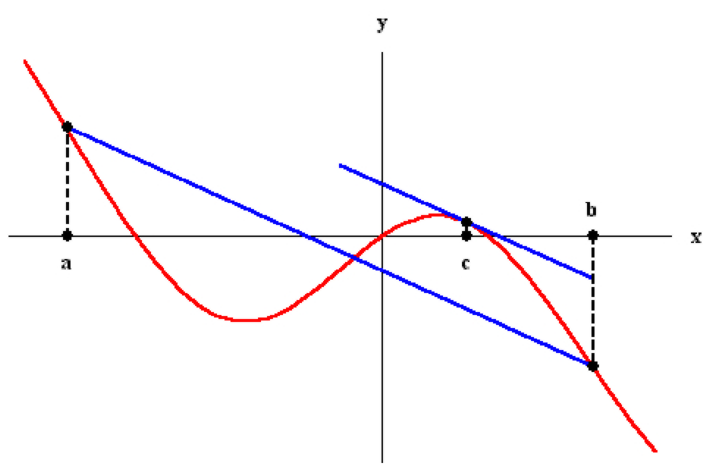
We can see its geometric meaning as follows:



Theorem 1.2. *(Lagrange's theorem) Let* $f : [a,b] \to \mathbb{R}$ *be a continuous function on* $[a,b]$ *and differentiable on* (a,b)*. Then, there is a point* $c \in (a,b)$ *such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

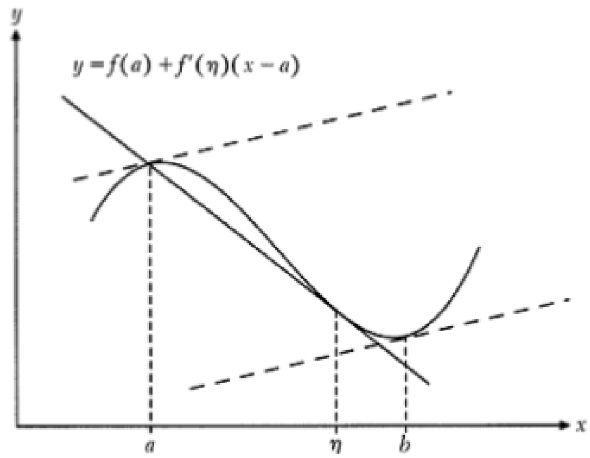
The geometric meaning:



Theorem 1.3. *(Flett's theorem) Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$, differentiable on (a, b), and such that $f'(a) = f'(b)$. Then, there is $\eta \in (a, b)$ such that*

$$f'(\eta) = \frac{f(\eta) - f(a)}{\eta - a}.$$

The geometric meaning:



The proof of these problems can be found in just about any Calculus textbook.

2 Main

In this section, we will solve some problems. Through solutions, we can find ideas or techniques to solve other problems or maybe create new ones. All functions considered in this section are real-valued.

We start with a simple problem.

Problem 2.1. *Let $f : [0, 1] \to \mathbb{R}$ be a continuous function on $[0, 1]$, differentiable on $(0, 1)$, and such that $f'(x) - f(x) \geq 0$, $\forall\, x \in [0, 1]$ and $f(0) = 0$. Prove that $f(x) \geq 0$, $\forall\, x \in [0, 1]$.*

Proof. Let $g(x) = e^{-x} f(x)$, then $g'(x) = e^{-x} \cdot (f'(x) - f(x)) \geq 0$, $\forall x \in [0, 1]$, hence g is increasing on $[0, 1]$. Thus, $g(x) \geq g(0) = 0$, $\forall x \in [0, 1]$ and the conclusion follows. □

Comment 2.2. *Why did we let $g(x) = e^{-x} f(x)$, involving the integrant factor e^{-x}? Because looking at $f'(x) - f(x) \geq 0$, we consider the equality $f'(x) - f(x) = 0 \iff f' = f \iff \frac{f'}{f} = 1$; by integrating both sides, we get $ln|f| = x + C$, and choosing C=0, we obtain $|f| = e^x \iff e^{-x}|f| = 1$, this is why we let $g(x) = e^{-x} f(x)$.*

Using this idea, we can solve more problems.

Problem 2.3. *Let $f : [0, 1] \to \mathbb{R}$ be a continuous function on $[0, 1]$ satisfying $\int\limits_0^1 f(x)dx = 0$. Prove that there is $c \in (0, 1)$ such that*

$$f(c) = \int\limits_0^c f(x)dx.$$

Proof.

Idea 2.4. *From* $f(c) = \int\limits_0^c f(x)dx$, *we consider* $f(x) = \int\limits_0^x f(t)dt$. *Let* $F(x) = \int\limits_0^x f(t)dt$. *Then* $f(x) = \int\limits_0^x f(t)dt \iff F'(x) = F(x) \iff \frac{F'(x)}{F(x)} = 1$; *by integrating both sides we obtain* $ln|F(x)| = x + C$, *and choosing C=0, we get* $|F(x)| = e^x \iff e^{-x}|F(x)| = 1$.
So we will let $g(x) = e^{-x}\int\limits_0^x f(t)dt$.

Returning to our problem:
Letting $g(x) = e^{-x}\int\limits_0^x f(t)dt$, g is continuous on $[0,1]$ and differentiable on (0, 1). We have $g(0) = g(1) = 0$, so by Rolle's theorem there is $c \in (0,1)$ such that $g'(c) = 0 \iff f(c) = \int\limits_0^c f(x)dx$. □

Problem 2.5. *Let* $f : [0,1] \to \mathbb{R}$ *be a continuous function [0, 1] satisfying* $\int\limits_0^1 f(x)dx = 0$. *Prove that there is* $c \in (0,1)$ *such that*

$$(1-c)f(c) = c\int\limits_0^c f(x)dx.$$

Proof.

Idea 2.6. *From* $(1-c)f(c) = c\int\limits_0^c f(x)dx$, *we consider* $(1-x)f(x) = x\int\limits_0^x f(t)dt$. *Let* $F(x) = \int\limits_0^x f(t)dt$. *Then* $(1-x)f(x) = x\int\limits_0^x f(x)dx \iff (1-x)F'(x) = xF(x) \iff \frac{x}{1-x} = \frac{F'(x)}{F(x)}$ *and by integrating both sides, we get* $-ln|1-x| - x = ln|F(x)| + C$; *choosing C=0,* $-ln|1-x| - x = ln|F(x)| \iff e^{-ln|1-x|-x} = |F(x)| \iff (|1-x|)^{-1}e^{-x} = |F(x)| \iff |1-x|e^x|F(x)| = 1$.
So we will let $g(x) = e^x(1-x)\int\limits_0^x f(t)dt$.

Coming back to our problem:
Letting $g(x) = e^x(1-x)\int\limits_0^x f(t)dt$, g is continuous on $[0,1]$ and differentiable on (0, 1). We have $g(0) = g(1)$, and by Rolle's theorem there is $c \in (0,1)$ such that $g'(c) = 0 \iff (1-c)f(c) = c\int\limits_0^c f(x)dx$. □

Problem 2.7. *(D.Andrica) Let* f *be continuous on* $[a,b]$, *differentiable on* (a,b) *and different from 0 for all* $x \in (a,b)$. *Prove that there exists* $c \in (a,b)$ *such that*

$$\frac{f'(c)}{f(c)} = \frac{1}{a-c} + \frac{1}{b-c}$$

Proof.

Idea 2.8. *From* $\frac{f'(c)}{f(c)} = \frac{1}{a-c} + \frac{1}{b-c}$*, we consider* $\frac{f'(x)}{f(x)} = \frac{1}{a-x} + \frac{1}{b-x}$*, and by integrating both sides we get* $ln|f(x)| = -ln|(a-x)(b-x)| + C$*. Choosing* $C = 0$*, we obtain* $ln|f(x)| = -ln|(a-x)(b-x)| \iff |f(x)||(a-x)(b-x)| = 1$*.*
So we let $g(x) = f(x)(a-x)(b-x)$*.*

Returning to our problem:
Let $g(x) = f(x)(a-x)(b-x)$. Then g is continuous on $[a, b]$, differentiable on (a, b) and $g(a) = g(b) = 0$, so by Rolle's theorem there is $c \in (a, b)$ such that $g'(c) = 0 \iff \frac{f'(c)}{f(c)} = \frac{1}{a-c} + \frac{1}{b-c}$ □

From the above, we now have a technique or idea to solve some problems related to mean value theorems. I will give an exercise for readers.

Exercise 2.9. *Let* $f : [0, 1] \to \mathbb{R}$ *be a continuous function on* $[0, 1]$ *satisfying* $\int\limits_0^1 f(x)dx = 0$*. Prove that there is* $c \in (0, 1)$ *such that*

1. $f(c) = f'(c)\int\limits_0^c f(x)dx$*, if* f *is differentiable on* $(0, 1)$*.*

2. $\frac{f(c)}{c} = \int\limits_0^c f(x)dx$*.*

3. $cf(c) = \int\limits_c^1 f(x)dx$*.*

Lemma 2.10. *Let f be continuous on [0, a]. If* $\int\limits_0^a f(x)dx = 0$*, then there is* $c \in (0, a)$ *such that* $\int\limits_0^c xf(x)dx = 0$*.*

Proof. Suppose that for all $c \in (0, a)$, $\int\limits_0^c xf(x)dx \neq 0$. Because the function $t \to \int\limits_0^t xf(x)dx$ is continuous on $[0, a]$, we can assume that $\int\limits_0^c xf(x)dx > 0$ for all $x \in (0, a)$.
Let $H(t) = \int\limits_0^t f(x)dx$, $t \in [0, a]$.
Then $\int\limits_0^t xf(x)dx = t\int\limits_0^t f(x)dx - \int\limits_0^t\int\limits_0^x f(t)dtdx = t.H(t) - \int\limits_0^t H(x)dx > 0$ for all $t \in (0, a)$ (1).
In (1), letting $t \to a$, we get $\int\limits_0^a H(x)dx \leq 0$ (2).
Now we consider the function:

$$g(t) = \begin{cases} \frac{\int\limits_0^t H(x)dx}{t}, t \neq 0 \\ 0, t = 0 \end{cases}$$

where $t \in [0,a]$.
g is continuous on $[0,a]$ and differentiable on $(0,a)$.
Clearly, we have $g'(t) = \frac{H(t).t - \int\limits_0^t H(x)dx}{t^2} > 0$, for all $t \in (0,a)$ (From (1)).
Applying Lagrange's theorem for g on $[0,a]$ we get there is $c \in (0,a)$ such that:
$g(a)-g(0) = g'(c).(a-0) > 0$, hence $g(a) > 0$ which means that $\int\limits_0^a H(x)dx > 0$ (contradicting (2)).
So we conclude that there is $c \in (0,a)$ such that $\int\limits_0^c xf(x)dx = 0$. □

Lemma 2.11. *Let f be a real-valued function, continuous on $[0,1]$ satisfying $\int\limits_0^1 f(x)dx = \int\limits_0^1 xf(x)dx$. Prove that there is $c \in (0,1)$ such that*

$$\int\limits_0^c f(x)dx = 0$$

Proof. Let $H(x) = \int\limits_0^x f(t)dt$. We have $\int\limits_0^1 H(x)dx = 1.H(1) - 0.H(0) - \int\limits_0^1 xf(x)dx = \int\limits_0^1 f(x)dx - \int\limits_0^1 xf(x)dx = 0$.
Let $G(x) = \int\limits_0^x H(t)dt$, G is continuous on $[0,1]$ and differentiable on $(0,1)$. Moreover $G(0) = G(1) = 0$, and by Rolle's theorem there is $c \in (0,1)$ such that $G'(c) = 0 \Longleftrightarrow \int\limits_0^c f(x)dx = 0$ □

Problem 2.12. *Let $f : [0,1] \to \mathbb{R}$ be continuous and $\int_0^1 f(x)dx = 0$.Prove that there is $c \in (0,1)$ such that*

$$\int\limits_0^c (x+x^2)f(x)dx = c^2 f(c). \quad (Duong\ Viet\ Thong, AMM, 2011)$$

Proof. Using *Lemma*2.10 there is $c_1 \in (0,1)$ such that $\int\limits_0^{c_1} xf(x)dx = 0$. Using *Problem*2.3 there is $c_2 \in (0,1)$ such that $c_2 f(c_2) = \int\limits_0^{c_2} xf(x)dx$.
Let $F(x) = \int\limits_0^x f(t).t[1-x+t]dt$. Then $F'(x) = xf(x) - \int\limits_0^x tf(t)dt$. We have $F'(c_2) = F'(0) = 0$ (Note: $F'(c_2) = 0$-proof above), so by Flett's Theorem we get $c \in (0,c_2) \subset (0,1)$ such that $F(c) - F(0) = c.F'(c)$, meaning that $\int\limits_0^c (x+x^2)f(x)dx = c^2 f(c)$. □

Comment 2.13.

1. *If we change the hypothesis* $\int_0^1 f(x)dx = 0$ *to* $\int\limits_0^1 f(x)dx = \int\limits_0^1 xf(x)dx$*, then we will get the same result //-Hint: Use Lemma*2.11

2. *Based on the lemma, we can create another problem:*

   Example 2.14. *Let* $f : [0,1] \to \mathbb{R}$ *be continuous and* $\int_0^1 f(x)dx = 0$*. Then there is* $c \in (0,1)$ *such that* $\int\limits_0^c xf(x)dx = 0$*. From here we can let* $G(x) = \int\limits_0^x tf(t)dt$*, and so* $G'(x) = xf(x)$*. We have* $G'(0) = 0$*, and using Flett's theorem, we need to find* $\alpha \in (0,1)$ *such that* $G'(\alpha) = 0$*. This is easy since we have* $G(c) = G(0) = 0$*, and by Rolle's theorem there is* $\alpha \in (0,1)$ *such that* $G'(\alpha) = 0$*.*
   Using Flett's theorem there is $\eta \in (0,\alpha)$ *such that* $G(\eta) - G(0) = \eta G'(\eta) \iff \int\limits_0^\eta tf(t)dt = \eta^2 f(\eta)$

We offer readers the following problem:

Problem 2.15.

Let $f : [0,1] \to \mathbb{R}$ *be continuous satisfying* $\int_0^1 f(x)dx = 0$ *and* $f(0) = 0$*. Prove that there is* $c \in (0,1)$ *such that*

$$c^2 f(c) = 2\int\limits_0^c xf(x)dx.$$