Junior problems

J349. Prove that for each positive integer n, $6^{n+1}+8^{n+1}+27^n-1$ has at least 11 proper positive divisors.

Proposed by Titu Andreescu, The University of Texas at Dallas, USA

Solution by Albert Stadler, Herrliberg, Switzerland

We put $f(n)=6^{n+1}+8^{n+1}+27^n-1$. Then

$$f(n)\equiv 0 \pmod 2, f(n)\equiv(-1)^{n+1}+1^{n+1}+(-1)^n-1\equiv 0 \pmod 7.$$

If n is even, then $f(n)\equiv 3^n-1=9^{\frac{n}{2}}-1\equiv 0 \pmod 8$.

If n is odd, then $f(n)\equiv(-1)^{n+1}-1\equiv 0 \pmod 9$.

Suppose n is odd. Then $f(n)$ is divisible by $2\cdot 7\cdot 9=126$, and 126 has $2\cdot 2\cdot 3=12$ divisors, so if n is odd then $f(n)$ has at least 11 proper positive divisors.

Suppose n is even. We note that $f(2)=1456=2^4\cdot 7\cdot 13$ which has 19 proper divisors. If n is even and $n\geq 4$ then $f(n)$ is divisible by $7\cdot 8=56$. Either $f(n)$ has a prime factor different from 2 or 7 or $f(n)$ has only the two prime factors 2 and 7. In the first case $f(n)$ is divisible by $56p$ where p is a prime different from 2 and 7. $56p$ has at least 11 proper divisors, so $f(n)$ has at least 11 proper divisors. In the second case $f(n)$ is of the form 2^a7^b with $a\geq 2$ and $b\geq 1$. Then $a+b\geq 6$, since $7^{a+b}>2^a7^b=f(n)\geq f(4)=571984$. However, 2^a7^b has $(a+1)(b+1)-1=ab+a+b\geq 6\cdot 1+6=12$ proper divisors.

Author's note: the expression factors as $a^3+b^3+c^3-3abc$, with $a=2^{n+1}, b=3^n, c=-1$, which helps with the second case.

Also solved by Arkady Alt, San Jose, CA, USA; Polyahedra, Polk State College, USA; Kyoung A Lee, The Hotchkiss School, Lakeville, CT, USA; Kwon Il Kobe Ko, Cushing Academy, Ashburnham, MA, USA; Adnan Ali, A.E.C.S-4, Mumbai, India; Daniel Lasaosa, Pamplona, Spain; Suparno Ghoshal; Prishtina Math Gymnasium Problem Solving Group, Republic of Kosova; Joel Schlosberg, Bayside, NY, USA.

J350. Let a, b, c be positive real numbers such that $ab + bc + ca = 1$. Prove that

$$\sqrt{a^4 + b^2} + \sqrt{b^4 + c^2} + \sqrt{c^4 + a^2} \geq 2.$$

Proposed by Titu Zvonaru, Comănești, România

Solution by Stefan Petrevski, Pearson College UWC, Metchosin, British Columbia, Canada
From Minkowski's inequality, we obtain the following:

$$\sqrt{a^4 + b^2} + \sqrt{b^4 + c^2} + \sqrt{c^4 + a^2} \geq \sqrt{(a^2 + b^2 + c^2)^2 + (a + b + c)^2}.$$

But from $(a-b)^2+(b-c)^2+(c-a)^2 \geq 0$, we obtain that $a^2+b^2+c^2 \geq ab+bc+ca$ and $(a+b+c)^2 \geq 3(ab+bc+ca)$. By using the fact that $ab + bc + ca = 1$, we get that

$$a^2 + b^2 + c^2 \geq 1 \text{ and } (a + b + c)^2 \geq 3.$$

Finally,

$$\sqrt{a^4 + b^2} + \sqrt{b^4 + c^2} + \sqrt{c^4 + a^2} \geq \sqrt{(a^2 + b^2 + c^2)^2 + (a + b + c)^2} \geq \sqrt{1 + 3} = 2$$

Equality holds if and only if $a - b = 0, b - c = 0, c - a = 0$, i.e. $a = b = c = \dfrac{1}{\sqrt{3}}$.

Also solved by Arkady Alt, San Jose, CA, USA; Ilyes Hamdi, Lycée du Parc, Lyon, France; Polyahedra, Polk State College, USA; Kyoung A Lee, The Hotchkiss School, Lakeville, CT, USA; Kwon Il Kobe Ko, Cushing Academy, Ashburnham, MA, USA; Adnan Ali, A.E.C.S-4, Mumbai, India; Duy Quan Tran, Nguyen Binh Khiem high school for the gifted, Vinhlong, Vietnam; Henry Ricardo, New York Math Circle, Tappan, NY, USA; Daniel Lasaosa, Pamplona, Spain; Madhurima Mondal, Kalyani University Experimental High School, India; Evgenidis Nikolaos, M.N.Raptou High School, Larissa, Greece; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Prasun De, South Point High School, Kolkata, India; Prishtina Math Gymnasium Problem Solving Group, Republic of Kosova; SooYoung Choi; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Yong Xi Wang, Affiliated High School of Shanxi University; Rao Yiyi, Wuhan, China; Vincelot Ravoson, Lycée Henri IV, France, Paris.

J351. Find the sum of all six-digit positive integers such that if a and b are adjacent digits of such an integer, then $|a-b| \geq 2$.

Proposed by Neelabh Deka, India

Solution by Polyahedra, Polk State College, USA

For $1 \leq k \leq 6$ and $0 \leq j \leq 9$, let $A_k(j)$ be the set of all k-digit integers $a_1 a_2 \cdots a_k$ such that $a_1 = j$ and for $2 \leq i \leq k$, $0 \leq a_i \leq 9$ and $|a_{i-1} - a_i| \geq 2$. Observe the symmetry $|A_k(j)| = |A_k(9-j)|$. Put $A_k = \bigcup_{j=0}^{9} A_k(j)$. Then $|A_1(j)| = 1$ for $0 \leq j \leq 4$, so $|A_1| = 10$.

Inductively, $|A_2(0)| = \sum_{i \neq 0,1} |A_1(i)| = 8$ and for $1 \leq j \leq 4$, $|A_2(j)| = \sum_{i \neq j, j\pm 1} |A_1(i)| = 7$. So $|A_2| = 72$.
$|A_3(0)| = 57$, $|A_3(1)| = 50$, and $|A_3(j)| = 51$ for $j = 2, 3, 4$. So $|A_3| = 520$.
$|A_4(0)| = 413$, $|A_4(1)| = 362$, $|A_4(2)| = 368$, and $|A_4(j)| = 367$ for $j = 3, 4$. So $|A_4| = 3754$.
$|A_5(0)| = 2979$, $|A_5(1)| = 2611$, $|A_5(2)| = 2657$, $|A_5(3)| = 2652$, and $|A_5(4)| = 2653$. So $|A_5| = 27104$.
$|A_6(0)| = 21514$, $|A_6(1)| = 18857$, $|A_6(2)| = 19184$, $|A_6(3)| = 19142$, and $|A_6(4)| = 19146$. So $|A_6| = 195686$.

Now for $0 \leq k \leq 5$, let B_k be the set of all six-digit integers $a_1a_2a_3a_4a_5a_6$ such that $a_1 = 0$, $|a_{i-1} - a_i| \in \{0, 1\}$ if $2 \leq i \leq 6-k$, and $a_{7-k}a_{8-k} \cdots a_6 \in A_k$. Let $s_6 = \sum_{x \in A_6} x$ and $s_k = \sum_{x \in B_k} x$ for $0 \leq k \leq 5$. Notice that if $x = a_1 \cdots a_k \in A_k$, then so is $x' = a_1' \cdots a_k'$ where $a_i' = 9 - a_i$ for $1 \leq i \leq k$. Summing by such pairing we see that $s_6 = \frac{999999}{2}|A_6| = 97842902157$. Similarly, since $B_5 = \{0x : x \in A_5\}$, we get $s_5 = \frac{99999}{2}|A_5| = 1355186448$. Next, $B_4 = \{00x : x \in A_4\} \cup \{01x : x \in A_4\}$, so

$$s_4 = \left(10000 + \frac{9999}{2} \times 2\right)|A_4| = 75076246.$$

Likewise,

$$B_3 = \{000x : x \in A_3\} \cup \{001x : x \in A_3\} \cup \{010x : x \in A_3\} \cup \{011x : x \in A_3\} \cup \{012x : x \in A_3\},$$

so

$$s_3 = \left(30000 + 4000 + \frac{999}{2} \times 5\right)|A_3| = 18978700.$$

Continuing in this fashion and with a tree diagram, we get

$$\begin{aligned} s_2 &= \left(80000 + 12000 + 1400 + \frac{99}{2} \times 13\right)|A_2| = 6771132; \\ s_1 &= \left(220000 + 34000 + 4200 + 460 + \frac{9}{2} \times 35\right)|A_1| = 2588175; \\ s_0 &= 610000 + 96000 + 12100 + 1380 + 147 = 719627. \end{aligned}$$

Finally, by the principle of inclusion-exclusion, the answer we are seeking is

$$s_6 - s_5 + s_4 - s_3 + s_2 - s_1 + s_0 = 96548715839.$$

J352. Let ABC be a triangle and let D be a point on side AC such that $\frac{1}{3}\angle BCA = \frac{1}{4}\angle ABD = \angle DBC$, and $AC = BD$. Find the angles of triangle ABC.

Proposed by Marius Stănean, Zalău, România

Solution by Daniel Lasaosa, Pamplona, Spain

Denote $\alpha = \angle DBC$, or $\angle BCA = 3\alpha$, $\angle ABD = 4\alpha$, and consequently $\angle ABC = 5\alpha$ and $\angle CAB = 180° - 8\alpha$. Applying the Sine Law to triangles ABC and BCD, we have

$$\frac{BC}{AC} = \frac{\sin(8\alpha)}{\sin(5\alpha)}, \qquad \frac{BC}{BD} = \frac{\sin(4\alpha)}{\sin(3\alpha)}.$$

Since $AC = BD$, we conclude that

$$\sin(5\alpha) = \frac{\sin(3\alpha)\sin(8\alpha)}{\sin(4\alpha)} = 2\sin(3\alpha)\cos(4\alpha) = \sin(7\alpha) - \sin\alpha.$$

Therefore,

$$\sin\alpha = \sin(7\alpha) - \sin(5\alpha) = 2\cos(6\alpha)\sin\alpha,$$

and since $\sin\alpha \neq 0$ (otherwise ABC would be degenerate), we must have

$$\cos(6\alpha) = \frac{1}{2}, \qquad 6\alpha = 60°, \qquad \alpha = 10°,$$

where we have used that, since $\angle CAB = 180° - 8\alpha$, we must have $\alpha < 30°$ or $6\alpha < 180°$. Consequently

$$\angle A = 100°, \qquad \angle B = 50°, \qquad \angle C = 30°.$$

Also solved by Arkady Alt, San Jose, CA, USA; Polyahedra, Polk State College, USA; Kyoung A Lee, The Hotchkiss School, Lakeville, CT, USA; Kwon Il Kobe Ko, Cushing Academy, Ashburnham, MA, USA; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Adnan Ali, A.E.C.S-4, Mumbai, India; Ercole Suppa, Teramo, Italy; Joel Schlosberg, Bayside, NY, USA; Prasun De, South Point High School, Kolkata, India; Tolibjon Ismoilov, academic lycuem S.H.Sirojiddinov, Tashkent, Uzbekistan; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

J353. Let a, b, c be nonnegative real numbers and let

$$A = \frac{1}{4a+1} + \frac{1}{4b+1} + \frac{1}{4c+1},$$
$$B = \frac{1}{3a+b+1} + \frac{1}{3b+c+1} + \frac{1}{3c+a+1},$$
$$C = \frac{1}{2a+b+c+1} + \frac{1}{2b+c+a+1} + \frac{1}{2c+a+b+1}.$$

Prove that $A \geq B \geq C$.

Proposed by Nguyen Viet Hung, Hanoi, Vietnam

Solution by Rao Yiyi, Wuhan, China

Since $\dfrac{1}{4a+1} = \displaystyle\int_0^1 t^{4a}$, $\dfrac{1}{3a+b+1} = \displaystyle\int_0^1 t^{3a+b}$, and $\dfrac{1}{2a+b+c+1} = \displaystyle\int_0^1 t^{2a+b+c}$, it remains to prove that

$$t^{4a} + t^{4b} + t^{4c} \geq t^{3a+b} + t^{3b+c} + t^{3c+a} \geq t^{2a+b+c} + t^{2b+c+a} + t^{2c+a+b}$$

Letting $t^a = x, t^b = y, t^c = z$, this becomes

$$x^4 + y^4 + z^4 \geq x^3y + y^3z + z^3x \geq x^2yz + y^2zx + z^2xy$$

Which is obvious by AM-GM.

Also solved by Arkady Alt, San Jose, CA, USA; Polyahedra, Polk State College, USA; Kyoung A Lee, The Hotchkiss School, Lakeville, CT, USA; Kwon Il Kobe Ko, Cushing Academy, Ashburnham, MA, USA; Yong Xi Wang, Affiliated High School of Shanxi University; Albert Stadler, Herrliberg, Switzerland; Adnan Ali, A.E.C.S-4, Mumbai, India; Duy Quan Tran, Nguyen Binh Khiem high school for the gifted, Vinhlong, Vietnam; Daniel Lasaosa, Pamplona, Spain; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Prasun De, South Point High School, Kolkata, India; Prishtina Math Gymnasium Problem Solving Group, Republic of Kosova; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

J354. Evaluate

$$\sum_{n\geq 1} \frac{3n+1}{2n+1}\binom{2n}{n}^{-1}$$

Proposed by Cody Johnson, Carnegie Mellon University, USA

Solution by Henry Ricardo, New York Math Circle, Tappan, NY, USA

We claim that the sum is 1.

First we note that

$$\begin{aligned}\frac{3n+1}{2n+1}\binom{2n}{n}^{-1} &= \left(2 - \frac{n+1}{2n+1}\right)\binom{2n}{n}^{-1} \\ &= 2\binom{2n}{n}^{-1} - \frac{n+1}{2n+1}\binom{2n}{n}^{-1} \\ &= \frac{2}{\binom{2n}{n}} - \frac{(n+1)!n!}{(2n+1)!} = \frac{2}{\binom{2n}{n}} - \frac{1}{\binom{2n+1}{n}} \\ &= \frac{2}{\binom{2n}{n}} - \frac{2}{\binom{2n+2}{n+1}}.\end{aligned}$$

Thus we have the telescoping partial sum

$$\begin{aligned}\sum_{n=1}^{N} \frac{3n+1}{2n+1}\binom{2n}{n}^{-1} &= 2\sum_{n=1}^{N}\left(\frac{1}{\binom{2n}{n}} - \frac{1}{\binom{2n+2}{n+1}}\right) \\ &= 2\left(\frac{1}{2} - \frac{1}{\binom{2N+2}{N+1}}\right) = 1 - \frac{2}{\binom{2N+2}{N+1}},\end{aligned}$$

which tends to 1 since $\binom{2N+2}{N+1} \to \infty$ as $N \to \infty$.

Also solved by Arkady Alt, San Jose, CA, USA; Polyahedra, Polk State College, USA; Kwon Il Kobe Ko, Cushing Academy, Ashburnham, MA, USA; Yong Xi Wang, Affiliated High School of Shanxi University; Adnan Ali, A.E.C.S-4, Mumbai, India; Angel Plaza, University of Las Palmas de Gran Canaria, Spain; Ercole Suppa, Teramo, Italy; G. C. Greubel, Newport News, VA; Daniel Lasaosa, Pamplona, Spain; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Albert Stadler, Herrliberg, Switzerland; Cody Johnson, Carnegie Mellon University, USA.

Senior problems

S349. Each face of eight unit cubes is colored in one of the k colors, where $k \in \{2, 3, 4, 5, 6, 8, 12, 24\}$, so that there are $\frac{48}{k}$ faces of each color. Prove that from these unit cubes, we can assemble a 2×2 cube that has on its surface equal amount of squares of each color.

Proposed by Nairi Sedrakyan, Armenia

Solution by Daniel Lasaosa, Pamplona, Spain

Note first that although 5 appears in the original problem statement, it must clearly be a typo since 5 does not divide 48. Note next that it suffices to solve the problem with $k = 24$: 16 is the only divisor of 48 which does not divide 24, and it is not on the list of possible values of k, or any $k < 24$ divides 24. For any color of the k needed, define $\frac{24}{k}$ shades of that color, and out of the $\frac{48}{k}$ faces painted in each one of the k colors, paint exactly 2 in each one of the $\frac{24}{k}$ shades of that color. Clearly, finding a solution of 24 then produces a solution for k, sufficing to consider all faces having a shade of a given color, as faces of that color. We henceforth solve only the problem for $k = 24$, which suffices for the whole problem.

Once the $2 \times 2 \times 2$ cube is assembled, note that for each one of the original unit cubes, three faces show and three faces are hidden, such that a face shows iff its opposite face does not show. Note that rotating a unit cube 180° around the line joining the midpoints of two opposite edges, two opposite faces get inverted, in the sense that the one that was showing is now hidden, and *vice versa*, whereas the two other faces that were showing, are still showing. We can therefore choose, for any unit cube, which three faces show, as long as no two of them are opposite, the other three being hidden.

Choose now any one of the 24 colors. If both faces painted in this color are opposite faces of a unit cube, clearly exactly one of them will show and one will be hidden, regardless of how we position the cubes. We will say that this color is *set*. Otherwise, proceed as follows: denote A_1, B_1 the faces which are painted in this color. Denote by A_2 the face opposite B_1, and find the face who shares the color of B_1 (which clearly cannot be opposite B_1), and call it B_2. If B_2 is opposite A_1, stop, otherwise denote by A_3 the face opposite B_2, and so on. The process cannot continue indefinitely since the number of faces in the cubes is finite, and at some point, since each choice of A_i results in two opposite faces being assigned a name, we must have B_i opposite A_1. Note therefore that for each one of these i colors, a face is labeled A and a face is labeled B, such that each face labeled A is opposite a face labeled B. It is therefore possible to choose a way to assemble the $2 \times 2 \times 2$ cube, in such a way that all faces A are showing, all faces B are hidden, and no face not involved in this process has to satisfy any particular requisite as to being shown or hidden. We then say that these i colors are *set*. We continue this process, which cannot go on indefinitely, and must finish with all colors *set*, which means that for each one of the 24 colors, a way has been determined to assemble the unit cubes such that exactly one of the faces of this color shows, and exactly one is hidden. The conclusion follows.

S350. Let $a_1, a_2, \ldots, a_{15}$ be positive integers such that

$$(a_1+1)(a_2+1)\cdots(a_{15}+1) = 2015a_1a_2\cdots a_{15}$$

Prove that there are at least six and at most ten numbers among $a_1, a_2, \ldots, a_{15}$ that are equal to 1.

Proposed by Titu Zvonaru, Comănești and Neculai Stanciu, Buzău, România

Solution by Adnan Ali, A.E.C.S-4, Mumbai, India

The given condition can be rewritten as

$$P = \left(1+\frac{1}{a_1}\right)\left(1+\frac{1}{a_2}\right)\cdots\left(1+\frac{1}{a_{15}}\right) = 2015.$$

Without loss of generality assume that $a_1 \le a_2 \le \cdots \le a_{15}$. For the sake of contradiction we assume that at most 5 are 1. Then we have $a_1 = \cdots = a_5 = 1 < a_6 \le \cdots \le a_{15}$. So,

$$P \le 2^5 \cdot \left(\frac{3}{2}\right)^{10} = \left(\frac{9}{2}\right)^5 = (4.5)^5 < 2015,$$

a contradiction. Thus, there must be at least 6 among $a_1, a_2, \cdots, a_{15}$ that are equal to 1. Now, again we may assume that at least 11 are 1. Then $a_1 = \cdots = a_{11} = 1$ and so,

$$P > 2^{11} \cdot 1^4 = 2048 > 2015,$$

a contradiction. Thus there are most 10 among $a_1, a_2, \cdots, a_{15}$ that are equal to 1. An example includes $a_1 = \cdots = a_{10} = 1$ and $(a_{11}, a_{12}, a_{13}, a_{14}, a_{15}) = (2, 8, 12, 24, 30)$.

Also solved by Kwon Il Kobe Ko, Cushing Academy, Ashburnham, MA, USA; Isroilov Mukhiddin, University of World Economy and Diplomacy, Tashkent, Uzbekistan; Joel Schlosberg, Bayside, NY, USA; Daniel Lasaosa, Pamplona, Spain.

S351. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$a+b+c \geq \frac{1}{a(b+1)} + \frac{1}{b(c+1)} + \frac{1}{c(a+1)} + \frac{3}{2}.$$

Proposed by Nguyen Viet Hung, Hanoi, Vietnam

Solution by Madhurima Mondal, Kalyani University Experimental High School, India

We have

$$a - \frac{1}{b(c+1)} = \frac{abc+ab-1}{b(c+1)} = \frac{abc+ab-abc}{b(c+1)} = \frac{a}{c+1},$$

since $abc = 1$. Therefore, we have

$$b - \frac{1}{c(a+1)} = \frac{b}{a+1}$$

and

$$c - \frac{1}{a(b+1)} = \frac{c}{b+1}$$

Also, since $abc = 1$ let

$$a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{x}$$

for some real numbers $x, y, z > 0$. Then

$$\frac{a}{c+1} + \frac{b}{a+1} + \frac{c}{b+1} = \frac{x^2}{y(z+x)} + \frac{y^2}{z(x+y)} + \frac{z^2}{x(y+z)}$$

By Titu's Lemma,

$$\frac{x^2}{y(z+x)} + \frac{y^2}{z(x+y)} + \frac{z^2}{x(y+z)} \geq \frac{(x+y+z)^2}{2(xy+yz+zx)} \geq \frac{3(xy+yz+zx)}{2(xy+yz+zx)} = \frac{3}{2},$$

since $(x+y+z)^2 \geq 3(xy+yz+zx)$ for real x, y, z. Therefore,

$$a+b+c - \frac{1}{b(c+1)} - \frac{1}{c(a+1)} - \frac{1}{a(b+1)} \geq \frac{3}{2}$$

Also solved by Arkady Alt, San Jose, CA, USA; Kwon Il Kobe Ko, Cushing Academy, Ashburnham, MA, USA; Rao Yiyi, Wuhan, China; Vincelot Ravoson, Lycée Henri IV, France, Paris; Anant Mudgal, Delhi Public School, Faridabad, India; Adnan Ali, A.E.C.S-4, Mumbai, India; Andrea Fanchini, Cantù, Italy; Duy Quan Tran, Nguyen Binh Khiem high school for the gifted, Vinhlong, Vietnam; Farras M. Hibban Faddila, Jakarta, Indonesia; Khakimboy Egamberganov, National University of Uzbekistan, Tashkent, Uzbekistan; Isroilov Mukhiddin, University of World Economy and Diplomacy, Tashkent, Uzbekistan; Daniel Lasaosa, Pamplona, Spain; Evgenidis Nikolaos, M.N.Raptou High School, Larissa, Greece; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Albert Stadler, Herrliberg, Switzerland; Tolibjon Ismoilov, academic lycuem S.H.Sirojiddinov, Tashkent, Uzbekistan; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

S352. In the triangle ABC, let ω denote its Brocard angle, and let φ satisfy the identity

$$\tan\varphi = \tan S + \tan B + \tan C.$$

Prove that

$$\frac{\cos 2A + \cos 2B + \cos 2C}{\sin 2A + \sin 2B + \sin 2C} = -\frac{1}{4}(\cot\omega + 3\cot\varphi).$$

Proposed by Oleg Faynshteyn, Leipzig, Germany

Solution by Kwon Il Kobe Ko, Cushing Academy, Ashburnham, MA, USA

We know that for any $\triangle ABC$:

$$\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B\cos C$$

$$\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$$

Then,

$$\text{LHS} = -\frac{1}{4}\csc A\csc B\csc C - \cot A\cot B\cot C$$

We also know:

$$\cot\omega = \cot A + \cot B + \cot C$$

Moreover:

$$\cot\varphi = \cot A\cot B\cot C,$$

since $\tan A + \tan B + \tan C = \tan A\tan B\tan C = \tan\varphi$. Therefore,

$$\text{RHS} = -\frac{1}{4}(\cot A + \cot B + \cot C + 3\cot A\cot B\cot C)$$

When we combine the LHS and RHS, it suffices to show

$$\csc A\csc B\csc C = \cot A + \cot B + \cot C - \cot A\cot B\cot C$$

However, $\cot A + \cot B = \dfrac{\cot A\cot B - 1}{\cot(A+B)}$; then

$$\cot A + \cot B + \cot C(1 - \cot A\cot B) = (\cot A + \cot B)(1 + \cot^2 C) = \frac{\sin(A+B)}{\sin A\sin B}\cdot\frac{1}{\sin^2 C} = \csc A\csc B\csc C.$$

This proves the equation above.

Also solved by Adnan Ali, A.E.C.S-4, Mumbai, India; Andrea Fanchini, Cantù, Italy; Joel Schlosberg, Bayside, NY, USA; Daniel Lasaosa, Pamplona, Spain; Prasun De, South Point High School, Kolkata, India; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

S353. Let a, b, c, x, y be positive real numbers such that $xy \geq 1$. Prove that

$$\frac{ab}{xa+yb+2c}+\frac{bc}{xb+yc+2a}+\frac{ca}{xc+ya+2b} \leq \frac{a+b+c}{x+y+2}$$

Proposed by Yong Xi Wang, HS of Shanxi University, China

Solution by the author
Using the Cauchy-Schwarz inequality, we have

$$\left(\frac{(x+1)^2}{c+ax}+\frac{(y+1)^2}{by+c}\right)(ax+by+2c) \geq (x+y+2)^2$$

$$\Longrightarrow \frac{ab}{xa+yb+2c} \leq \frac{1}{(x+y+2)^2}\left(\frac{(x+1)^2ab}{c+ax}+\frac{(y+1)^2ab}{by+c}\right)$$

then we have

$$\begin{aligned}\sum_{cyc}\frac{ab}{xa+yb+2c} &\leq \frac{1}{(x+y+2)^2}\left(\sum_{cyc}\left(\frac{(x+1)^2ab}{c+ax}+\frac{(y+1)^2ab}{by+c}\right)\right)\\ &= \frac{1}{(x+y+2)^2}\left(\sum_{cyc}\left(\frac{(x+1)^2ab}{c+ax}+\frac{(y+1)^2bc}{yc+a}\right)\right)\\ &= \frac{1}{(x+y+2)^2}\left(\sum_{cyc}b\left(\frac{(x+1)^2a}{c+ax}+\frac{(y+1)^2c}{yc+a}\right)\right)\end{aligned}$$

by the other hand, we have the following identity:

$$x+y+2=\frac{(x+1)^2z}{t+xz}+\frac{(y+1)^2t}{yt+z}+\frac{(t-z)^2(xy-1)}{(t+xz)(yt+z)}$$

so we have

$$\sum_{cyc}\frac{ab}{xa+yb+2c} \leq \frac{1}{(x+y+2)^2}\sum_{cyc}b(x+y+2)=\frac{a+b+c}{x+y+2}$$

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

S354. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x, y,

$$(f(x+y))^2 = (f(x))^2 + 2f(xy) + (f(y))^2.$$

Proposed by Oleksiy Klurman, Université de Montréal, Canada

Solution by Navid Safei, Sharif University of Technology, Iran

First put $(-x, x)$ to receive the equation $f^2(0) = f^2(x) + f^2(-x) + 2f(-x^2)$. Then put $(x+y, -x)$ to receive that $f^2(y) = f^2(x+y) + 2f(-x^2-xy) + f^2(-x)$. By comparing this with the first equation and the problem statement we receive the following equation:

$$f(-x^2) = f(xy) + f(-x^2 - xy) + \frac{1}{2}f^2(0)$$

Now define the function $g(x) = f(x) + \frac{1}{2}f^2(0)$; then we receive the following equality:

$$g(-x^2) = g(xy) + g(-x^2 - xy)$$

Now set the system $xy = a, x^2 + xy = -b$, so that $x = \pm\sqrt{-(a+b)}$ and $y = \pm\frac{a}{\sqrt{-(a+b)}}$. Indeed, the system has solutions for all a, b such that $a + b \le 0$. Thus for all real values for which $a + b \le 0$ we have $g(a+b) = g(a) + g(b)$. Now if $a + b > 0$ there exists a real number c such that $c + a + b, c + a \le 0$. Then we have

$$g(c) + g(a+b) = g(a+b+c) = g(b) + g(a+c) = g(b) + g(a) + g(c)$$

Thus g is additive for all reals a, b. Now we need to prove that it is bounded. Consider the relation

$$f(-x^2) = \frac{f^2(0) - f^2(x) - f^2(-x)}{2} \le \frac{1}{2}f^2(0)$$

Thus $g(-x^2) \le f^2(0)$, which implies that g is bounded from above on the left side of the real line. Thus we have $g(x) = cx$ for some real number c, so $f(x) = cx + d$ for some real numbers c, d. By checking this with our original equation we find the solutions $f(x) = 0, -2, x, x - 2$.

U349. Let $0 < x, y, z < 1$. Prove that

$$\frac{1}{1-x^4}+\frac{1}{1-y^4}+\frac{1}{1-z^4}+\frac{1}{1-x^2yz}+\frac{1}{1-y^2zx}+\frac{1}{1-z^2xy} \geq$$
$$\frac{1}{1-x^3y}+\frac{1}{1-xy^3}+\frac{1}{1-y^3z}+\frac{1}{1-yz^3}+\frac{1}{1-z^3x}+\frac{1}{1-zx^3}.$$

Proposed by Mehtaab Sawhney, Commack High School, USA

Solution by Li Zhou, Polk State College, USA

Using geometric series, we see that it suffices to prove that for all $n \geq 0$,

$$x^{4n}+y^{4n}+z^{4n}+x^{2n}y^nz^n+y^{2n}z^nx^n+z^{2n}x^ny^n \geq$$
$$x^{3n}y^n+x^ny^{3n}+y^{3n}z^n+y^nz^{3n}+z^{3n}x^n+z^nx^{3n}.$$

But this is equivalent to

$$x^{2n}(x^n-y^n)(x^n-z^n)+y^{2n}(y^n-x^n)(y^n-z^n)+z^{2n}(z^n-y^n)(z^n-x^n) \geq 0,$$

which is true by Schur's inequality.

Also solved by Arkady Alt, San Jose, CA, USA; Yong Xi Wang, HS of Shanxi University, China; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Khakimboy Egamberganov, National University of Uzbekistan, Tashkent, Uzbekistan; Moubinool Omarjee, Lycée Henri IV, Paris, France; Navid Safei, Sharif University of Technoogy, Iran; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Shohruh Ibragimov, National University of Uzbekistan, Uzbekistan; Albert Stadler, Herrliberg, Switzerland; Tolibjon Ismailov, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Nicusor Zlota ,"Traian Vuia" Technical College, Focsani, Romania.

U350. Let a and b be real numbers such that $a \geq 1$ and $b > a^2 - a + 1$. Prove that the equation $x^5 - ax^3 + a^2x - b = 0$ has a unique real solution x_0, and $2b - a^3 < {x_0}^6 < b^2 + a - a^3$.

Proposed by Corneliu Mănescu-Avram, Ploeşti, România

Solution by Daniel Lasaosa, Pamplona, Spain

Denote $p(x) = x^5 - ax^3 + a^2x - b$. We look for its extrema by setting $p'(x) = 0$, ie $5x^4 - 3ax^2 + a^2 = 0$. Considering this equation as quadratic on x^2, its discriminant is $9a^2 - 20a^2 = -11a^2 < 0$, or $p(x)$ is always increasing and may have no more than one real root. Moreover, $\lim_{x\to-\infty} p(x) = -\infty$ and $\lim_{x\to+\infty} p(x) = +\infty$, or by intermediate value theorem, $p(x)$ has at least one root. Hence the proposed equation has exactly one real solution, which we will denote by x_0.

Now, ${x_0}^5 - a{x_0}^3 + a^2x_0 = b > a^2 - a + 1$, or since the LHS is strictly increasing with x_0, and equality would hold for $x_0 = 1$, it follows that $x_0 > 1$. Note next that, after adding b to both sides and then multiplying both sides of the proposed equation by $(x^2 + a)$, it rewrites in the equivalent form

$$({x_0}^2 + a)b = x_0({x_0}^6 + a^3).$$

Now, if ${x_0}^6 \geq b^2 + a - a^3$, then

$$({x_0}^2 + a)b \geq x_0(b^2 + a), \qquad (bx_0 - a)(x_0 - b) \geq 0.$$

Since $b - a > (a - 1)^2 \geq 0$, and $x_0 > 1$, then the first term in the LHS is positive, hence $x_0 > b$, or since again $p(x)$ is strictly increasing, we have $0 = b^5 - ab^3 + a^2b - b = (b^2 - a)b^3 + (a^2 - 1)b$, absurd since $b^2 - a > b - a > 0$ and $a^2 - 1 \geq 0$. We have reached a contradiction, or ${x_0}^6 < b^2 + a - a^3$.

Next, if $2b - a^3 \geq {x_0}^6$, then

$$({x_0}^2 + a)b = x_0({x_0}^6 + a^3) \leq 2bx_0 < b\left({x_0}^2 + 1\right) \leq b\left({x_0}^2 + a\right),$$

where the strict inequality is so because $x_0 > 1$, hence no equality is reached in the AM-GM inequality between ${x_0}^2$ and 1. We have again reached a contradiction, hence $2b - a^3 < {x_0}^6$.

Also solved by Yong Xi Wang, HS of Shanxi University, China; Moubinool Omarjee, Lycée Henri IV, Paris, France; Nicusor Zlota ,"Traian Vuia" Technical College, Focsani, Romania; Li Zhou, Polk State College, USA.

U351. Let $a \geq 0$. Evaluate

$$\lim_{n\to\infty} \frac{1}{2^n n^a} \sum_{k=0}^{n} \binom{n}{k} k^a.$$

Proposed by Albert Stadler, Herrliberg, Switzerland

Solution by G. C. Greubel, Newport News, VA, USA

By making use of the series

$$\sum_{k=0}^{n} \binom{n}{k} t^k = (1+t)^n \tag{1}$$

derivatives with respect to t can be taken and then evaluated at $t = 1$. The first few are

$$\begin{aligned} \sum_{k=0}^{\infty} \binom{n}{k} k\, t^k &= n\, t\, (1+t)^{n-1} \\ \sum_{k=0}^{n} \binom{n}{k} k^2\, t^k &= n^2\, t\, (1+t)^{n-1} - n(n-1)\, t\, (1+t)^{n-2} \end{aligned} \tag{2}$$

for which

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} &= 2^n \\ \sum_{k=0}^{n} \binom{n}{k} k &= 2^{n-1}\, n \\ \sum_{k=0}^{n} \binom{n}{k} k^2 &= 2^{n-2}\, n(n+1). \end{aligned} \tag{3}$$

By continuing the process it is seen that

$$\sum_{k=0}^{n} \binom{n}{k} k^a \approx 2^{n-a}\, n^a \left(1 + \frac{\alpha_1}{n} + \frac{\alpha_2}{n^2} + \mathcal{O}\left(\frac{1}{n^3}\right)\right). \tag{4}$$

With this in mind then

$$\lim_{n\to\infty} \frac{1}{2^n\, n^a} \sum_{k=0}^{n} \binom{n}{k} k^a = \lim_{n\to\infty} \left[\frac{1}{2^a} \left(1 + \frac{\alpha_1}{n} + \frac{\alpha_2}{n^2} + \mathcal{O}\left(\frac{1}{n^3}\right)\right)\right] \tag{5}$$

which yields the desired result, namely,

$$\lim_{n\to\infty} \frac{1}{2^n\, n^a} \sum_{k=0}^{n} \binom{n}{k} k^a = \frac{1}{2^a}. \tag{6}$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Shohruh Ibragimov, National University of Uzbekistan, Uzbekistan; Moubinool Omarjee, Lycée Henri IV, Paris, France.

U352. Evaluate

$$\sum_{n=1}^{\infty} \frac{n-1}{\binom{2n}{n}}.$$

Proposed by Cody Johnson, Carnegie Mellon University, USA

Solution by Adnan Ali, Student in A.E.C.S-4, Mumbai, India

For all positive integers n,

$$(3n-3)\tbinom{2n}{n}^{-1} - n\tbinom{2n-2}{n-1}^{-1} = n\tbinom{2n-2}{n-1}^{-1}\left(\frac{3n(n-1)}{(2n-1)(2n)} - 1\right) = -(n+1)\cdot\frac{n!(n-1)!}{2(2n-1)!}$$
$$= -(n+1)\cdot\frac{(n!)(n!)}{2n(2n-1)!} = -(n+1)\tbinom{2n}{n}^{-1}.$$

So, for all positive integers N, we have a telescoping sum,

$$\textstyle\sum_{n=1}^{N} \frac{n-1}{\binom{2n}{n}} = \sum_{n=1}^{N} \frac{1}{3}\left(n\tbinom{2n-2}{n-1}^{-1} - (n+1)\tbinom{2n}{n}^{-1}\right) = \frac{1}{3}\left(1-(N+1)\tbinom{2N}{N}^{-1}\right).$$

Since $\frac{1}{m+1}\binom{2m}{m} \sim \infty$,

$$\textstyle\sum_{n=1}^{N} \frac{n-1}{\binom{2n}{n}} = \lim_{N\to\infty} \frac{1}{3}\left(1-(N+1)\tbinom{2N}{N}^{-1}\right) = \frac{1}{3}.$$

Also solved by Arkady Alt, San Jose, CA, USA; Daniel Lasaosa, Pamplona, Spain; Yong Xi Wang, HS of Shanxi University, China; Kwon Il Kobe Ko, Cushing Academy, Ashburnham, MA, USA; Li Zhou, Polk State College, USA; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Ercole Suppa, Teramo, Italy; G. C. Greubel, Newport News, VA, USA; Henry Ricardo, New York Math Circle, NY, USA; Moubinool Omarjee, Lycée Henri IV, Paris, France; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Shohruh Ibragimov, National University of Uzbekistan, Uzbekistan; Albert Stadler, Herrliberg, Switzerland; Nicusor Zlota ,"Traian Vuia" Technical College, Focsani, Romania.

U353. Let $a \in \mathbb{R}$ and let $f : (-1, 1) \to \mathbb{R}$ be a function differentiable at 0. Evaluate

$$\lim_{n\to\infty} \left[an - \sum_{k=1}^{n} f\left(\frac{k}{n^2}\right) \right].$$

Proposed by Dorin Andrica, Babeş-Bolyai University, România

Solution by Daniel Lasaosa, Pamplona, Spain
Clearly $\frac{k}{n^2} \leq \frac{1}{n} \to 0$ when $n \to \infty$, or to within $\frac{1}{n^2}$, we may approximate

$$f\left(\frac{k}{n^2}\right) = f(0) + \frac{k}{n^2} f'(0) + O\left(\frac{1}{n^2}\right).$$

Therefore,

$$\sum_{k=1}^{n} f\left(\frac{k}{n^2}\right) = nf(0) + \frac{f'(0)}{n^2} \sum_{k=1}^{n} k + O\left(\frac{1}{n}\right),$$

or since the sum over k equals $\frac{n(n+1)}{2}$, then

$$an - \sum_{k=1}^{n} f\left(\frac{k}{n^2}\right) = (a - f(0))\, n - \frac{f'(0)}{2} + O\left(\frac{1}{n}\right).$$

Now, the limit of the third term in the RHS is clearly 0. The second term is constant, and the first term either diverges when $a \neq f(0)$, or is zero when $a = f(0)$. We conclude that

$$\lim_{n\to\infty} \left[an - \sum_{k=1}^{n} f\left(\frac{k}{n^2}\right) \right] = \begin{cases} -\infty & \text{when} \quad f(0) > a, \\ +\infty & \text{when} \quad f(0) < a, \\ -\frac{f'(0)}{2} & \text{when} \quad f(0) = a. \end{cases}$$

Also solved by Kwon Il Kobe Ko, Cushing Academy, Ashburnham, MA, USA; Stanescu Florin, Gaesti, Serban Cioculescu HS, Romania; Shohruh Ibragimov, National University of Uzbekistan, Uzbekistan; Moubinool Omarjee, Lycée Henri IV, Paris, France; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Yong Xi Wang, HS of Shanxi University, China.

U354. Let $f, g : [-1, 1] \to \mathbb{R}$ be increasing functions. Prove that if $f(-x) = -f(x)$, for all $x \in [-1, 1]$, then

$$\int_{-1}^{1} f(x)g(x)dx \geq 0.$$

Proposed by Marcel Chiriţă, Bucharest, România

Solution by Albert Stadler, Herrliberg, Switzerland

We claim that $f(x) \geq 0$ for $0 \leq x \leq 1$. Suppose that $f(a) < 0$ for some $a > 0$. Then $f(-a) = -f(a) > 0$. But $f(-a) \leq f(a)$, since $f(x)$ is increasing. This is a contradiction.Therefore,

$$\int_{-1}^{1} f(x)g(x)dx = \int_{0}^{1} f(x)g(x)dx + \int_{0}^{1} f(-x)g(-x)dx = \int_{0}^{1} \underbrace{f(x)}_{\geq 0}\underbrace{(g(x) - g(-x))}_{\geq 0}dx \geq 0.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Kwon Il Kobe Ko, Cushing Academy, Ashburnham, MA, USA; Yong Xi Wang, HS of Shanxi University, China; Li Zhou, Polk State College, USA; Stanescu Florin, Gaesti, Serban Cioculescu HS, Romania; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Daniel López Aguayo, Centro de Ciencias Matemáticas UNAM, Morelia, Mexico; Maxim Ignatiuc, University of Texas at Dallas, USA; Khakimboy Egamberganov, National University of Uzbekistan, Tashkent, Uzbekistan; Henry Ricardo, New York Math Circle, NY, USA; Joel Schlosberg, Bayside, NY, USA; Moubinool Omarjee, Lycée Henri IV, Paris, France; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Shohruh Ibragimov, student at National University of Uzbekistan.

O349. Find all positive integers n such that

$$\sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor$$

is an even integer.

Proposed by Dorin Andrica, Babeş-Bolyai University, România

Solution by Joel Schlosberg, Bayside, NY, USA

Note that

$$\left\lfloor \frac{n}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor = \begin{cases} 0 & \text{if } k \nmid n \\ 1 & \text{if } k \mid n. \end{cases}$$

Therefore,

$$\sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor - \sum_{k=1}^{n-1} \left\lfloor \frac{n-1}{k} \right\rfloor = 1 + \sum_{k=1}^{n-1} \left(\left\lfloor \frac{n}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor \right) = \sum_{\substack{k \in \{1,\ldots,n\} \\ k \mid n}} 1 = \tau(n),$$

the number of divisors of n. Therefore, $\sum_{k=1}^{n-1} \left\lfloor \frac{n-1}{k} \right\rfloor$ and $\sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor$ are of opposite parity iff $\tau(n)$ is odd. Since the pairing $d \leftrightarrow n/d$ matches divisors which are distinct unless $d^2 = n$, that occurs iff n is a perfect square. Since for $n = 1$, $\sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor = 1$ is odd, the parity of $\sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor$ for $n = 1, 2, 3, \ldots$ in sequence switches to even at even perfect squares and to odd at odd perfect squares. Thus, $\sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor$ is even iff $m^2 \le n < (m+1)^2$ for even m, that is iff $\lfloor \sqrt{n} \rfloor$ is even.

Also solved by Daniel Lasaosa, Pamplona, Spain; Kwon Il Kobe Ko, Cushing Academy, Ashburnham, MA, USA; Rao Yiyi, Wuhan, China; Henri Godefroy, Lycée Louis le Grand, Paris; G. C. Greubel, Newport News, VA, USA; Johannes Hosle, South Bend, IN, USA; Li Zhou, Polk State College, USA.

O350. Find all triples (x, y, z) of integers satisfying the equation $x^3 + 3xy + y^3 = 2^z + 1$.

Proposed by Titu Andreescu, The University of Texas at Dallas, USA

Solution by Li Zhou, Polk State College, USA

Clearly, $(x, y, z) = (1, 1, 2)$, $(3, -1, 4)$, $(-1, 3, 4)$ are solutions. We show that they are the only solutions.

Observe that $2^{z+1} = 2(x^3+3xy+y^3-1) = (x+y-1)f(x, y)$, where $f(x, y) = (x-y)^2+(x+1)^2+(y+1)^2$. So $x + y - 1 = 2^m$ for some $m \geq 0$ and

$$f(x, y) \geq (x - y)^2 + 2\left(\frac{x + y + 2}{2}\right)^2 \geq 8.$$

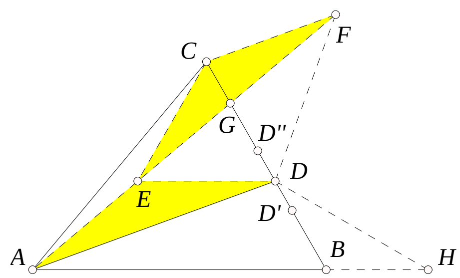
Also, if x and y are not both odd, then exactly two of $x - y$, $x + 1$, $y + 1$ are odd, so $f(x, y) \equiv 2 \pmod 4$, which implies that $f(x, y)$ cannot be a power of 2. Therefore, x and y must be both odd. Then $m = 0$, and thus $x = 2k + 1$ and $y = -2k + 1$. Consequently, $f(x, y) = 8(3k^2 + 1)$. Now $\{1, 4, 5\}$ is the complete set of residues of $3k^2 + 1$ modulo 8, and $3k^2 + 1 > 8$ when $|k| \geq 2$. Hence $3k^2 + 1$ is a power of 2 if and only if $k = 0$ or ± 1. This completes the proof.

Also solved by Daniel Lasaosa, Pamplona, Spain; Ilyes Hamdi, Lycée du Parc, Lyon, France; Kwon Il Kobe Ko, Cushing Academy, Ashburnham, MA, USA; Rao Yiyi, Wuhan, China; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Isroilov Mukhiddin, University of World Economy and Diplomacy, Tashkent, Uzbekistan; Navid Safei, Sharif University of Technoogy, Iran; Evgenidis Nikolaos, M.N.Raptou High School, Larissa, Greece; Nicusor Zlota ,"Traian Vuia" Technical College, Focsani, Romania.

O351. Let ABC be a triangle with $\angle ABC = 60^\circ$ and $\angle BCA = 70^\circ$, and let point D lie on side BC. Prove that $\angle BAD = 20^\circ$ if and only if $AB + BD = AD + DC$.

Proposed by Mircea Lascu and Titu Zvonaru, România

Solution by Li Zhou, Polk State College, USA



Suppose that $\angle BAD = 20^\circ$. Locate point G on BC such that $\angle GAC = 10^\circ$. Let E, F be points on AG such that $AE = EC = CF$, as in the figure. Then $\angle CFG = \angle CEG = 20^\circ$, so $\angle CGF = 80^\circ = \angle GCF$. Hence $GF = CF = AE$, thus $EF = AG = AD$. By SAS, $\triangle CEF \cong \triangle EAD$. Therefore, $ED = CF = EC$, that is, $\triangle DCE$ is equilateral. Consequently, $AD + DC = AG + GF = AF$ and $\angle CDF = \angle CFD = 50^\circ$. Now locate point H on AB such that $BH = BD$, as in the figure. Then $\angle ADH = 130^\circ = \angle ADF$, so by ASA, $\triangle ADH \cong \triangle ADF$. Hence $AH = AF$, establishing $AB + BD = AD + DC$.

Conversely, if D', D'' are points on BC different from D, as in the figure, then

$$AB + BD' < AB + BD = AD + DC < (AD' + D'D) + DC = AD' + D'C,$$

$$AB + BD'' > AB + BD = AD + DC > (AD'' - DD'') + DC = AD'' + D''C,$$

completing the proof.

Also solved by Daniel Lasaosa, Pamplona, Spain; Rao Yiyi, Wuhan, China; Isroilov Mukhiddin, University of World Economy and Diplomacy, Tashkent, Uzbekistan; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.

O352. Solve in positive integers the system of equations

$$\begin{cases} \frac{1}{x}+\frac{2}{y}+\frac{3}{z}=1, \\ x+2y+3z=\frac{50yz}{8+yz}. \end{cases}$$

Proposed by Titu Andreescu, The University of Texas at Dallas, USA

Solution by Li Zhou, Polk State College, USA

It is easy to verify that $(x, y, z) = (3, 12, 6), (8, 4, 8)$ are solutions. We show that they are the only solutions.

First, by the Cauchy-Schwarz inequality,

$$x+2y+3z=(x+2y+3z)\left(\frac{1}{x}+\frac{2}{y}+\frac{3}{z}\right)\geq 36.$$

Next, since $x+2y+3z=50-\frac{400}{8+yz}$, $8+yz$ must divide 400. Hence

$$(x+2y+3z, yz) \in \{(49,392),(48,192),(46,92),(45,72),(42,42),(40,32)\}.$$

For the pair $(49, 392)$, we have

$$49=x+2y+3z>2y+3z\geq 2\sqrt{6yz}=2\sqrt{6\cdot 392}>49,$$

a contradiction. Likewise, the pairs $(48, 192)$ and $(46, 92)$ are eliminated.

If $(x+2y+3z, yz) = (45, 72)$, then $(y, z) \in \{(8, 9), (9, 8), (12, 6)\}$, and only $(12, 6)$ yields a solution $(x, y, z) = (3, 12, 6)$.

If $(x + 2y + 3z, yz) = (42, 42)$, then $(y, z) \in \{(6, 7), (7, 6)\}$, none of which yields x satisfying $\frac{1}{x}+\frac{2}{y}+\frac{3}{z}=1$.

Finally, if $(x+2y+3z, yz) = (40, 32)$, then $(y, z) \in \{(4, 8), (8, 4)\}$, and only $(4, 8)$ yields a solution $(x, y, z) = (8, 4, 8)$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Ercole Suppa, Teramo, Italy; Rao Yiyi, Wuhan, China.

O353. Let a, b, c, d be nonnegative real numbers such that $a \geq b \geq 1 \geq c \geq d$ and $a+b+c+d=4$. Prove that $4\left(a^2+b^2+c^2+d^2\right) \geq 12+a^3+b^3+c^3+d^3$.

Proposed by Marius Stănean, Zalău, România

Solution by the author

Let be $f(a,b,c,d) = 4(a^2+b^2+c^2+d^2)-(a^3+b^3+c^3+d^3)$, we have

$$f(a,b,c,d) \geq f\left(a, b, \frac{c+d}{2}, \frac{c+d}{2}\right) \Longleftrightarrow$$

$$4(c^2+d^2)-c^3-d^3-2(c+d)^2+\frac{(c+d)^3}{4} \Longleftrightarrow$$

$$\frac{(c-d)^2}{4}\left[8-3(c+d)\right] \geq 0$$

which is true because $c+d \leq 2 \Longrightarrow 8-3(c+d) \geq 0$. Hence it's enough to show that

$$f\left(a, b, \frac{c+d}{2}, \frac{c+d}{2}\right) \geq 12 \Longleftrightarrow f\left(a, b, \frac{4-a-b}{2}, \frac{4-a-b}{2}\right) \geq 12.$$

Using Vieta's formulas we have

$$4(a^2+b^2+c^2+d^2)-(a^3+b^3+c^3+d^3) = 4(ab+bc+cd+da+ac+bd)-3(abc+bcd+cda+dab). \quad (7)$$

Let be $a = x+1$, $b = y+1$ so $x \geq y \geq 0$, $x+y \leq 2$ and the last inequality, using (1), becomes

$$4\left[(1+x)(1+y)+(2-x-y)(x+y+2)+\frac{(2-x-y)^2}{4}\right]-$$

$$3\left[(x+1)(y+1)(2-x-y)-\frac{(2-x-y)^2}{4}(2+x+y)\right] \geq 12 \Longleftrightarrow$$

$$4xy-4(2-x-y)+4(2-x-y)(2+x+y)+(2-x-y)^2-$$

$$3(x+1)(y+1)(2-x-y)-\frac{3}{4}(2-x-y)^2(2+x+y) \geq 0 \Longleftrightarrow$$

$$4xy+(2-x-y)\left[-4+8+4(x+y)+2-(x+y)-3xy-3(x+y)-3-\frac{3}{4}(4-(x+y)^2)\right] \geq 0 \Longleftrightarrow$$

$$4xy+\frac{3}{4}(2-x-y)(x-y)^2 \geq 0$$

which is true, with equality when $x=y=0$ or $x=2$, $y=0$ which means that $a=b=c=d=1$ or $a=3$, $b=1$, $c=d=0$.

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Daniel Lasaosa, Pamplona, Spain; Kwon Il Kobe Ko, Cushing Academy, Ashburnham, MA, USA; Rao Yiyi, Wuhan, China; Li Zhou, Polk State College, USA; Nicusor Zlota ,"Traian Vuia" Technical College, Focsani, Romania.

O354. Find all primes p such that

$$\frac{p^2}{1+\frac{1}{2}+\cdots+\frac{1}{p-1}}$$

is an integer.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Li Zhou, Polk State College, USA

It is easy to verify that the expression is an integer for $p = 2, 3, 5, 7$. We show that they are the only such primes. Consider a prime $p \geq 11$. Let $H_{p-1} = 1+\frac{1}{2}+\cdots+\frac{1}{p-1} = \frac{a}{b}$, where $(a,b) = 1$. Then by Wolstenholm's theorem, $p^2|a$. Hence, for p^2/H_{p-1} to be an integer, $a = p^2$.

If $p = 11$, then $2^3 \cdot 3^2 \cdot 7$ divides b. So $1 < H_{10} \leq \frac{11^2}{8\cdot 9\cdot 7} < 1$, a contradiction.

If $13 \leq p \leq 53$, then there are two primes q_1, q_2 such that $\frac{p+1}{2} \leq q_1 < q_2 < p-1$. Since $2^3 q_1 q_2 | b$, we have $1 < H_{p-1} \leq \frac{p^2}{8q_1q_2} < \frac{p^2}{2(p+1)^2} < 1$, a contradiction.

Finally, if $p \geq 59$, then there exists an $m \geq 5$ such that $2^m \leq p-1 < 2^{m+1}$. Also, by Chebyshev's theorem (a.k.a. Bertrand's postulate), there exists a prime q such that $\frac{p+1}{2} \leq q < p-1$. Since $2^m q | b$,

$$4.07 < \log 59 < H_{p-1} \leq \frac{p^2}{2^m q} < \frac{(2^{m+1}+1)p}{2^{m-1}(p+1)} < 4 + \frac{1}{16} < 4.07.$$

This contradiction completes the proof.

Also solved by Daniel Lasaosa, Pamplona, Spain; Navid Safei, Sharif University of Technoogy, Iran; Rao Yiyi, Wuhan, China.