Junior Problems

J355. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$4(a^2+b^2+c^2)-(a^3+b^3+c^3)\geq 9.$$

Proposed by Anant Mudgal, India

J356. Find all positive integers n such that

$$2(6+9i)^n-3(1+8i)^n=3(7+4i)^n.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J357. Prove that for any $z\in\mathbb{C}$ such that $\left|z+\frac{1}{z}\right|=\sqrt{5}$,

$$\left(\frac{\sqrt{5}-1}{2}\right)^2\leq|z|\leq\left(\frac{\sqrt{5}+1}{2}\right)^2.$$

Proposed by Mihály Bencze, Brașov, România

J358. Prove that for $x\in\mathbb{R}$, the equations,

$$2^{2^{x-1}}=\frac{1}{2^{2^x}-1}\quad\text{and}\quad 2^{2^{x+1}}=\frac{1}{2^{2^{x-1}}-1},$$

are equivalent.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J359. The midline of triangle ABC, parallel to side BC, intersects the triangle's circumcircle at B' and C'. Evaluate the length of segment $B'C'$ in terms of triangle ABC's side-lengths.

Proposed by Dorin Andrica and Dan Ștefan Marinescu, România

J360. In triangle ABC, let AA' and BB' be the angle bisectors of $\angle A$ and $\angle B$. Prove that

$$\frac{A'B'}{ab\sin\dfrac{C}{2}}\leq\frac{1}{(b+c)\sin\left(A+\dfrac{B}{2}\right)}+\frac{1}{(c+a)\sin\left(\dfrac{A}{2}+B\right)}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Senior Problems

S355. Let a, b, c be nonnegative real numbers such that $ab + bc + ca = 1$ and $\min(a, b, c) \leq \sqrt{2}\,|a + b + c - 2|$. Prove that

$$(a + b + c - 2)^4 \geq 16(a + b + c - 1)(abc + a + b + c - 2).$$

Proposed by Marcel Chiriță, Bucharest, România

S356. Let a, b, c, d, e be real numbers such that $\sin a + \sin b + \sin c + \sin d + \sin e \geq 3$. Prove that $\cos a + \cos b + \cos c + \cos d + \cos e \leq 4$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S357. Prove that in any triangle,

$$\sum \sqrt{\frac{a\,(h_a - 2r)}{(3a + b + c)\,(h_a + 2r)}} \leq \frac{3}{4}.$$

Proposed by Mihály Bencze, Brașov, România

S358. Prove that for each integer n, there are eighteen integers such that both their sum and the sum of their fifth powers are equal to n.

Proposed by Nairi Sedrakyan, Armenia

S359. Prove that in any triangle,

$$m_a\left(\frac{1}{2r_a} - \frac{R}{bc}\right) + m_b\left(\frac{1}{2r_b} - \frac{R}{ca}\right) + m_c\left(\frac{1}{2r_c} - \frac{R}{ab}\right) \geq 0.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S360. Let ABC be a triangle with orthocenter H and circumcenter O. The parallels through B and C to AO intersect the external angle bisector of $\angle BAC$ at D and E, respectively. Prove that BE, CD, AH are concurrent.

Proposed by Iman Munire Bilal, University of Cambridge and Marius Stănean, România

Undergraduate Problems

U355. Let a be a real number such that $a \neq 0$ and $a \neq \pm 1$, and let n be an integer greater than 1. Find all polynomials $P(X)$ with real coefficients such that

$$\left(a^2 X^2 + 1\right) P\left(aX\right) = \left(a^{2n} X^2 + 1\right) P\left(X\right).$$

Proposed by Marcel Chiriță, Bucharest, România

U356. Let $(x_n)_{n\geq 1}$ be a monotonic sequence, and let $a \in (-1, 0)$. Find

$$\lim_{n\to\infty} \left(x_1 a^{n-1} + x_2 a^{n-2} + \cdots + x_n\right).$$

Proposed by Mihai Piticari and Sorin Rădulescu, România

U357. Evaluate

$$\lim_{n\to\infty} \left[\frac{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)\cdots\left(1+\frac{n}{n^2}\right)}{\sqrt{e}}\right]^n.$$

Proposed by Dorin Andrica, Babeș-Bolyai University, Cluj Napoca, România

U358. Let $(x_n)_{n\geq 1}$ be an increasing sequence of real numbers for which there is a real number $a > 2$ such that

$$x_{n+1} \geq a x_n - (a-1) x_{n-1},$$

for all $n \geq 1$. Prove that $(x_n)_{n\geq 1}$ is divergent.

Proposed by Mihai Piticari and Sorin Rădulescu, România

U359. Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ be sequences of nonnegative real numbers. Furthermore, let $c_1, \ldots, c_n$ and $d_1, \ldots, d_m$ be sequences of real numbers. Prove that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \min(a_i, a_j) + \sum_{i=1}^{m}\sum_{j=1}^{m} d_i d_j \min(b_i, b_j) \geq 2\sum_{i=1}^{n}\sum_{j=1}^{m} c_i d_j \min(a_i, b_j).$$

Proposed by Mehtaab Sawhney, Commack High School, New York, USA

U360. Let $f : [-1, 1] \to [0, \infty)$ be a C^1 increasing function. Prove that

$$\int_{-1}^{1} (f'(x))^{\frac{1}{2015}}\, dx \leq 2015 \int_{-1}^{1} \left(\frac{f(x)}{1-x}\right)^{\frac{1}{2015}} dx.$$

Proposed by Oleksiy Klurman, University College London

Olympiad Problems

O355. Let ABC be a triangle with incenter I. Prove that

$$\frac{(IB+IC)^2}{a(b+c)}+\frac{(IC+IA)^2}{b(c+a)}+\frac{(IA+IB)^2}{c(a+b)}\le 2.$$

Proposed by Nguyen Viet Hung, High School for Gifted Students, Hanoi University of Science, Vietnam

O356. We take out an even number from the set $\{1, 2, 3, \ldots, 25\}$. Find this number knowing that the remaining set has precisely 124 subsets with three elements that form an arithmetic progression.

Proposed by Marian Teler, Costești and Marin Ionescu, Pitești, România

O357. Prove that in any triangle

$$\frac{ab+4m_am_b}{c}+\frac{bc+4m_bm_c}{a}+\frac{ca+4m_cm_a}{b}\ge\frac{16K}{R}.$$

Proposed by Titu Andreescu, USA and Oleg Mushkarov, Bulgaria

O358. Let a, b, c, d be nonnegative real numbers such that $a\ge 1\ge b\ge c\ge d$ and $a+b+c+d=4$. Prove that

$$abcd+\frac{15}{2(ab+bc+cd+da+ac+bd)}\ge\frac{9}{a^2+b^2+c^2+d^2}.$$

Proposed by Marius Stănean, Zalău, România

O359. Solve, in positive integers, the equation

$$x^6+x^3y^3-y^6+3xy\left(x^2-y^2\right)^2=1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O360. Find the least positive integer n with the following property: for any polynomial $P(x)\in\mathbb{C}[x]$, there are polynomials $f_1(x),\ f_2(x),\ \ldots,\ f_n\in\mathbb{C}[x]$ and $g_1(x),\ g_2(x),\ \ldots,\ g_n(x)\in\mathbb{C}[x]$ such that

$$P(x)=\sum_{i=1}^{n}\left(f_i^2(x)+g_i^3(x)\right).$$

Proposed by Oleksiy Klurman, University College London