Concyclicities in Tucker-like configurations

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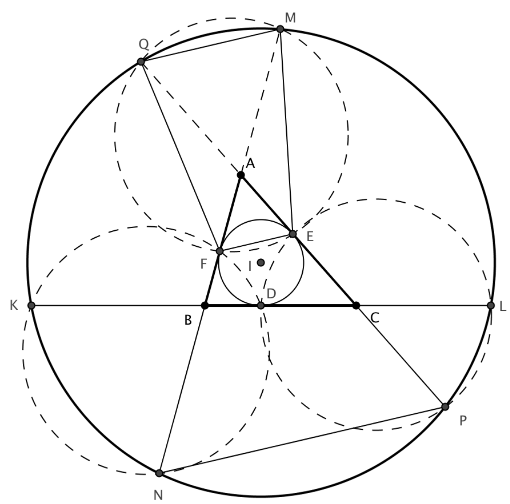
Abstract. In this paper we will present a few theorems, followed by some problems in Tucker-like configurations.

Theorems

1) Conway Circle

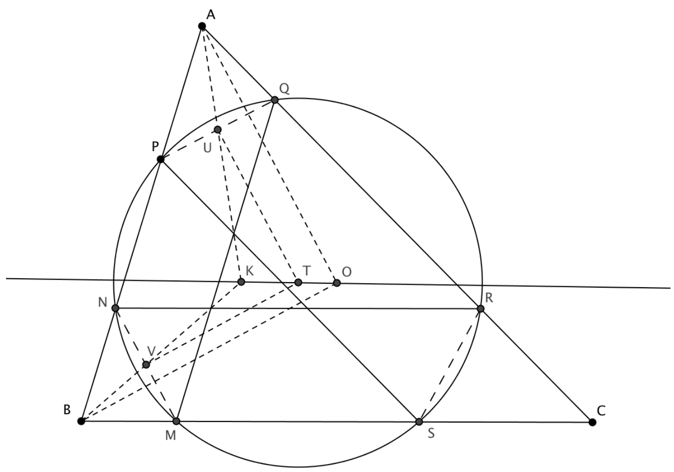
In triangle ABC, let I be its incenter and D, E, F the contact point of the incircle with the sides. On sides AB and AC, consider points M and Q such that $AM = AQ = BC$ and $B - A - M$, $C - A - Q$ are collinear in this order. Analogously, define N, K and L, P. Then Q, M, L, P, N, K are concyclic on **Conway Circle** with the center of the circle I.



Proof:
Considering the isosceles triangles AQM, CQK, BKN it is not difficult to see that $\widehat{MNK} + \widehat{QMN} = 180^\circ \implies K, N, M, Q$ are concyclic. As $AQ \cdot AP = AM \cdot AN$, by power of a point $\implies Q, M, P, N$ are concyclic. So we obtain that K, N, M, Q, P are concyclic and analogously, we obtain that Q, M, L, P, N, K are concyclic. As $BD = p - b$ and $CD = p - c$, it follows that D is the midpoint of KL so I lies on the perpendicular bisector of KL. Similarly, we obtain that I is the center of $QMLPNK$. Moreover, from the right triangle IFN, the radius of the **Conway Circle** is $R_{\mathcal{C}} = \sqrt{p^2 + r^2}$, where p denotes the semiperimeter and r is the radius of the incircle.

2) Tucker Circle

A hexagon inscribed in triangle ABC with sides alternating parallel and antiparallel has its vertices on a circle $\mathcal{C}$ with center T collinear with O and K, where O and K denote the circumcenter, and the symmedian point respectively of triangle ABC



Proof:

First, we will prove the existence of such a hexagon and the concyclicity of its vertices. Starting with a point P on AB draw successively an antiparrallel to side BC intersecting AC at Q, through Q a parallel to side AB intersecting BC at M, through M an antiparallel to side AC intersecting AB at N, through N a parallel to side BC intersecting AC at R and through R an antiparallel to side AB intersecting BC at S. We have to prove that $PS \parallel AC$ and P, Q, R, S, M, N are concyclic.

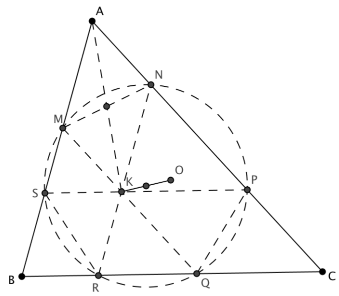
It is easy to see that $\left\{\begin{array}{c} \angle\text{PQM}=\angle\text{APQ}=\angle\text{ACB}=\angle\text{BNM} \\ \angle\text{ARN}=\angle\text{ACB}=\angle\text{APQ} \\ \angle\text{MQR}=\angle\text{BAC}=\angle\text{RSC} \end{array}\right\| \Longrightarrow P, Q, R, S, M, N$ are concyclic. Noting that $\angle ACB = \angle PQM = \angle PSM$ implies $PS \parallel AC$.

Denote by U and V the midpoints of PQ and MN.

As PQ is antiparallel to $BC \Longrightarrow AU$ is the A-symmedian passing through K and also $AO \perp PQ$. Let the perpendicular bisector of PQ intersect OK at T' and the perpendicular bisector of MN intersect OK at T''. Then $\frac{KT'}{T'O} = \frac{KU}{UA}$ and $\frac{KT''}{T''O} = \frac{KV}{VB}$. But from the isosceles trapezoid $PQMN$, as $PN \parallel UV \parallel QM$, we obtain $\frac{KU}{UA} = \frac{KV}{VB}$ $\Longrightarrow \frac{KT'}{T'O} = \frac{KT''}{T''O} \Longrightarrow T' = T''$, meaning that the perpendicular bisectors of two chords of $\mathcal{C}$ intersect on $OK \Longrightarrow$ the center of $\mathcal{C}$ lies on OK.

3) Lemoine Circles

i) First Lemoine circle: The lines passing through the symmedian point K parallel to the sides, intersect the sides at six points lying on a circle called **the First Lemoine Circle**. Its center is the midpoint of OK.



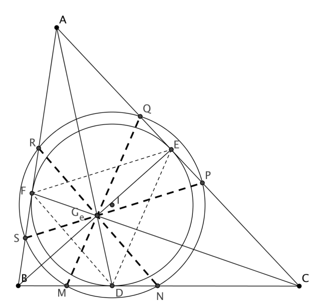
ii) Second Lemoine Circle: The lines passing through the symmedian point K antiparallel to the sides, intersect the sidelines at six points lying on a circle called **the Second Lemoine Circle**. Its center is K.

Proof of i):
As $AMKN$ is a parallelogram $\Longrightarrow AK$ passes through the midpoint of MN. At the same time AK is the $A-symmedian \Longrightarrow MN$ is antiparallel to BC. Analogously, SR and PQ are antiparallel to the corresponding sides. Some direct angle-chasing yields that M, N, P, Q, R, S are concyclic. As in the proof of **Tucker Circle**, it is easy to see that the perpendicular bisectors of MN, SR, PQ pass through the midpoint of OK, so the center of **the First Lemoine Circle** is the midpoint of OK.

Remark: These are applications of the **Tucker Circle**, when all the parallel sides, as well as the antiparallel sides pass through K.

4) Adams' Circle

The parallel lines drawn through the Gergonne point G_e parallel to the sides of the contact triangle of triangle ABC intersect the sidelines of triangle ABC at six points. These six points lie on a circle called **the Adams' circle** and its center is the incenter I of ABC.



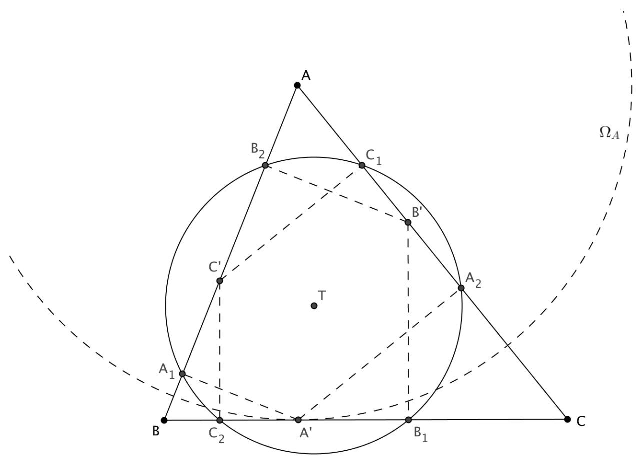
This is a consequence of the **First Lemoine circle** as G_e is the symmedian point of the contact triangle.

5) Taylor Circle

The orthogonal projections of the feet of the altitudes on the sidelines of a triangle ABC are conyclic on a circle called **Taylor circle**.
This is a direct consequence of **Tucker Circle**.

We present a problem which involves a nice property of the center of the **Taylor circle**:

Let ABC be a triangle and let A', B', C' be the feet of the altitudes from A, B, C, respectively. Let A_1 be the foot of the perpendicular from A' to AB and let A_2 be the foot of the perpendicular from A' to AC. Furthermore, let Ω_A be the circle centered at vertex A having radius AA'. Analogously, define points B_1, B_2, C_1, C_2 and circles Ω_B and Ω_C . Prove that the center of the **Taylor circle** has equal powers with respect to Ω_A, Ω_B and Ω_C.

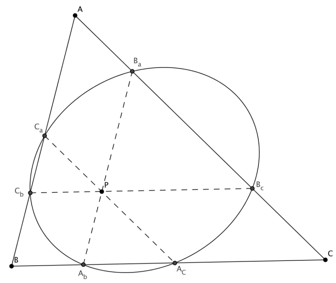


Proof:
Let the center of the Taylor circle be T and let its radius be R_1, The power of T with respect to Ω_A is $|R_1^2 - AT^2| = |A'A^2 - AT^2|$ We have that $AT^2 - R_1^2 = AB_2 \cdot AA_1$. But $AA_1 \cdot AB = A'A^2$, so $AA_1 = \frac{A'A^2}{AB}$ and $AB_2 = AB' \cdot \cos\angle A = AB \cdot \cos^2\angle A$, so $AB_2 \cdot AA_1 = A'A^2 \cdot \cos^2\angle A$. It follows that $AT^2 = R_1^2 + A'A^2 \cdot \cos^2\angle A \implies$ the power of T with respect to Ω_A is $|A'A^2 - (R_1^2 + A'A^2 \cdot \cos^2\angle A)| = |A'A^2 \cdot (1 - \cos^2\angle A - R_1^2)| = |\frac{2^2 \cdot S^2 \cdot \sin^2\angle A}{BC^2} - R_1| = |\frac{S^2}{R^2} - R_1^2|$, where S is the area of triangle ABC and R is its circumradius. We applied the law of sines at the last step. We see that this relation is independent of the sides of triangle ABC, so the power of T with respect to Ω_A is equal to the power of T with respect to Ω_B and to the power of T with respect to Ω_C.

Bonus configuration

We present a lemma in the style of the aforementioned theorems.

Lemma ($\star$)**:** Let P be a point inside a triangle ABC. Through P draw parallels to the sides of the triangle, intersecting the sides in six points. These six points lie on a conic.



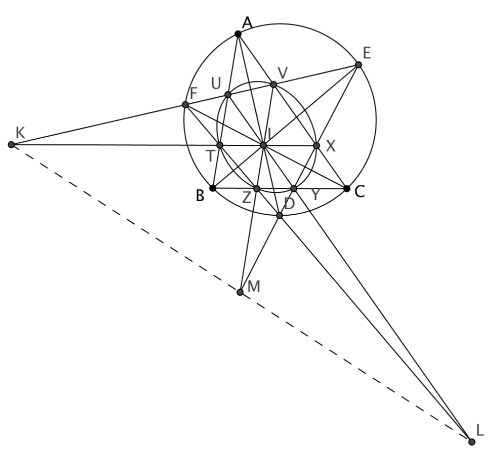
Proof:

Recall the well-known **Carnot's Theorem** stating that six points lying on the sides of a triangle ABC, $A_b, A_c, B_c, B_a, C_a, C_b$, lie on a conic if and only if $\boxed{\dfrac{AC_a}{BC_a} \cdot \dfrac{AC_b}{BC_b} \cdot \dfrac{BA_b}{CA_b} \cdot \dfrac{BA_c}{CA_c} \cdot \dfrac{CB_c}{AB_c} \cdot \dfrac{CB_a}{AB_a} = 1}$

Returning to the proof of the lemma, using parallel lines we obtain $\frac{AC_b}{BC_b} = \frac{AB_c}{CB_c}$, $\frac{BC_a}{AC_a} = \frac{BA_c}{CA_c}$, $\frac{CA_b}{BA_b} = \frac{CB_a}{AB_a} \Longrightarrow$ $\frac{AC_a}{BC_a} \cdot \frac{AC_b}{BC_b} \cdot \frac{BA_b}{CA_b} \cdot \frac{BA_c}{CA_c} \cdot \frac{CB_c}{AB_c} \cdot \frac{CB_a}{AB_a} = 1$ and from **Carnot's Theorem**, it follows that $A_b, A_c, B_c, B_a, C_a, C_b$, lie on a conic.

Now we solve a problem from the **2015 Balkan Mathematical Olympiad (BMO)** using the lemma.

2015 BMO: Let ABC be a scalene triangle with incenter I and circumcircle ω. Lines AI, BI, CI intersect ω for the second time at points D, E, F, respectively. The parallel lines from I to the sides BC, AC, AB intersect EF, DF, DE at points K, L, M, respectively. Prove that points K, L, M are collinear.



Proof: Let U, V be the intersections of E, F with AB, AC. It is well-known that EF is the perpendicular bisector of AI and because triangle AUV is isosceles. It follows that $AU = AV = IV = IU \Longrightarrow AUIV$ is a parallelogram so $IV \parallel AB$ and $IU \parallel AC \Longrightarrow V, U$ are the intersections of the parallel through I to AB, AC with AC, AB. analogously X, Y, Z, T. From **Lemma** $(\star)$ we get that U, V, X, Y, Z, T lie on a conic. Now, applying **Pascal's Theorem** on $UVZTXY$ we get the desired result.

Proposed problems

1. Prove the statement of **Second Lemoine Circle**.

2. Prove that the center of **the Adams' circle** is the incenter of triangle ABC.

3. In triangle ABC, let M, N, P be the midpoints of the major arcs BAC, CBA, ACB. Prove that the Simson lines of M, N, P with respect to $\triangle ABC$ are concurrent at the center of the **Taylor circle** of the excentral triangle.

4. In triangle ABC, let M, N, P be the midpoints of the major arcs BAC, CBA, ACB. Prove that the Simson lines of A, B, C with respect to $\triangle MNP$ are concurrent at the center of the **Taylor circle** of the excentral triangle.

5. In triangle ABC, let H_a, H_b, H_c be the midpoints of the segments HA, HB, HC, where H is the orthocenter of $\triangle ABC$. Prove that the Simson lines of H_a, H_b, H_c with respect to the orthic triangle of $\triangle ABC$ form a triangle having as orthocenter the center of the **Taylor circle** of the $\triangle ABC$.

References

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