Junior Problems

J361. Solve in positive integers the equation

$$\frac{x^2 - y}{8x - y^2} = \frac{y}{x}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J362. Let a, b, c, d be real numbers such that $abcd = 1$. Prove that the following inequality holds:

$$ab + bc + cd + da \le \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2}$$

Proposed by Mircea Becheanu, University of Bucharest, România

J363. Solve in integers the system of equations

$$\begin{aligned} x^2 + y^2 - z(x + y) &= 10 \\ y^2 + z^2 - x(y + z) &= 6 \\ z^2 + x^2 - y(z + x) &= -2 \end{aligned}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J364. Consider a triangle ABC with circumcircle ω. Let O be the center of ω and let D, E, F be the midpoints of minor arcs BC, CA, AB respectively. Let DO intersect ω again at a point A'. Define B' and C' similarly. Prove that

$$\frac{[ABC]}{[A'B'C']} \le 1.$$

Note that $[X]$ denotes the area of figure X.

Proposed by Taimur Khalid, Coral Academy of Science, Las Vegas, USA

J365. Let x_1, x_2, $\ldots$, x_n be nonnegative real numbers such that $x_1 + x_2 + \cdots + x_n = 1$. Find the minimum possible value of

$$\sqrt{x_1 + 1} + \sqrt{2x_2 + 1} + \cdots + \sqrt{nx_n + 1}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J366. Prove that in any triangle ABC,

$$\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \le \sqrt{6 + \frac{r}{2R}} - 1.$$

Proposed by Florin Stănescu, Găești, România

Senior Problems

S361. Find all integers n for which there are integers a and b such that $(a+bi)^4 = n + 2016i$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S362. Let $0 < a, b, c, d \leq 1$. Prove that

$$\frac{1}{a+b+c+d} \geq \frac{1}{4} + \frac{64}{27}(1-a)(1-b)(1-c)(1-d).$$

Proposed by An Zhen-ping, Xianyang Normal University, China

S363. Determine if there are distinct positive integers $n_1, n_2, \ldots, n_{k-1}$ such that

$$\left(3n_1^2 + 4n_2^2 + \cdots + (k+1)n_{k-1}^2\right)^3 = 2016\left(n_1^3 + n_2^3 + \cdots + n_{k-1}^3\right)^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S364. Let a, b, c be nonnegative real numbers such that $a \geq 1 \geq b \geq c$ and $a+b+c=3$. Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{2\left(a^2+b^2+c^2\right)}{3\left(ab+bc+ca\right)} + \frac{5}{6}.$$

Proposed by Marius Stănean, Zalău, România

S365. Let

$$a_k = \frac{\left(k^2+1\right)^2}{k^4+4}, \quad k = 1, 2, 3, \ldots.$$

Prove that for every positive integer n,

$$a_1^n a_2^{n-1} a_3^{n-2} \cdots a_n = \frac{2^{n+1}}{n^2+2n+2}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S366. Let a, b, c, d be positive real numbers such that $a+b+c+d=4$. Prove that

$$9 + \frac{1}{6}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)^2 \geq \frac{70}{ab+bc+cd+da+ac+bd}.$$

Proposed by Marius Stănean, Zalău, România

Undergraduate Problems

U361. Consider all possible ways one can assign the numbers 1 through 10 with a nonnegative probability so that the probabilities sum to 1. Let X be the number selected. Suppose that $E\left[X\right]^k = E\left[X^k\right]$ for a given integer $k \geq 2$. Find the number of possible ways of assigning these probabilities.

Proposed by Mehtaab Sawhney, Commack High School, New York, USA

U362. Let

$$S_n = \sum_{1 \leq i < j < k \leq n} q^{i+j+k},$$

where $q \in (-1, 0) \bigcup (0, 1)$. Evaluate $\lim_{n\to\infty} S_n$.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U363. Let a be a positive number. Prove that there is a number $\theta = \theta(a)$, $1 < \theta < 2$, such that

$$\sum_{j=0}^{\infty} \left| \binom{a}{j} \right| = 2^a + \theta \left| \binom{a-1}{[a]+1} \right|,$$

where $[a]$ denotes the integral part of a. Furthermore, prove that

$$\left| \binom{a-1}{[a]+1} \right| \leq \frac{|\sin \pi a|}{\pi a}.$$

Proposed by Albert Stadler, Herrliberg, Switzerland

U364. Evaluate

$$\int \frac{5x^2 - x - 4}{x^5 + x^4 + 1}\, dx.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U365. Let n be a positive integer. Evaluate

(a) $\displaystyle\int_0^n e^{[x]}\, dx,$

(b) $\displaystyle\int_0^n \lfloor e^x \rfloor\, dx,$

where $\lfloor a \rfloor$ denotes the integer part of a.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U366. If $f : [0, 1] \to \mathbb{R}$ is a convex and integrable function with $f(0) = 0$, prove that

$$\int_0^1 f(x)\, dx \geq 4 \int_0^{\frac{1}{2}} f(x)\, dx.$$

Proposed by Florin Stănescu, Găești, România

Olympiad Problems

O361. Determine the least integer $n > 2$ such that there are n consecutive integers whose sum of squares is a perfect square.

Proposed by Alessandro Ventullo, Milan, Italy

O362. Let (F_n), $n \geq 0$, with $F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$ for all $n \geq 1$. Prove that the following identities hold:

(a) $\dfrac{F_{3n}}{F_n} = 2\left(F_{n-1}^2 + F_{n+1}^2\right) - F_{n-1}F_{n+1}$.

(b) $\dbinom{2n+1}{0}F_{2n+1} + \dbinom{2n+1}{1}F_{2n-1} + \dbinom{2n+1}{2}F_{2n-3} + \cdots + \dbinom{2n+1}{n}F_1 = 5^n$.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

O363. Solve in integers the system of equations

$$x^2 + y^2 + z^2 + \frac{xyz}{3} = 2\left(xy + yz + zx + \frac{xyz}{3}\right) = 2016.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O364. (a) If $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, where p_i are distinct primes, find the value of

$$\sum_{d|n} \frac{n\phi(d)}{d}$$

as a function of $\{p_i\}$ and $\{e_i\}$.

(b) Find the number of integral solutions to $x^x \equiv 1 \pmod{97}$, $1 \leq x \leq 9312$.

Proposed by Mehtaab Sawhney, Commack High School, New York, USA

O365. Prove or disprove the following statement: there is a non-vanishing polynomial $P(x, y, z)$ with integer coefficients such that $P(\sin u, \sin v, \sin w) = 0$ whenever $u + v + w = \dfrac{\pi}{3}$.

Proposed by Albert Stadler, Herrliberg, Switzerland

O366. In triangle ABC, let A_1, A_2 be two arbitrary isotomic points on BC. We define points B_1, $B_2 \in CA$ and C_1, $C_2 \in AB$ similarly. Let ℓ_a be the line passing through the midpoints of segments (B_1C_2) and (B_2C_1). We define lines ℓ_b and ℓ_c similarly. Prove that all three of these lines are concurrent.

Proposed by Marius Stănean, Zalău, România