Junior problems

J355. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$4(a^2+b^2+c^2)-(a^3+b^3+c^3)\geq 9$$

Proposed by Anant Mudgal, India

Solution by Michael Tang, Edina High School, MN, USA

By AM-GM, $a^2b+ab^2+ac^2+a^2c+b^2c+bc^2\geq 6abc$, so

$$(a+b+c)^3-6abc+(a^2b+ab^2+ac^2+a^2c+b^2c+bc^2)\geq (a+b+c)^3.$$

Expanding $(a+b+c)^3$ on the left-hand side, we get

$$(a^3+b^3+c^3)+4(a^2b+ab^2+ac^2+a^2c+b^2c+bc^2)\geq (a+b+c)^3$$

or

$$4(a^3+b^3+c^3+a^2b+ab^2+ac^2+a^2c+b^2c+bc^2)-3(a^3+b^3+c^3)\geq (a+b+c)^3.$$

Factoring the first term, we have

$$4(a+b+c)(a^2+b^2+c^2)-3(a^3+b^3+c^3)\geq (a+b+c)^3$$

Since $a+b+c=3$, this is equivalent to

$$12(a^2+b^2+c^2)-3(a^3+b^3+c^3)\geq 27.$$

Therefore,

$$4(a^2+b^2+c^2)-(a^3+b^3+c^3)\geq 9$$

as requested. Equality holds if and only if $a=b=c=1$.

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA, USA; Seung Hwan An, The Taft School; YunJin Jeong, Emma Willard School, Troy, NY, USA; Yeonjune Kang, Peddie School; Kyoung A Lee, The Hotchkiss School, Lakeville, CT, USA; Joseph Lee, Loomis Chattee School, Windsor, CT, USA; Ji Eun Kim, Tabor Academy, MA, USA; Kwonil Ko, Cushing Academy, Ashburham, MA, USA; Hyun Jin Kim, Stuyvesant High School, New York, NY, USA; Hyunsun (Heidi) Kim, Trinity School, NY, USA; Jamshid Yakhshivev, Academic Lyceum No. 3, Bukhara, Uzbekistan; Zafar Ahmed, BARC, India; Nikos Kalapodis, Patras, Greece; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Edgar Wang; George Gavrilopoulos; Henry Ricardo, New York Math Circle, Tappan, NY, USA; Lucie Wang, Lycée Louis le Grand, Paris, France; Vincelot Ravoson, Lycée Henri IV, Paris, France; Arkady Alt, San Jose, California, USA; David E. Manes, Oneonta, NY, USA; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Georgios Tsapakidis, Panagia Prousiotissa Private School, Agrinio, Greece; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Duy Quan Tran, Nguyen Binh Khiem high school for the gifted, Vinhlong, Vietnam; Sewon Park, Peddie School, Hightstown, NJ, USA; Hyun Min Victoria Woo, Northfield Mount Hermon School, Mount Hermon, MA,USA; Madhurima Mondal, Kalyani University Experimental High School, India; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Albert Stadler, Herrliberg, Switzerland; Neculai Stanciu and Titu Zvonaru, Romania; Paul Revenant; Prishtina Math Gymnasium Problem Solving Group, Republic of Kosova; SooYoung Choi, ChungDam Middle School, Seoul, South Korea; Joel Schlosberg, Bayside, NY; Daniel Lasaosa, Pamplona, Spain; Jorge Ledesma, Faculty of Sciences UNAM, Mexico City, Mexico; Polyahedra, Polk State College, FL, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Eeshan Banerjee and Ajay Kumar Banerjee, West Bengal, India; Tristan Shin, San Diego, California, USA; Stefan Petrevski, Pearson College UWC, Metchosin, British Columbia, Canada.

J356. Find all positive integers n such that

$$2(6+9i)^n - 3(1+8i)^n = 3(7+4i)^n.$$

Proposed by Titu Andreescu, University if Texas at Dallas, USA

Solution by Gabriel Chicas Reyes, El Salvador

Rearranging the given equation and taking absolute values yields

$$|2(6+9i)^n| = |3(1+8i)^n + 3(7+4i)^n|.$$

But $|3(1+8i)^n + 3(7+4i)^n| \leq |3(1+8i)^n| + |3(7+4i)^n$ by the triangle inequality. Thus we have

$$|2(6+9i)^n| \leq |3(1+8i)^n| + |3(7+4i)^n|$$

and computing the absolute values yields

$$2\left(\sqrt{6^2+9^2}\right)^n \leq 3\left(\sqrt{1^2+8^2}\right)^n + 3\left(\sqrt{7^2+4^2}\right)^n$$

which further simplifies to

$$9^{n-1} \leq 5^n.$$

But this inequality is satisfied only for $n = 1, 2, 3$. Indeed, as $9^{4-1} = 729 > 625 = 5^4$ and $9^m \geq 5^m$ for all $n \geq 0$, we have $9^{m+3} > 5^{m+4}$ for all $m \geq 0$. Hence $9^{n-1} > 5^n$ for every $n \geq 4$.
It remains to verify whether $2(6+9i)^n = 3(1+8i)^n + 3(7+4i)^n$ holds for $n = 1, 2, 3$. For $n = 1, 3$ both sides of the given equation respectively become

$$12 + 18i \neq 24 + 36i$$
$$-2484 + 486i \neq -552 + 108i.$$

For $n = 2$ both sides equal $-90 + 216i$, so this is the only solution.

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA; Seung Hwan An, The Taft School; Hyun Min Victoria Woo, Northfield Mount Hermon School, Mount Hermon, MA; Ji Eun Kim, Tabor Academy, MA; Kyoung A Lee, The Hotchkiss School, Lakeville, CT; Yeonjune Kang, Peddie School; YunJin Jeong, Emma Willard School, Troy, NY, USA; Joseph Lee, Loomis Chattee School, Windsor, CT; Kwonil Ko, Cushing Academy, Ashburham, MA; Hyunsun (Heidi) Kim, Trinity School, NY, USA; Albert Stadler, Herrliberg, Switzerland; Moubinool Omarjee, Lycée Henri IV, Paris, France; Hyun Min Victoria Woo, Northfield Mount Hermon School, Mount Hermon, MA; Sewon Park, Peddie School, Hightstown, NJ, USA; Michael Tang, Edina High School, MN, USA; Nikos Kalapodis, Patras, Greece; Jamshid Yakhshivev, Academic Lyceum No. 3, Bukhara, Uzbekistan; Polyahedra, Polk State College, FL, USA; Tristan Shin, San Diego, California, USA; Arkady Alt, San Jose, California, USA; Joel Schlosberg, Bayside, NY; Jorge Ledesma, Faculty of Sciences UNAM, Mexico City; Daniel Lasaosa, Pamplona, Spain; Prishtina Math Gymnasium Problem Solving Group, Republic of Kosova.

J357. Prove that for any $z \in \mathbb{C}$ such that $|z + \frac{1}{z}| = \sqrt{5}$,

$$\left(\frac{\sqrt{5}-1}{2}\right)^2 \leq |z| \leq \left(\frac{\sqrt{5}+1}{2}\right)^2$$

Proposed by Mihály Bencze, Braşov, România

Solution by Tristan Shin, San Diego, CA, USA

The condition is equivalent to $\left|z^2+1\right| = \sqrt{5}\,|z|$. By the Triangle Inequality, $\left|z^2+1\right| + |-1| \geq \left|z^2\right|$, so $\sqrt{5}\,|z| = \left|z^2+1\right| \geq \left|z^2\right| - 1$. Solving this inequality gives

$$|z| \leq \frac{3+\sqrt{5}}{2} = \left(\frac{\sqrt{5}+1}{2}\right)^2.$$

Now, by the Triangle Inequality again, $\left|z^1+1\right| + \left|-z^2\right| \geq |1|$, so $\sqrt{5}\,|z| = \left|z^2+1\right| \geq 1 - z^2$. Solving this inequality gives

$$|z| \geq \frac{3-\sqrt{5}}{2} = \left(\frac{\sqrt{5}-1}{2}\right)^2.$$

Combining these two inequality gives the desired result.

Also solved by Jamshid Yakhshivev, Academic Lyceum No. 3, Bukhara, Uzbekistan; Arkady Alt, San Jose, California, USA; Joel Schlosberg, Bayside, NY; Daniel Lasaosa, Pamplona, Spain; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Polyahedra, Polk State College, FL, USA; Nikos Kalapodis, Patras, Greece; Prishtina Math Gymnasium Problem Solving Group, Republic of Kosova; Jorge Ledesma, Faculty of Sciences UNAM, Mexico City; Michael Tang, Edina High School, MN, USA; Zafar Ahmed, BARC, India; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Seung Hwan An, The Taft School; Sewon Park, Peddie School, Hightstown, NJ, USA; Albert Stadler, Herrliberg, Switzerland; Ji Eun Kim, Tabor Academy, MA; Hyun Jin Kim, Stuyvesant High School, New York, NY, USA; Kwonil Ko, Cushing Academy, Ashburham, MA; Yeonjune Kang, Peddie School; Kyoung A Lee, The Hotchkiss School, Lakeville, CT; Hyunsun (Heidi) Kim, Trinity School, NY, USA; YunJin Jeong, Emma Willard School, Troy, NY, USA; Joseph Lee, Loomis Chattee School, Windsor, CT, USA.

J358. Prove that for $x \in \mathbb{R}$, the equations,

$$2^{2^{x-1}} = \frac{1}{2^{2x} - 1} \text{ and } 2^{2^{x+1}} = \frac{1}{2^{2^{x-1}} - 1}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Tolibjon Ismoilov, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan

First of all we denote $n = 2^{2^{x-1}}$, $(n > 1)$ and rewrite the given equations:

$$n^3 - n - 1 = 0 \qquad \text{and} \qquad n^5 - n^4 - 1 = 0 \Leftrightarrow (n^2 - n + 1)(n^3 - n - 1) = 0.$$

$n^2 - n + 1 = 0$ has no solutions in real numbers, and $n^3 - n - 1 = 0$ is equivalent to the first equation. Then

$$2^{2^{x-1}} = \frac{1}{2^{2^x} - 1} \quad \text{and} \quad 2^{2^{x+1}} = \frac{1}{2^{2^{x-1}} - 1}$$

are equivalent.

Also solved by Jamshid Yakhshivev, Academic Lyceum No. 3, Bukhara, Uzbekistan; Duy Quan Tran, Nguyen Binh Khiem high school for the gifted, Vinhlong, Vietnam; Seung Hwan An, The Taft School; Sewon Park, Peddie School, Hightstown, NJ, USA; Hyun Jin Kim, Stuyvesant High School, New York, NY, USA; Moubinool Omarjee, Lycee Henri IV, Paris, France; Albert Stadler, Herrliberg, Switzerland; Edgar Wang; Lucie Wang, Lycée Louis le Grand, Paris, France; YunJin Jeong, Emma Willard School, Troy, NY, USA; Hyunsun (Heidi) Kim, Trinity School, NY, USA; Yeonjune Kang, Peddie School; Joseph Lee, Loomis Chattee School, Windsor, CT; Kyoung A Lee, The Hotchkiss School, Lakeville, CT; Kwonil Ko, Cushing Academy, Ashburham, MA; Ji Eun Kim, Tabor Academy, MA; Vincelot Ravoson, Lycée Henri IV, Paris, France; Polyahedra, Polk State College, FL, USA; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Arkady Alt, San Jose, California, USA; Michael Tang, Edina High School, MN, USA; Joel Schlosberg, Bayside, NY; Tristan Shin, San Diego, California, USA; Jorge Ledesma, Faculty of Sciences UNAM, Mexico City; Latofat Bobojonova, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Prishtina Math Gymnasium Problem Solving Group, Republic of Kosova; Daniel Lasaosa, Pamplona, Spain; David E. Manes, Oneonta, NY, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Khurshid Juraev, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzekistan.

J359. The midline of triangle ABC, parallel to side BC, intersects the triangle's circumcircle at B' and C'. Evaluate the length of segment $B'C'$ in terms of triangle ABC's side-lengths.

Proposed by Dorin Andrica and Dan Ştefan Marinescu, România

Solution by Polyahedra, Polk State College, FL, USA

Let M and N be the midpoints of AB and AC. Let $x = B'M$ and $y = NC'$. Then by Power of a Point, $x\left(\frac{a}{2}+y\right) = \frac{c^2}{4}$ and $\left(x+\frac{a}{2}\right)y = \frac{b^2}{4}$. Subtracting the two equations we get $x - y = \frac{c^2-b^2}{2a}$; adding the two equations we get $\frac{a}{2}(x+y) + 2xy = \frac{c^2+b^2}{4}$. Since $4xy = (x+y)^2 - (x-y)^2$, we then have

$$\frac{c^2+b^2}{2} = a(x+y) + (x+y)^2 - (x-y)^2 = \left(\frac{a}{2}+x+y\right)^2 - \frac{a^2}{4} - \left(\frac{c^2-b^2}{2a}\right)^2.$$

Hence,

$$B'C' = \frac{a}{2}+x+y = \sqrt{\frac{2a^2(c^2+b^2)+a^4+(c^2-b^2)^2}{4a^2}} = \frac{\sqrt{\left[a^2+(b+c)^2\right]\left[a^2+(b-c)^2\right]}}{2a}.$$

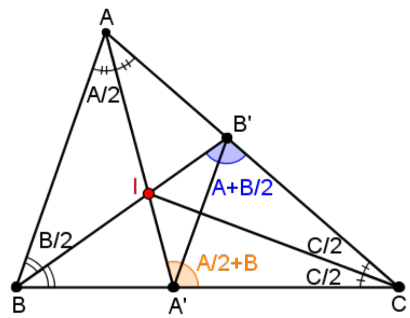
Also solved by Sewon Park, Peddie School, Hightstown, NJ, USA; Neculai Stanciu and Titu Zvonaru, Romania; Yeonjune Kang, Peddie School; YunJin Jeong, Emma Willard School, Troy, NY, USA; Hyunsun (Heidi) Kim, Trinity School, NY, USA; Joseph Lee, Loomis Chattee School, Windsor, CT; Kyoung A Lee, The Hotchkiss School, Lakeville, CT; Jamshid Yakhshivev, Academic Lyceum No. 3, Bukhara, Uzbekistan; Kwonil Ko, Cushing Academy, Ashburham, MA; Ji Eun Kim, Tabor Academy, MA; Hyun Jin Kim, Stuyvesant High School, New York, NY, USA; Seung Hwan An, The Taft School; Nikos Kalapodis, Patras, Greece; Tristan Shin, San Diego, California, USA; Daniel Lasaosa, Pamplona, Spain; Prishtina Math Gymnasium Problem Solving Group, Republic of Kosova; Tolibjon Ismoilov, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Arkady Alt, San Jose, California, USA; Lucie Wang, Lycée Louis le Grand, Paris, France.

J360. In triangle ABC, let AA' and BB' be the angle bisectors of $\angle A$ and $\angle B$. Prove that

$$\frac{A'B'}{ab\sin\frac{C}{2}} \leq \frac{1}{(b+c)\sin\left(A+\dfrac{B}{2}\right)} + \frac{1}{(c+a)\sin\left(\dfrac{A}{2}+B\right)}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Nikos Kalapodis, Patras, Greece



Let I be the incenter of triangle ABC. Applying Ptolemy's Inequality to quadrilateral $IB'CA'$ we obtain

$$A'B' \cdot IC \leq IB' \cdot A'C + IA' \cdot B'C, \quad \text{i.e.} \quad A'B' \leq \frac{IB'}{IC} \cdot A'C + \frac{IA'}{IC} \cdot B'C \quad (1)$$

From the law of sines in triangles $IB'C$ and $IA'C$ we obtain

$$\frac{IB'}{IC} = \frac{\sin\dfrac{C}{2}}{\sin\left(A+\dfrac{B}{2}\right)}, \quad \frac{IA'}{IC} = \frac{\sin\dfrac{C}{2}}{\sin\left(\dfrac{A}{2}+B\right)} \quad (2)$$

Also, by the angle bisector theorem we have $A'C = \dfrac{ab}{b+c}$, $B'C = \dfrac{ab}{c+a}$ (3)

Substituting (2), (3) to (1) we have

$$A'B' \leq \frac{\sin\dfrac{C}{2}}{\sin\left(A+\dfrac{B}{2}\right)} \cdot \frac{ab}{b+c} + \frac{\sin\dfrac{C}{2}}{\sin\left(\dfrac{A}{2}+B\right)} \cdot \frac{ab}{c+a}$$

i.e.

$$\frac{A'B'}{ab\sin\dfrac{C}{2}} \leq \frac{1}{(b+c)\sin\left(A+\dfrac{B}{2}\right)} + \frac{1}{(c+a)\sin\left(\dfrac{A}{2}+B\right)}.$$

Equality holds iff quadrilateral $IB'CA'$ is cyclic, i.e. $\angle C = 60^o$.

Also solved by Kyoung A Lee, The Hotchkiss School, Lakeville, CT; Yeonjune Kang, Peddie School; Joseph Lee, Loomis Chattee School, Windsor, CT; YunJin Jeong, Emma Willard School, Troy, NY, USA; Hyunsun (Heidi) Kim, Trinity School, NY, USA; Neculai Stanciu and Titu Zvonaru, Romania; Hyun Jin Kim, Stuyvesant High School, New York, NY, USA; Sewon Park, Peddie School, Hightstown, NJ, USA; Seung Hwan An, The Taft School; Kwonil Ko, Cushing Academy, Ashburham, MA; Ji Eun Kim, Tabor Academy, MA; Polyahedra, Polk State College, FL, USA; Tristan Shin, San Diego, California, USA; Latofat Bobojonova, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Daniel Lasaosa, Pamplona, Spain; Tolibjon Ismoilov, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Adnan Ali, Student in A.E.C.S-4, Mumbai, India.

Senior problems

S355. Let a, b, c be nonnegative real numbers such that $ab + bc + ca = 1$ and $\min(a, b, c) \le \sqrt{2}|a + b + c - 2|$. Prove that

$$(a + b + c - 2)^4 \ge 16(a + b + c - 1)(abc + a + b + c - 2).$$

Proposed by Marcel Chiriţă, Bucharest, România

Solution by Li Zhou, Polk State College, USA

The "$\min(a, b, c)$" should be "$\max(a, b, c)$", otherwise the conclusion is false for $(a, b, c) = (3, \frac{1}{5}, \frac{1}{8})$. Without loss of generality, assume that $0 \le c \le b \le a \le \sqrt{2}|a + b + c - 2|$. The condition $ab + bc + ca = 1$ implies $a + b + c \ge \sqrt{3(ab + bc + ca)} > 1$ and $abc + a + b + c - 2 = (a - 1)(b - 1)(c - 1)$. If $a \le 1$, then $(a + b + c - 1)(a - 1)(b - 1)(c - 1) \le 0$. Hence it suffices to consider $a > 1$. Then $b < 1$, and by the AM-GM inequality,

$$\begin{aligned} & 16(a + b + c - 1)(a - 1)(1 - b)(1 - c) \\ \le\ & 16\left(a - 1 + \frac{b + c}{2}\right)^2\left(1 - \frac{b + c}{2}\right)^2 = (2a - 2 + b + c)^2(2 - b - c)^2 \\ =\ & \left[a^2 - (a + b + c - 2)^2\right]^2 \le \left[2(a + b + c - 2)^2 - (a + b + c - 2)^2\right]^2 = (a + b + c - 2)^4. \end{aligned}$$

S356. Let a, b, c, d, e be real numbers such that $\sin a + \sin b + \sin c + \sin d + \sin e \geq 3$. Prove that $\cos a + \cos b + \cos c + \cos d + \cos \leq 4$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

Using the Cauchy-Schwartz inequality, we have:

$$\begin{aligned}\sum_{cyc} \cos a &\leq \sqrt{\sum_{cyc} 1^2 \cdot \sum_{cyc} \cos^2 a} = \sqrt{5 \cdot \sum_{cyc}(1 - \sin^2 a)} \\ &= \sqrt{25 - 5 \cdot \sum_{cyc} \sin^2 a} = \sqrt{25 - \sum_{cyc} 1^2 \cdot \sum_{cyc} \sin^2 a} \\ &\leq \sqrt{25 - \left(\sum_{cyc} \sin a\right)^2} \leq \sqrt{25 - 9} = 4.\end{aligned}$$

Equality holds only when $\sin a = \sin b = \sin c = \sin d = \sin e = \dfrac{3}{5}$.

Also solved by Ioan Viorel Codreanu, Satulung, Maramures, Romania; Moubinool Omarjee, Lycee Henri IV, Paris, France; Nicusor Zlota, Traian Vuia Technical College, Focsani ,Romania; Albert Stadler, Herrliberg, Switzerland; Seung Hwan An, The Taft School; Sewon Park, Peddie School, Hightstown, NJ, USA; Hyun Jin Kim, Stuyvesant High School, New York, NY, USA; Kwonil Ko, Cushing Academy, Ashburham, MA; Yeonjune Kang, Peddie School; Joseph Lee, Loomis Chattee School, Windsor, CT; YunJin Jeong, Emma Willard School, Troy, NY, USA; Kyoung A Lee, The Hotchkiss School, Lakeville, CT; Ji Eun Kim, Tabor Academy, MA; SooYoung Choi, ChungDam Middle School, Seoul, South Korea; Nikos Kalapodis, Patras, Greece; Li Zhou, Polk State College, USA; Brian Bradie, Christopher Newport University, Newport News, VA; Jorge Ledesma, Faculty of Sciences UNAM, Mexico City; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Daniel Lasaosa, Pamplona, Spain; Latofat Bobojonova, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Khurshid Juraev, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Tolibjon Ismoilov, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan.

S357. Prove that in any triangle,

$$\sum \sqrt{\frac{a(h_a - 2r)}{(3a+b+c)(h_a+2r)}} \le \frac{3}{4}$$

Proposed by Mihály Bencze, Braşov, România

Solution by Arkady Alt, San Jose, California, USA

Since $h_a = \dfrac{2sr}{a}$, then

$$\frac{h_a - 2r}{h_a + 2r} = \frac{\dfrac{2sr}{a} - 2r}{\dfrac{2sr}{a} + 2r} = \frac{s-a}{s+a}\ ,$$

and by AM-GM,

$$\frac{a\,(h_a - 2r)}{(3a+b+c)\,(h_a+2r)} = \frac{a\,(s-a)}{(2a+2s)\,(s+a)} = \frac{a\,(s-a)}{2\,(s+a)^2} = \frac{2a\,(s-a)}{4\,(s+a)^2} \le \frac{\left(\dfrac{2a+(s-a)}{2}\right)^2}{4\,(s+a)^2} = \frac{1}{16}.$$

Hence, $\displaystyle\sum_{cyc} \sqrt{\frac{a\,(h_a - 2r)}{(3a+b+c)\,(h_a+2r)}} \le \sum_{cyc} \sqrt{\frac{1}{16}} = \frac{3}{4}.$

Also solved by Edgar Wang; Albert Stadler, Herrliberg, Switzerland; Seung Hwan An, The Taft School; Neculai Stanciu and Titu Zvonaru, Romania; Hyun Jin Kim, Stuyvesant High School, New York, NY, USA; Ji Eun Kim, Tabor Academy, M; Joseph Lee, Loomis Chattee School, Windsor, CT; Yeonjune Kang, Peddie School; Kwonil Ko, Cushing Academy, Ashburham, MA; Kyoung A Lee, The Hotchkiss School, Lakeville, CT; SooYoung Choi, ChungDam Middle School, Seoul, South Korea; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Li Zhou, Polk State College, USA; Khurshid Juraev, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Brian Bradie, Christopher Newport University, Newport News, VA; Latofat Bobojonova, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Daniel Lasaosa, Pamplona, Spain; Tolibjon Ismoilov, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Nikos Kalapodis, Patras, Greece; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania.

S358. Prove that for each integer n, there are eighteen integers such that both their sum and the sum of their fifth powers are equal to n.

Proposed by Nairi Sedrakyan, Armenia

Solution by Li Zhou, Polk State College, USA

First, consider $n = 2m - 1$. Notice that $(2m-1)^5 - (2m-1) = 8m(m-1)(2m-1)(2m^2-2m+1)$. If $m \equiv 1 \pmod 3$ then $m - 1 \equiv 0 \pmod 3$; if $m \equiv -1 \pmod 3$ then $2m - 1 \equiv 0 \pmod 3$. If $m \equiv 1 \pmod 5$ then $m - 1 \equiv 0 \pmod 5$; if $m \equiv 2, -1 \pmod 5$ then $2m^2 - 2m + 1 \equiv 0 \pmod 5$; if $m \equiv -2 \pmod 5$ then $2m - 1 \equiv 0 \pmod 5$. Since $120 = 8 \cdot 3 \cdot 5$, we conclude that $(2m-1)^5 \equiv 2m - 1 \pmod{120}$. So we can write $(2m-1)^5 = 120q + 2m - 1$.

Let $a_1 = 2m-1$, $a_2 = 0$, $a_3 = -(q-2)$, $a_4 = \cdots = a_7 = q - 1$, $a_8 = \cdots = a_{13} = -q$, $a_{14} = \cdots = a_{17} = q+1$, and $a_{18} = -(q+2)$. Then $a_3 + \cdots + a_{18} = 0$ and

$$a_3^5 + \cdots + a_{18}^5 = -(q-2)^5 + 4(q-1)^5 - 6q^5 + 4(q+1)^5 - (q+2)^5 = -120q.$$

Hence $a_1 + \cdots + a_{18} = 2m - 1 = a_1^5 + \cdots + a_{18}^5$.

Now if $n = 2m$, then we simply change a_2 from 0 to 1, and it is obvious that $a_1 + \cdots + a_{18} = 2m = a_1^5 + \cdots + a_{18}^5$.

S359. Prove that in any triangle,

$$m_a\left(\frac{1}{2r_a}-\frac{R}{bc}\right)+m_b\left(\frac{1}{2r_b}-\frac{R}{ca}\right)+m_c\left(\frac{1}{2r_c}-\frac{R}{ab}\right)\geq 0.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Brian Bradie, Christopher Newport University, Newport News, VA

With

$$r_a=\frac{\Delta}{s-a},\quad r_b=\frac{\Delta}{s-b},\quad r_c=\frac{\Delta}{s-c},\quad \text{and}\quad R=\frac{abc}{4\Delta},$$

where Δ is the area of the triangle and s is the semi-perimeter, it follows that

$$\begin{aligned}\frac{1}{2r_a}-\frac{R}{bc} &= \frac{2(s-a)-a}{4\Delta},\\ \frac{1}{2r_b}-\frac{R}{ac} &= \frac{2(s-b)-b}{4\Delta},\ \text{and}\\ \frac{1}{2r_c}-\frac{R}{ab} &= \frac{2(s-c)-c}{4\Delta}.\end{aligned}$$

Thus,

$$m_a\left(\frac{1}{2r_a}-\frac{R}{bc}\right)+m_b\left(\frac{1}{2r_b}-\frac{R}{ac}\right)+m_c\left(\frac{1}{2r_c}-\frac{R}{ab}\right)\geq 0$$

is equivalent to

$$(a+b+c)(m_a+m_b+m_c)\geq 3(am_a+bm_b+cm_c). \tag{1}$$

Without loss of generality, suppose that $a\geq b\geq c$. Then $m_a\leq m_b\leq m_c$, and (1) follows from Chebyshev's sum inequality.

Also solved by Sewon Park, Peddie School, Hightstown, NJ, USA; Hyun Min Victoria Woo, Northfield Mount Hermon School, Mount Hermon, MA; YunJin Jeong, Emma Willard School, Troy, NY, USA; Kyoung A Lee, The Hotchkiss School, Lakeville, CT; Kwonil Ko, Cushing Academy, Ashburham, MA; Ji Eun Kim, Tabor Academy, MA; Hyun Jin Kim, Stuyvesant High School, New York, NY, USA; Seung Hwan An, The Taft School; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Nikos Kalapodis, Patras, Greece; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Li Zhou, Polk State College, Winter Haven, FL; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Arkady Alt, San Jose, California, USA; Jorge Ledesma, Faculty of Sciences UNAM, Mexico City; Daniel Lasaosa, Pamplona, Spain; Khurshid Juraev, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Latofat Bobojonova, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan.

S360. Let ABC be a triangle with orthocenter H and circumcenter O. The parallels through B and C to AO intersect the external angle bisector of $\angle BAC$ at D and E, respectively. Prove that BE, CD, AH are concurrent.

Proposed by Iman Munire Bilal, University of Cambridge and Marius Stanean, România

Solution by SooYoung Choi, ChungDam Middle School, Seoul, South Korea

Let H' be the second intersection of AH with the circumcircle of $\triangle ABC$. Since AE is the external bisector of $\angle BAC$, $\angle DAB = \angle EAC$. Since $DB \parallel AO$ and AH and AO are isogonal,

$$\angle DBA = \angle OAB = \angle H'AC = \angle H'BC,$$

using the fact that $H'ABC$ is cyclic. In a similar way, $\angle ACD = \angle H'CB$. Therefore,

$$\angle DAB = \angle EAC, \angle DBA = \angle H'BC, \angle ACE = \angle BCH',$$

so by Jacobi's theorem, AH, CD, and BE are concurrent.

Also solved by Li Zhou, Polk State College, Winter Haven, FL; Daniel Lasaosa, Pamplona, Spain; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Gabriel Chicas Reyes, El Salvador; Khurshid Juraev, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Philippe Fondanaiche, Paris, France; George Gavrilopoulos.

Undergraduate problems

U355. Let a be a real number such that $a \neq 0$ and $a \neq \pm 1$, and let n be an integer greater than 1. Find all polynomials $P(X)$ with real coefficients such that

$$\left(a^2X^2+1\right)P(aX) = \left(a^{2n}X^2+1\right)P(X).$$

Proposed by Marcel Chiriţă, Bucharest, România

Solution by Li Zhou, Polk State College, USA

Notice that for $1 \leq j < k \leq n$, $a^{2j}X^2+1$ has complex zeros $\pm i/a^j$, which are not zeros of $a^{2k}X^2+1$. Hence, the given functional equation forces $P(X) = (a^2X^2+1)Q_1(X)$. Then $P(aX) = (a^4X^2+1)Q_1(aX)$, which in turn forces $Q_1(X) = (a^4X^2+1)Q_2(X)$. Inductively, we see that

$$P(X) = (a^2X^2+1)(a^4X^2+1)\cdots(a^{2n-2}X^2+1)Q_{n-1}(X).$$

Now the functional equation is satisfied if and only if $Q_{n-1}(aX) = Q_{n-1}(X)$, thus $Q_{n-1}(X)$ must be a real constant c. Therefore,

$$P(X) = c(a^2X^2+1)(a^4X^2+1)\cdots(a^{2n-2}X^2+1), \quad c \in \mathbb{R}.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Albert Stadler, Herrliberg, Switzerland; Shohruh Ibragimov, National University of Uzbekistan, Tashkent, Uzbekistan; Jorge Ledesma, Faculty of Sciences UNAM, Mexico City, Mexico; Kwonil Ko, Cushing Academy, Ashburham, MA; Ji Eun Kim, Tabor Academy, MA; Hyun Jin Kim, Stuyvesant High School, New York, NY, USA; Seung Hwan An, The Taft School.

U356. Let $(x_n)_{n\geq 1}$ be a monotonic sequence, and let $a \in (-1,0)$. Find

$$\lim_{n\to\infty}\left(x_1a^{n-1}+x_2a^{n-2}+\cdots+x_n\right).$$

Proposed by Mihai Piticari and Sorin Rădulescu, România

Solution by Daniel Lasaosa, Pamplona, Spain

Denote by y_n the expression whose limit we are asked to find. Note first that

$$y_n - ay_{n-1} = x_1a^{n-1}+x_2a^{n-2}+\cdots+x_n - a\left(x_1a^{n-2}+x_2a^{n-3}+\cdots+x_{n-1}\right) = x_n.$$

Now, if y_n converges to limit L, for any $\epsilon > 0$ there is an N such that for all $n \geq N$ we have $|y_n - L| < \epsilon$. Then, $x_{n+1} = y_{n+1} - ay_{n-1} \in ((1-a)L - 2\epsilon, (1-a)L + 2\epsilon)$, or sequence (x_n) needs to converge (in fact, it must converge to $(1-a)L$ where L is the limit of (y_n)) for $(x_n)_{n\geq 1}$ to converge. Since (x_n) is monotonic, it converges iff it is bounded.

Case 1: If $(x_n)_{n\geq 1}$ is bounded, then it is convergent with some limit L. Consider any $\epsilon > 0$, and note that since $|a| < 1$, then $|a|^n \to 0$ when $n \to \infty$, ie., there exists some N_1 such that, for all $n \geq N_1$, $|a^n| < \left|\frac{(1-a)\epsilon}{3L}\right|$. At the same time, since the sequence (x_n) is convergent with limit L, there exists some M_1 such that, for all $n \geq M_1$, we have $|x_n - L| < \frac{(1-|a|)\epsilon}{3}$. On the other hand, since (x_n) is monotonic and bounded, there exists a difference $D = |x_1 - L| \geq |x_n - L|$ for all n. Using again that $|a|^n$ converges to 0, there exists some M_2 such that, for all $n \geq M_2$, we have $|a|^{n+1} < \frac{(1-|a|)\epsilon}{3D}$. Taking $M = \max\{M_1, M_2\}$ and $N = \max\{N_1, M+2\}$, we have for all $n \geq N$ that

$$\left|\frac{L}{1-a} - L\left(1+a+a^2+\cdots+a^{n-1}\right)\right| = \left|\frac{a^nL}{1-a}\right| < \frac{\epsilon}{3},$$

while

$$\left|x_n + ax_{n-1} + a^2x_{n-2} + \cdots + a^Mx_{n-M} - L\left(1+a+a^2+\cdots+a^M\right)\right| \leq$$

$$\leq |x_n - L| + |a|\,|x_{n-1}-L| + \cdots + |a|^M\,|x_{n-M}-L| < \frac{\left(1-|a|^{M+1}\right)\epsilon}{3} < \frac{\epsilon}{3},$$

and

$$\left|a^{M+1}x_{n-M-1} + \cdots + a^{n-1}x_1 - L\left(a^{M+1}+\cdots+a^{n-1}\right)\right| \leq$$

$$\leq |a|^{M+1}\left(|x_{n-M-1}-L| + \cdots + |a|^{n-M-2}\,|x_1 - L|\right) \leq$$

$$\leq |a|^{M+1}D\left(1+|a|+\cdots+|a|^{n-M-2}\right) < \frac{\left(1-|a|^{n-M-1}\right)\epsilon}{3} < \frac{\epsilon}{3},$$

or using these three equations,

$$\left|y_n - \frac{L}{1-a}\right| \leq |x_n - L| + |a|\,|x_{n-1}-L| + \cdots + |a|^{n-1}\,|x_1 - L| + \left|\frac{a^nL}{1-a}\right| < \epsilon.$$

It then follows that, if $(x_n)_{n\geq 1}$ is bounded hence convergent to some limit L, then

$$\lim_{n\to\infty}\left(x_1a^{n-1}+x_2a^{n-2}+\cdots+x_n\right) = \frac{L}{1-a}.$$

If $(x_n)_{n\geq 1}$ is increasing but not bounded, note that $x_i > x_{i-1}$ for all $i \geq 2$, whereas for some $N \geq 0$ we have $x_n \geq 0$, and for some $M \geq 0$ we have $x_n \geq |x_1 + x_2 + \cdots + x_{N-1}|$ for all $n \geq M$, or for all even $n \geq M+2$, and since $0 < a^2 < 1$, we have

$$x_1a^{n-1}+x_2a^{n-2}+\cdots+x_n \geq (1+a)\left(x_n + a^2x_{n-2}+\cdots+a^{n-2}x_2\right) \geq (1+a)x_n,$$

whereas for all odd $n \geq M+2$, we have

$$x_1 a^{n-1} + x_2 a^{n-2} + \cdots + x_n \geq (1+a)\left(x_n + a^2 x_{n-2} + \cdots + a^{n-2} x_2\right) + a^{n-1} x_1 \geq (1+a)x_n + a^{n-1} x_1,$$

both of which diverge to $+\infty$ because a^{n-1} converges to 0, or

$$\lim_{n\to\infty} \left(x_1 a^{n-1} + x_2 a^{n-2} + \cdots + x_n\right) = +\infty.$$

An analogous argument allows us to conclude that, if $(x_n)_{n\geq 1}$ is decreasing but not bounded, then

$$\lim_{n\to\infty} \left(x_1 a^{n-1} + x_2 a^{n-2} + \cdots + x_n\right) = -\infty.$$

Note: The previous manipulations in Case 1 assume that $D > 0$ and $L \neq 0$. If $D = 0$, then (x_n) is constant with value L,

$$y_n = L\left(1 + a + \cdots + a^{n-1}\right) = \frac{L}{1-a} - \frac{La^n}{1-a},$$

and the second term in the RHS converges to 0 because $|a| < 1$, with identical result as in the general case. If $D \neq 0$ but $L = 0$, we may skip the definition of N_1, define M as in the general case, and take $N = M+2$, or for all $n \geq N$, we have $|y_n| < \frac{2\epsilon}{3} < \epsilon$, and (y_n) converges to $0 = \frac{L}{a-1}$, again as in the general case.

Also solved by Moubinool Omarjee, Lycée Henri IV, Paris, France; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy.

U357. Evaluate

$$\lim_{n\to\infty}\left[\frac{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)\cdots\left(1+\frac{n}{n^2}\right)}{\sqrt{e}}\right]^n$$

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj Napoca, România

Solution by Arkady Alt, San Jose, California, USA

Let $a_n := \ln\left(\dfrac{\left(1+\dfrac{1}{n^2}\right)\left(1+\dfrac{2}{n^2}\right)...\left(1+\dfrac{n}{n^2}\right)}{\sqrt{e}}\right)^n = n\left(\sum\limits_{k=1}^n \ln\left(1+\dfrac{k}{n^2}\right)-\dfrac{1}{2}\right), n\in\mathbb{N}.$

Since $\ln(1+x)=x-\dfrac{x^2}{2}+\dfrac{x^3}{3}+o\left(x^3\right)$ then $\ln\left(1+\dfrac{k}{n^2}\right)=\dfrac{k}{n^2}-\dfrac{k^2}{2n^4}+\dfrac{k^3}{3n^6}+o\left(\dfrac{1}{n^3}\right)$ for $k=1,2,...,n$.

Hence,

$$a_n = n\left(\sum_{k=1}^n\left(\frac{k}{n^2}-\frac{k^2}{2n^4}+\frac{k^3}{3n^6}+o\left(\frac{1}{n^3}\right)\right)-\frac{1}{2}\right)=$$

$$\sum_{k=1}^n\left(\frac{k}{n}-\frac{k^2}{2n^3}+\frac{k^3}{3n^5}+no\left(\frac{1}{n^3}\right)\right)-\frac{n}{2}=\sum_{k=1}^n\frac{k}{n}-\frac{n}{2}-\sum_{k=1}^n\frac{k^2}{2n^3}+\sum_{k=1}^n\frac{k^3}{3n^5}+o\left(\frac{1}{n^2}\right)=$$

$$\frac{n(n+1)}{2n}-\frac{n}{2}-\frac{n(n+1)(2n+1)}{12n^3}+\frac{n^2(n+1)^2}{12n^5}+o\left(\frac{1}{n^2}\right)=$$

$$\frac{1}{2}-\frac{(n+1)(2n+1)}{12n^2}+o\left(\frac{1}{n}\right)$$

and, therefore, $\lim\limits_{n\to\infty} a_n=\dfrac{1}{2}-\dfrac{1}{6}=\dfrac{1}{3}$.

Since

$$\left(\frac{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)...\left(1+\frac{n}{n^2}\right)}{\sqrt{e}}\right)^n = e^{a_n}$$

then

$$\lim_{n\to\infty}\left(\frac{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)...\left(1+\frac{n}{n^2}\right)}{\sqrt{e}}\right)^n = e^{\lim\limits_{n\to\infty} a_n}=\sqrt[3]{e}.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Albert Stadler, Herrliberg, Switzerland; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Moubinool Omarjee, Lycée Henri IV, Paris, France; Joshua Siktar,Carnegie Mellon University, USA; Jorge Ledesma, Faculty of Sciences UNAM, Mexico City; Hyun Min Victoria Woo, Northfield Mount Hermon School, Mount Hermon, MA; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Li Zhou, Polk State College, USA; Zafar Ahmed, BARC, India; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Kwonil Ko, Cushing Academy, Ashburham, MA, USA; Ji Eun Kim, Tabor Academy, MA, USA; Hyun Jin Kim, Stuyvesant High School, New York, NY, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA.

U358. Let $(x_n)_{n\geq 1}$ be an increasing sequence of real numbers for which there is a real number $a > 2$ such that

$$x_{n+1} \geq ax_n - (a-1)x_{n-1},$$

for all $n \geq 1$. Prove that $(x_n)_{n\geq 1}$ is divergent.

Proposed by Mihai Piticari and Sorin Rădulescu, România

Solution by Daniel Lasaosa, Pamplona, Spain

Note that

$$x_{n+1} - x_{n-1} \geq a\left(x_n - x_{n-1}\right) > 2\left(x_n - x_{n-1}\right), \qquad x_{n+1} - x_n > x_n - x_{n-1}.$$

Denote $y_n = x_{n+1} - x_n$, or $(y_n)_{n\geq 1}$ is an increasing sequence of positive reals, hence it either converges to a positive real, or it diverges. In either case, there is a positive real r and a positive integer N such that, for all $n \geq N$, then $y_n \geq r$, where r can be taken as half the limit of y_n in the case it converges, and r can take any positive real value in the case that y_n diverges. In either case, for every $n \geq N$, we have $x_n \geq (n-N)r$, which clearly diverges. The conclusion follows.

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Moubinool Omarjee, Lycée Henri IV, Paris, France; Jorge Ledesma, Faculty of Sciences UNAM, Mexico City; Francisco Javier Martínez Aguinaga, Universidad Complutense de Madrid, Spain; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Arkady Alt, San Jose, CA, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Kwonil Ko, Cushing Academy, Ashburham, MA, USA; Ji Eun Kim, Tabor Academy, MA, USA; Hyun Jin Kim, Stuyvesant High School, New York, NY, USA; Seung Hwan An, The Taft School.

U359. Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ be sequences of nonnegative real numbers. Furthermore, let $c_1, \ldots, c_n$ and $d_1, \ldots, d_m$ be sequences of real numbers. Prove that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \min(a_i, a_j) + \sum_{i=1}^{m}\sum_{j=1}^{m} d_i d_j \min(b_i, b_j) \geq 2\sum_{i=1}^{n}\sum_{j=1}^{m} c_i d_j \min(a_i, b_j).$$

Proposed by Mehtaab Sawhney, Commack High School, New York, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Note first that if $a_k = 0$ for some $k \in \{1, 2, \ldots, n\}$, then all terms where c_k appear do vanish, since $\min(a_i, a_k) = \min(a_k, b_j) = a_k = 0$ because $a_i, b_j \geq 0$ for all i, j, hence the problem is equivalent to eliminating a_k, c_k, reducing n by 1 and renumbering the i's in the a_i's and c_i's from 1 to $n-1$. In other words, we may assume wlog that all a_i's and b_j's are positive, and all c_i, d_j such that $a_i = 0$ or $b_j = 0$ are irrelevant since they do not participate because the terms in which they appear, vanish.

Let u be the number of distinct positive values in $\{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m\}$. We will prove by induction on u the following

Claim: The proposed inequality always holds, with equality iff the a_i's and b_j's can be partitioned in groups such that their values are equal, and such that the sum of the c_i's and the sum of the d_j's corresponding to each group of equal-valued a_i's and b_j's are equal.

Proof: When $u = 1$, $a_1 = a_2 = \cdots = a_n = b_1 = b_2 = \cdots = b_m = K$ for some positive real K, and denoting $S_c = c_1 + c_2 + \cdots + c_n$ and $S_d = d_1 + d_2 + \cdots + d_m$, the inequality rewrites as

$$0 \leq K\left(\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j + \sum_{i=1}^{m}\sum_{j=1}^{m} d_i d_j - 2\sum_{i=1}^{n}\sum_{j=1}^{m} c_i d_j\right) = K\left({S_c}^2 + {S_d}^2 - 2S_cS_d\right) =$$

$$= K(S_c - S_d)^2,$$

and the Claim is clearly true in this case. Assume that the result is true for $1, 2, \ldots, u-1$, and let $\epsilon = \min\{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m\}$. Define $a_i{}' = a_i - \epsilon$ and $b_j{}' = b_j - \epsilon$. Note that the inequality then rewrites as

$$\epsilon\left(\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j + \sum_{i=1}^{m}\sum_{j=1}^{m} d_i d_j - 2\sum_{i=1}^{n}\sum_{j=1}^{m} c_i d_j\right) + \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \min(a_i{}', a_j{}') +$$

$$+ \sum_{i=1}^{m}\sum_{j=1}^{m} d_i d_j \min(b_i{}', b_j{}') \geq 2\sum_{i=1}^{n}\sum_{j=1}^{m} c_i d_j \min(a_i{}', b_j{}'),$$

where there are u distinct values of the $a_i{}'$'s and $b_j{}'$'s, but one of them is zero, and all zero $a_i{}'$'s and $b_j{}'$'s and their corresponding c_i's and d_j's can be removed, yielding $u-1$ distinct nonzero values of the $a_i{}'$'s and $b_j{}'$'s. Therefore, by hypothesis of induction for $u-1$, the last two terms in the LHS are no less than the RHS. At the same time, by hypothesis of induction for $u = 1$, the first term in the LHS is non-negative, being zero iff the sum of all c_i's and the sum of all d_j's are equal. The Claim follows for all u. Note that for equality to hold in each step in the induction process, the sum of each set of c_i's and d_j's that remain after removing each ϵ must be equal, or the sum of the c_i's and d_j's that disappear after each ϵ is removed must also be equal.

The conclusion follows directly from the Claim, equality holds if, for every nonzero value taken by some of the a_i's and b_j's, the sum of the c_i's and the sum of the d_j's corresponding to the a_i's and b_j's with equal value, are equal, and the c_i's and the d_j's corresponding to zero a_i's and b_j's are irrelevant.

Also solved by Moubinool Omarjee, Lycée Henri IV, Paris, France.

U360. Let $f : [-1, 1] \to [0, \infty)$ be C^1 monotone increasing function. Prove that

$$\int_{-1}^{1} (f'(x))^{\frac{1}{2015}} \le 2015 \int_{-1}^{1} \left(\frac{f(x)}{1-x} \right)^{\frac{1}{2015}} dx.$$

Proposed by Oleksiy Klurman, University College London

Solution by the author

Let $q = \dfrac{1}{2015}$. Changing $f(x) \to f(x) - f(0)$ we can assume $f(-1) = 0$. Integration by parts yields

$$S = \int_{-1}^{1} \frac{f^q(x)}{(1-x)^q} dx = \frac{1}{q-1} f^q(x)(1-x)^{1-q}|_{-1}^{1} + \frac{q}{1-q} \int_{-1}^{1} f'(x) f^{q-1}(x)(1-x)^{1-q} dx.$$

Since $f(-1) = 0$ we have

$$S_1 = \frac{1-q}{q} S = \int_{-1}^{1} f'(x) f^{q-1}(x)(1-x)^{1-q} dx.$$

We now estimate $S_1 + S$ to get the result:

$$S_1 + S = \frac{1}{q} S = \int_{-1}^{1} \left[\frac{f^q}{(1-x)^q} + f'(x) f^{q-1}(x)(1-x)^{1-q} \right] dx \ge \int_{-1}^{1} [f'(x)]^q dx$$

since

$$\frac{f^q(x)}{(1-x)^q} + f(x) f^{q-1}(x)(1-x)^{1-q} \ge [f'(x)]^q.$$

pointwise. Indeed, if

$$\frac{f^q(x)}{(1-x)^q} \ge [f(x)]^q$$

the inequality clearly holds. In the other case, if

$$\frac{f^q(x)}{(1-x)^q} < [f'(x)]^q,$$

then

$$[f'(x)]^{q-1} < (1-x)^{1-q} f^{q-1}(x)$$

and the second term dominates the right-hand side.

O355. Let ABC be a triangle with incenter I. Prove that

$$\frac{(IB+IC)^2}{a(b+c)}+\frac{(IC+IA)^2}{b(c+a)}+\frac{(IA+IB)^2}{c(a+b)}\leq 2.$$

Proposed by Nguyen Viet Hung, High School for Gifted Students, Hanoi University of Science, Vietnam

Solution by Arkady Alt, San Jose, CA, USA

Since

$$l_a=\frac{2\sqrt{bcs\,(s-a)}}{b+c},\frac{2\sqrt{cas\,(s-b)}}{c+a}$$

then

$$IA+IB=\frac{l_a\,(b+c)}{a+b+c}+\frac{l_b\,(c+a)}{a+b+c}=$$

$$\frac{\sqrt{bcs\,(s-a)}}{s}+\frac{\sqrt{cas\,(s-b)}}{s}=\sqrt{\frac{c}{s}}\left(\sqrt{b\,(s-a)}+\sqrt{a\,(s-b)}\right)$$

and

$$(IA+IB)^2=\frac{c}{s}\left(\sqrt{b\,(s-a)}+\sqrt{a\,(s-b)}\right)^2.$$

By Cauchy Inequality

$$\left(\sqrt{b\,(s-a)}+\sqrt{a\,(s-b)}\right)^2\leq(b+a)\,(s-a+s-b)=c\,(a+b)\,.$$

Thus,

$$(IA+IB)^2\leq\frac{c^2}{s}\,(a+b)\iff\frac{(IA+IB)^2}{c(a+b)}\leq\frac{c}{s}$$

and, therefore,

$$\sum_{cyc}\frac{(IA+IB)^2}{c(a+b)}\leq\sum_{cyc}\frac{c}{s}=2.$$

Also solved by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Khurshid Juraev, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Latofat Bobojonova, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Nicusor Zlota ,Traian Vuia Technical College, Focsani, Romania; Neculai Stanciu and Titu Zvonaru, Romania; Tolibjon Ismoilov, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Li Zhou, Polk State College, USA.

O356. We take out an even number from the set $\{1, 2, 3, \ldots, 25\}$. Find this number knowing that the remaining set has precisely 124 subsets with three elements that form an arithmetic progression.

Proposed by Marian Teler, Costeşti and Marin Ionescu, Piteşti, România

Solution by Adnan Ali, Student in A.E.C.S-4, Mumbai, India

First we prove by iteration that the set $\{1, 2, \cdots, 25\}$ has 144 subsets with three elements that form an AP. Let $f(m)$ denote the number of such subsets for the set $\{1, 2, \cdots, m\}$ $(m \geq 3)$. Consider the sets

$$A = \{1, \cdots, 2n+1\}$$
$$B = \{1, \cdots, 2n+1, 2n+2, 2n+3\},$$

where $n \in \mathbb{Z}^+$. Consider $2n+2$ in B. Any three-term AP with $2n+2$ as a member is of the form $\{2i, i+n+1, 2n+2\}$ for any integer $1 \leq i \leq n$, with another three-term AP $\{2n+1, 2n+2, 2n+3\}$. Since there are n choices for i in total, and one extra AP of which $2n+2$ is a member, there are exactly $n+1$ three-term AP's of which $2n+2$ can be a member. Next we consider $2n+3$ in B. Any AP with $2n+3$ as a member is of the form $\{2j-1, n+j+1, 2n+3\}$ for any integer $1 \leq j \leq n+1$. Thus j having $n+1$ choices, there are only $n+1$ three-term AP's of which $2n+3$ can be a member. But here we have over-counted the three-term AP $\{2n+1, 2n+2, 2n+3\}$, and so we must subtract it once from the total. Thus we have

$$f(2n+3) = f(2n+1) + n + 1 + n + 1 - 1 = f(2n+1) + 2n + 1.$$

Knowing that $f(3) = 1$, we conclude that $f(25) = 12^2 = 144$. Now consider any any even number $2k \in \{1, 2, \cdots, 25\}$. Since its removal has left the remaining set with only 124 such subsets, we conclude that $2k$ is a member of exactly $144 - 124 = 20$ three-element AP's. But any three-element AP involving $2k$ is of the form $\{2u, u+k, 2k\}$ for any integer $1 \leq u \leq 12, u \neq k$, (which implies that there are 11 choices for u) and also of the form $\{2k-v, 2k, 2k+v\}$ for $1 \leq 2k-v \leq 2k-1$ and $2k+1 \leq 2k+v \leq 25$ which gives $1 \leq v \leq \min.(2k-1, 25-2k)$. In case $\min.(2k-1, 25-2k) = 2k-1$, v has $2k-1$ choices and so in total, $11 + 2k - 1 = 20$ which yields $2k = 10$, (and clearly $10 - 1 < 25 - 10$) and the other case when $\min.(2k-1, 25-2k) = 25-2k$, there are $25-2k$ choices for v. Thus $11 + 25 - 2k = 20$ which yields $2k = 16$ (and clearly $25 - 16 < 16 - 1$). Thus we conclude that the even number removed is either 10 or 16.

Also solved by Marian Teler, Costesti and Marin Ionescu, Pitesti, Romania; Li Zhou, Polk State College, USA; Prishtina Math Gymnasium Problem Solving Group, Republic of Kosova; Daniel Lasaosa, Pamplona, Spain; Khurshid Juraev, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzekistan; Jorge Ledesma, Faculty of Sciences UNAM, Mexico City; Adithya Bhaskar, Atomic Energy Central School-2, Mumbai, India; Kwonil Ko, Cushing Academy, Ashburham, MA, USA; Ji Eun Kim, Tabor Academy, MA, USA; Hyun Jin Kim, Stuyvesant High School, New York, NY, USA.

O357. Prove that in any triangle

$$\frac{ab+4m_am_b}{c}+\frac{bc+4m_bm_c}{a}+\frac{ca+4m_cm_a}{b}\geq\frac{16K}{R}.$$

Proposed by Titu Andreescu, USA and Oleg Mushkarov, Bulgaria

Solution by Nikos Kalapodis, Patras, Greece

By the AM-GM inequality we have that $a=(s-b)+(s-c)\geq 2\sqrt{(s-b)(s-c)}$.
Similarly $b\geq 2\sqrt{(s-c)(s-a)}$ and $c\geq 2\sqrt{(s-a)(s-b)}$.

We also have $m_a^2=\dfrac{2(b^2+c^2)-a^2}{4}\geq\dfrac{(b+c)^2-a^2}{4}=s(s-a)$.
Thus, $m_a\geq\sqrt{s(s-a)}$, and similarly $m_b\geq\sqrt{s(s-b)}$ and $m_c\geq\sqrt{s(s-c)}$.

Therefore

$$\sum_{cyc}\frac{ab+4m_am_b}{c}\geq\sum_{cyc}\frac{4(s-c)\sqrt{(s-a)(s-b)}+4s\sqrt{(s-a)(s-b)}}{c}=$$

$$4\sum_{cyc}\frac{a+b}{c}\sqrt{(s-a)(s-b)}\geq 4\cdot 3\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{abc}(s-a)(s-b)(s-c)}\geq$$

$$24\sqrt[3]{(s-a)(s-b)(s-c)}\geq 24\cdot\frac{3}{\dfrac{1}{s-a}+\dfrac{1}{s-b}+\dfrac{1}{s-c}}=\frac{72}{\dfrac{r_a+r_b+r_c}{K}}=$$

$$72K\cdot\frac{1}{r_a+r_b+r_c}\geq 72K\cdot\frac{2}{9R}=\frac{16K}{R}.$$

Where we used successively the AM-GM Inequality, the well-known inequality $(a+b)(b+c)(c+a)\geq 8abc$, the AM-HM Inequality, and finally the inequality $r_a+r_b+r_c=r+4R\leq\frac{9R}{2}$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Li Zhou, Polk State College, USA; Albert Stadler, Herrliberg, Switzerland; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Latofat Bobojonova, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Jorge Ledesma, Faculty of Sciences UNAM, Mexico City; Hyun Min Victoria Woo, Northfield Mount Hermon School, Mount Hermon, MA, USA; Arkady Alt, San Jose, California, USA; Ioan Viorel Codreanu, Satulung, Maramures, Romania; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Kwonil Ko, Cushing Academy, Ashburham, MA, USA; Ji Eun Kim, Tabor Academy, MA, USA; Hyun Jin Kim, Stuyvesant High School, New York, NY, USA.

O358. Let a,b,c,d be non-negative real numbers such that $a \geq 1 \geq b \geq c \geq d$ and $a+b+c+d=4$. Prove that

$$abcd + \frac{15}{2(ab+bc+cd+da+db+ac)} \geq \frac{9}{a^2+b^2+c^2+d^2}$$

Proposed by Marius Stanean, Zalau, Romania

Solution by M.A.Prasad, Mumbai, Maharashtra, India

Let $S_2 = a^2+b^2+c^2+d^2$. Then $2(ab+bc+cd+da+db+ac) = (a+b+c+d)^2 - S_2 = 16 - S_2$. Therefore, we need to prove that

$$\begin{aligned}
&abcd + \frac{15}{2(ab+bc+cd+da+db+ac)} \geq \frac{9}{a^2+b^2+c^2+d^2}\\
&\Leftrightarrow abcd(16-S_2)S_2 + 24S_2 \geq 144 \qquad\qquad (*)
\end{aligned}$$

Let $a = 1+\delta, b = 1-\delta_1, c = 1-\delta_2, d = 1-\delta_3$ with $0 \leq \delta_1 \leq \delta_2 \leq \delta_3, \leq 1$ and $\delta = \delta_1+\delta_2+\delta_3....$

$$\begin{aligned}
S_2 &= (1+\delta)^2 + (1-\delta_1)^2 + (1-\delta_2)^2 + (1-\delta_3)^2\\
&= 4(1+\frac{x}{4}) \text{ where } x = \delta^2+\delta_1^2+\delta_2^2+\delta_3^2\\
abcd &= (1+\delta)(1-\delta_1)(1-\delta_2)(1-\delta_3)\\
&= 1-\delta^2+\delta_1\delta_2+\delta_2\delta_3+\delta_3\delta_1+\delta(\delta_1\delta_2+\delta_2\delta_3+\delta_3\delta_1)-\delta_1\delta_2\delta_3-\delta\delta_1\delta_2\delta_3\\
&= 1-\delta^2+\frac{\delta^2}{2}-\frac{\delta_1^2+\delta_2^2+\delta_3^2}{2}+\delta(\delta_1\delta_2+\delta_2\delta_3+\delta_3\delta_1)-\delta_1\delta_2\delta_3-\delta\delta_1\delta_2\delta_3\\
&\geq 1-\frac{x}{2}\\
16-S_2 &= 16-4-x = 12(1-\frac{x}{12})
\end{aligned}$$

We note that if $S_2 \geq 6$, then $\frac{15}{16-S_2} \geq \frac{9}{S_2}$. Therefore, we assume $0 \leq x \leq 2$ Using these in the LHS of (*), we get

$$\begin{aligned}
&abcdS_2(16-S_2)+24S_2 \geq 48(1-\frac{x}{2})(1+\frac{x}{4})(1-\frac{x}{12})+96(1+\frac{x}{4})\\
&= 48(3+\frac{x}{6}-\frac{5x^2}{48}+\frac{x^3}{96}) = 144+\frac{x(x^2-10x+16)}{2}\\
&= 144+\frac{x(x-2)(x-8)}{2}\\
&\geq 144 \text{ for } 0 \leq x \leq 2
\end{aligned}$$

Also solved by Kwonil Ko, Cushing Academy, Ashburham, MA, USA; Ji Eun Kim, Tabor Academy, MA, USA; Hyun Jin Kim, Stuyvesant High School, New York, NY, USA.

O359. Solve, in positive integers, the equation

$$x^6 + x^3y^3 - y^6 + 3xy\left(x^2 - y^2\right)^2 = 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Li Zhou, Polk State College, USA

Notice that $x^6 + x^3y^3 - y^6 + 3xy(x^2 - y^2)^2 = (x^2 - y^2 + xy)^3$. So $x^2 - y^2 + xy = 1$. Multiplying by 4 we get the well-known Pell equation $(2x + y)^2 - 5y^2 = 4$.

If $y = 2z$, then $(x + z)^2 - 5z^2 = 1$, with minimal solution $(x_1 + z_1, z_1) = (9, 4)$ and all solutions given by $x_n + z_n + z_n\sqrt{5} = (9 + 4\sqrt{5})^n$ for $n \in \mathbb{N}$. It is then easy to see that $(x_1, y_1) = (5, 8)$ and $(x_{n+1}, y_{n+1}) = (5x_n + 8y_n, 8x_n + 13y_n)$.

If y is odd, then we have the minimal solution $(2x_1 + y_1, y_1) = (3, 1)$ and all solutions given by $2x_n + y_n + y_n\sqrt{5} = (3 + \sqrt{5})(9 + 4\sqrt{5})^{n-1}$ for $n \in \mathbb{N}$. Again, it is easy to see that this second family of solutions are $(x_1, y_1) = (1, 1)$ and $(x_{n+1}, y_{n+1}) = (5x_n + 8y_n, 8x_n + 13y_n)$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Tolibjon Ismoilov, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Navid Safei, University of Technoogy in Policy Making of Science and Technology, Iran; M.A.Prasad, Mumbai, Maharashtra, India; Khurshid Juraev, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzekistan; Jorge Ledesma, Faculty of Sciences UNAM, Mexico City, Mexico; Arkady Alt, San Jose, California, USA; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Lucie Wang, Lycée Louis le Grand, Paris, France.

O360. Find the least positive integer n with the following property: for any polynomial $P(x) \in C[x]$, there exist polynomials $f_1(x), f_2(x), ..., f_n(x) \in C[x]$ and $g_1(x), g_2(x), ..., g_n(x) \in C[x]$ such that

$$P(x) = \sum_{i=1}^{n} (f_i(x)^2 + g_i(x)^3)$$

Proposed by Oleksiy Klurman, University College London

Solution by M.A.Prasad, Mumbai, Maharashtra, India

We note that every polynomial $P(x) \in C[x]$ can be expressed as the sum of the squares of two polynomials $f_1(x) = \frac{P(x)+1}{2}$ and $f_2(x) = i\frac{P(x)-1}{2}$. Choosing $g_1(x) = -\omega g_2(x)$ where ω is a cube root of unity and $g_2(x)$ is any arbitrary polynomial, the polynomials $f_1(x), f_2(x), g_1(x), g_2(x)$ will satisfy the desired relation.
Now, we show by a counterexample that there exist polynomials which cannot be expressed as $(f_1(x)^2 + g_1(x)^3)$. Let $P(x) = x^2$ and assume to the contrary that there exist polynomials $f_1(x)$ and $g_1(x)$ such that

$$f_1(x)^2 + g_1(x)^3 = x^2$$

Therefore, $g_1(x)^3 = (x - f_1(x))(x + f_1(x))$. The only polynomial which can be a divisor of $(x - f_1(x))$ as well as $(x + f_1(x))$ is x. There are two cases
(i) x divides $(x - f_1(x))$ and divides $(x + f_1(x))$. In this case x will also divide $g_1(x)$. Let $f_1(x) = xs_1(x)$ and $g_1(x) = xt_1(x)$. We, then get

$$xt_1(x)^3 = (1 - s_1(x))(1 + s_1(x))$$

Now, $(1 - s_1(x))$ and $(1 + s_1(x))$ are coprime. Therefore,

$$1 - s_1(x) = xt_3(x)^3 \text{ and } 1 + s_2(x) = t_4(x)^3 \Rightarrow t_4(x)^3 - xt_3(x)^3 = 2$$

Clearly, there is no solution to this equation since the polynomials $t_4(x)^3$ and $xt_3(x)^3$ are of different degree.
(ii) x does not divide $(x - f_1(x))$ or does not divide $(x + f_1(x))$. In this case, we have

$$x - f_1(x) = t_3(x)^3 \text{ and } x + f_2(x) = t_4(x)^3 \Rightarrow t_4(x)^3 - t_3(x)^3 = 2x$$

This yields

$$t_4(x) - t_3(x) = 2k \text{ and } t_4(x)^2 + t_4(x)t_3(x) + t_3(x)^2 = \frac{x}{k}$$

Therefore,

$$(t_4(x) + t_3(x))^2 = \frac{4}{3}(t_4(x)^2 + t_4(x)t_3(x) + t_3(x)^2) - \frac{1}{3}(t_4(x) - t_3(x))^2 = \frac{4x}{3k} - \frac{4k^2}{3}$$

Clearly, this has no solution.

Also solved by Navid Safei, University of Technoogy in Policy Making of Science and Technology, Iran.