Szemerédi-Trotter: Polynomials and Incidences

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1 Preliminaries

Definition 1 We let O signify an asymptotic upper bound. Say $f(n)$ and $g(n)$ are two functions with codomain $\mathbb{R}$. To assert that $f(n) = O(g(n))$ is to assert that there exists a positive constant c such that, for all n, $f(n) < c \cdot g(n)$.

Definition 2 Similarly, we let Ω signify an asymptotic lower bound. To assert that $f(n) = \Omega(g(n))$ is to assert that there exists a positive constant c such that, for all n, $f(n) > c \cdot g(n)$.

Definition 3 For a point p and a curve c, we say p is "incident" to c if p lies on c. For a set of points P and a set of curves C, we define the incidence graph of $P \times C$ to be the following bipartite graph: a vertex for each point in P on one side, a vertex for each curve in C on the other, and an edge between a point and a curve if that point is incident to that curve.

2 The Szemerédi-Trotter Theorem and Its Extensions

The Szemerédi-Trotter Theorem states:

(Szemerédi, Trotter [1]) Let P be a set of m points and let L be a set of n lines, both in $\mathbb{R}^2$. Then, the number of incidences between points in P and lines in $L = I(P, L) = O(m^{2/3}n^{2/3} + m + n)$

The original proof provided by Szemerédi and Trotter used a technique known as cell decomposition. Later, Székely came up with a simple combinatorial proof of Szemerédi-Trotter *[2]*. His argument converted the points, lines, and incidences into a graph theory diagram, and used a crossing number inequality *[3]*.

With the help of Székely's simpler proof, Pach and Sharir were able to generalize his arguments using a multigraph. They derived a theorem that bounded incidences between points and algebraic curves in general, not just incidences between points and lines:

(Pach, Sharir [4]) Let P be a set of m points and let C be a set of n constant-degree algebraic curves, both in $\mathbb{R}^2$, such that the incidence graph of $P \times C$ does not contain a copy of $K_{s,t}$. Then, $I(P, C) = O(m^{\frac{s}{2s-1}} n^{\frac{2s-2}{2s-1}} + m + n)$.

3 Using Szemerédi-Trotter with Polynomials

Let A and B be sets of real numbers. We define the set $A + B = \{a + b : a \in A, b \in B\}$. We define the set $A \cdot B = \{ab : a \in A, b \in B\}$. Let $f(x)$ be a polynomial. We define the set $f(A) = \{f(a) : a \in A\}$.

Say we want to lower bound $|A + A|$ in terms of $|A|$. Unfortunately, the tightest bound we have is $|A+A| = \Omega(|A|)$, which comes from the trivial fact that $|A+A| > |A|$ since $2a \in A+A$ for every $a \in A$. To show that this is the strongest bound we have, we look at the case when the terms of $|A|$ form an arithmetic sequence (say $A = \{x, 2x, \cdots, nx\}$). For all A of this form, we have $|A + A| = 2|A| - 1$. However, we can get non-trivial bounds when we try to minimize the number of elements in 2 different sets at the same time, such as $max(|A+A|, |A*A|)$. A famous unproven conjecture of Erdős and Szemerédi states:

(Erdős, Szemerédi [5]) For every set of reals A, $max(|A+A|, |A*A|) = \Omega(|A|^{2-\epsilon})$ *for every positive* ϵ *arbitrarily close to 0.*

Elekes *[6]* was able to prove the bound $max(|A+A|, |A*A|) = \Omega(|A|^{5/4})$ using an argument involving Szemerédi-Trotter. The following result also serves to bound two sets at the same time, except we will be considering sets constructed with polynomials. We use a Szemerédi-Trotter argument similar to that of Elekes.

Theorem 1. [1] *Let A be a set of real numbers and let* $f(x), g(x)$ *be polynomials with* $d = deg(f) \leq deg(g)$. *Then,* $max(|A+A|, |f(A)+g(A)|) = \Omega(|A|^{\frac{2d+1}{2d}})$.

Proof. Let P be the set of points in $\mathbb{R}^2$: $\{(x,y) : x \in A+A, y \in f(A)+g(A)\}$. Therefore, $|P| = |A+A| \cdot |f(A)+g(A)|$. Let C be the set of polynomial curves in $\mathbb{R}^2$: $\{y = f(x-u)+g(v) : u, v \in A\}$. $|C| = O(|A|^2)$ since there are $|A|^2$ choices for u, v. We may have duplicate polynomials in C if $g(v)$ takes on the same value for different v in A. However, for any real r, there are at most $deg(g)$ reals v such that $g(v) = r$. Therefore, each polynomial in $\{y = f(x-u)+g(v) : u, v \in A\}$ is duplicated at most $deg(g)$ times over the $|A|^2$ choices for u, v. Thus, $|C| \geq \frac{|A|^2}{deg(g)} \Rightarrow |C| = \Omega(|A|^2)$.

We want to bound the number of incidences $I(P,C)$ between points in P and curves in C. For each curve $y = f(x-u)+g(v)$ in C, the point $(k+u, f(k)+g(v))$ is incident to it for every real k. For every $k \in A$, the point $(k+u, f(k)+g(v)) \in P$. Therefore, each curve in C must have at least $|A|$ points in P incident to it: $(k+u, f(k)+g(v)) \in y = f(x-u)+g(v)$ for each of the $|A|$ ways to pick k. Consequently, $I(P,C) > |A| \cdot |C| = \Omega(|A|^3)$.

We finish by using the generalized version of Szemerédi-Trotter for curves:

(Pach, Sharir [4]) Let P be a set of m points and let C be a set of n constant-degree algebraic curves, both in $\mathbb{R}^2$, *such that the incidence graph of* $P \times C$ *does not contain a copy of* $K_{s,t}$. *Then,* $I(P,C) = O(m^{\frac{s}{2s-1}} n^{\frac{2s-2}{2s-1}} + m + n)$.

All of the polynomials in C have degree d and the same leading coefficient. Therefore, any two distinct curves in C intersect in at most $d-1$ points. Thus, the incidence graph of $P \times C$ does not contain a copy of $K_{d,2}$. Consequently, $I(P,C) = O(|P|^{\frac{d}{2d-1}} \cdot |C|^{\frac{2s-2}{2s-1}} + |P| + |C|) = O(|P|^{\frac{d}{2d-1}} \cdot |A|^{\frac{4s-4}{2s-1}} + |P| + |A|^2)$. Since $I(P,C)$ is asymptotically lower bounded by $|A|^3$ and asymptotically upper bounded by $|P|^{\frac{d}{2d-1}} \cdot |A|^{\frac{4s-4}{2s-1}} + |P| + |A|^2$, we must have $|P|^{\frac{d}{2d-1}} \cdot |A|^{\frac{4s-4}{2s-1}} + |P| + |A|^2 = \Omega(|A|^3)$. Therefore, $|P| = \Omega(|A|^{\frac{2d+1}{d}})$ since one of the terms in the sum $|P|^{\frac{d}{2d-1}} \cdot |A|^{\frac{4s-4}{2s-1}} + |P| + |A|^2$ must be asymptotically greater than $|A|^3$. And since $|P| = |A+A| \cdot |f(A)+g(A)| = \Omega(|A|^{\frac{2d+1}{d}})$, we must have $max(|A+A|, |f(A)+g(A)|) = \Omega(\sqrt{|A|^{\frac{2d+1}{d}}}) = \Omega(|A|^{\frac{2d+1}{2d}})$.

□

Note: The bound $\Omega(|A|^{\frac{2d+1}{2d}})$ seems relatively weak, as it is close to $\Omega(|A|)$. However, we realize that d is solely determined by the minimum degree between our two polynomials, meaning we can get decent bounds from this generalization as long as one of the two polynomials has relatively small degree.

[1] From my 2016 Intel STS submission

4 References

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