Coefficients and Roots of Polynomials

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In this paper, we study the behavior of a polynomial's roots as a function of its coefficients. Our key tool is Cauchy's residue formula in the form

$$\oint_{S_R} f(z)dz = 2\pi i \sum_{i=1}^{n} \text{Res} f(z)|_{z=x_i}.$$

Problem. Let $P(x)$ be a polynomial with real coefficients of degree n, having n distinct real roots $x_1, x_2 \ldots, x_n$. Prove that for any non-negative integer $k \leq n-2$,

$$\sum_{i=1}^{n} \frac{x_i^k}{P'(x_i)} = 0.$$

(IMC longlist 2005)

Solution. Consider the rational function $r(z) = \dfrac{z^k}{P(z)}$ in complex variable z. Since $\deg P(z) \geq k+2$, we have $r(z) \leq \dfrac{C}{|z|^2}$, for some constant C and large enough $|z|$. So for the integral over the circle S_R of radius R centred at the origin, we have

$$\left| \oint_{S_R} r(z)dz \right| \leq \frac{2\pi C}{R}$$

for large enough R. Additionally, for large enough R, Cauchy's residues theorem implies

$$\oint_{S_R} r(z)dz = 2\pi i \sum_{i=1}^{n} \text{Res}_{x_i} r(z) = 2\pi i \sum_{i=1}^{n} \frac{x_i^k}{P'(x_i)}.$$

Therefore,

$$\left| \sum_{i=1}^{n} \frac{x_i^k}{P'(x_i)} \right| \leq \frac{C}{R},$$

for large R. Letting $R \to +\infty$ implies that the left hand side of the last expression is zero.

Problem. Let $P(x) = \sum_{i=0}^{n} a_i x^i$ be a polynomial with real coefficients, and suppose that all roots $x_1, x_2, \ldots, x_n$ of $P(x)$, are simple. Prove that

$$\sum_{i=1}^{n} \frac{x_i^{n-1}}{P'(x_i)} = \frac{1}{a_n}.$$

(IMC proposals 2007)

Solution. We consider the integral

$$I = \frac{1}{2\pi i} \oint_{S_r} \frac{x^{n-1}}{P(x)} dx,$$

where S_r is a circle centered at the origin and radius $r > \max_{1\le i\le n}\{|x_i|\}$. Applying Cauchy's integral formula, we obtain

$$I = \frac{1}{2\pi i}\oint_{S_r}\frac{x^{n-1}}{P(x)}dx = \sum_{i=1}^{n}\text{Res}\frac{x^{n-1}}{P(x)}|_{x=x_i} = \frac{1}{a_n}\sum_{i=1}^{n}x_i^{n-1}\prod_{j=1,j\neq i}^{n}\frac{1}{x_i-x_j} = \sum_{i=1}^{n}\frac{x_i^{n-1}}{P'(x_i)}.$$

Furthermore, evaluating the integral outside the S_r contour, we obtain

$$I = \frac{1}{2\pi i}\oint_{S_r}\frac{x^{n-1}}{P(x)dx} = -\text{Res}\frac{x^{n-1}}{P(x)}|_{x=\infty} = \text{Res}\frac{1}{x^2}f\left(\frac{1}{x}\right)|_{x=0},$$

where $f(x) = \frac{x^{n-1}}{P(x)}$. Therefore,

$$I = \text{Res}\frac{1}{x^2}\frac{1}{x^{n-1}P(1/x)}|_{x=0} = \text{Res}\frac{1}{x^{n+1}P(1/x)}|_{x=0} = \text{Res}\frac{1}{x(a_n + a_{n-1}x + \cdots + a_0x^n)}|_{x=0} = \frac{1}{a_n}.$$

Comparing the two sides yields the result.

Problem. Let $P(x) = \sum_{i=0}^{n} a_ix^i$ be a polynomial with real coefficients and simple nonzero roots $x_1, x_2, \ldots, x_n$. Prove that

$$\sum_{i=1}^{n}\frac{1}{x_iP'(x_i)} = -\frac{1}{a_0}.$$

(IMC proposals 2007)

Solution. We consider the integral

$$I = \frac{1}{2\pi i}\oint_{S_r}\frac{dx}{xP(x)},$$

where S_r is a circle centered at the origin, and radius $r < \min_{1\le i\le n}\{|x_i|\}$. By Cauchy's resiude formula with $x_{n+1} = \infty$, we have

$$I = \frac{1}{2\pi i}\oint_{S_r}\frac{dx}{xP(x)} = -\sum_{i=1}^{n+1}\text{Res}\frac{1}{xP(x)}|_{x=x_i} = -\frac{1}{a_n}\sum_{i=1}^{n}\frac{1}{x_i}\prod_{j=1,j\neq i}^{n}\frac{1}{x_i-x_j} = -\sum_{i=1}^{n}\frac{1}{x_iP'(x_i)}.$$

We also have,

$$I = \frac{1}{2\pi i}\oint_{S_r}\frac{dx}{xP(x)} = \text{Res}\frac{1}{xP(x)}|_{x=0} = \frac{1}{P(0)} = \frac{1}{a_0}.$$

The result follows by comparing the two expressions for I.

Problem. Let $P(z) = \sum_{i=0}^{n} a_iz^i$ be a non-constant polynomial with $a_i \neq 0$, for all $0 \le i \le n$. Show that all of its zeros lie in the annulus $C = \{z \in \mathbb{C} : r_1 \le |z| \le r_2\}$, where

$$r_1 = \min_{1\le i\le n}\left\{\frac{3^i\binom{n}{i}}{4^n-1}\left|\frac{a_0}{a_i}\right|\right\}^{\frac{1}{i}} \quad \text{and} \quad r_2 = \max_{1\le i\le n}\left\{\frac{4^n-1}{3^i\binom{n}{i}}\left|\frac{a_{n-i}}{a_n}\right|\right\}^{\frac{1}{i}}.$$

(IMC longlist 2005)

Solution. Suppose that $|z| < r_1$. We have

$$|P(z)| = \left|\sum_{i=0}^{n}a_iz^i\right| \ge |a_0| - \sum_{i=1}^{n}|a_i||z|^i > |a_0| - \sum_{i=1}^{n}|a_i|r_1^i = |a_0|\left(1 - \sum_{i=1}^{n}\left|\frac{a_i}{a_0}\right|r_1^i\right).$$

Taking into account the identity

$$\sum_{i=0}^{n} 3^i \binom{n}{i} = 4^n$$

by the definition of r_1, we arrive at

$$\left|\frac{a_i}{a_0}\right| r_1^i \leq \frac{3^i \binom{n}{i}}{4^n - 1}, \tag{1}$$

for all $1 \leq i \leq n$. Substituting (2) into (1) yields

$$|P(z)| > |a_0| \left(1 - \sum_{i=1}^{n} \left|\frac{a_i}{a_0}\right| r_1^i\right) \geq |a_0| \left(1 - \sum_{i=1}^{n} \frac{3^i \binom{n}{i}}{4^n - 1}\right) = 0.$$

Consequently, $P(z)$ does not have zeros in $\{z \in \mathbb{C} : |z| < r_1\}$. To prove the second inequality we will use the following well-known (and easy to prove!) fact: all the zeros of $P(z)$ have a modulus less than or equal to the unique positive root of the equation

$$G(z) = |a_n|z^n - |a_{n-1}|z^{n-1} - \cdots - |a_1|z - |a_0| = 0.$$

Therefore, the second part of our statement will be proven if we show that $G(r_2) \geq 0$. From the expression for r_2, we immediately deduce the estimate

$$\left|\frac{a_{n-i}}{a_n}\right| \leq \frac{3^i \binom{n}{i}}{4^n - 1} r_2^i,$$

for all $1 \leq i \leq n$. Moreover,

$$G(r_2) = |a_n| \left(r_2^n - \sum_{i=1}^{n} \left|\frac{a_{n-i}}{a_n}\right| r_2^{n-i}\right) \geq |a_n| \left(r_2^n - \sum_{i=1}^{n} \left(\frac{3^i \binom{n}{i}}{4^n - 1} r_2^i\right) r_2^{n-i}\right) = |a_n| r_2^n \left(1 - \sum_{i=1}^{n} \frac{3^i \binom{n}{i}}{4^n - 1}\right) = 0.$$

This completes the proof.

Problem. Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial with nonzero complex coefficients. Prove that all of its zeros lie in the region $C = \{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\}$, where

$$r_1 = \frac{5}{2} \min_{1 \leq i \leq n} \sqrt[i]{\frac{2^n \binom{n}{i} P_i}{P_{3n}} \left|\frac{a_0}{a_i}\right|}, \tag{2}$$

and

$$r_2 = \frac{2}{5} \max_{1 \leq i \leq n} \sqrt[i]{\frac{P_{3n}}{2^n \binom{n}{i} P_i} \left|\frac{a_{n-i}}{a_n}\right|}, \tag{3}$$

where P_n is the n^{th} Pell number defined by $P_0 = 0, P_1 = 1$; and, for all $n \geq 2$, $P_n = 2P_{n-1} + P_{n-2}$.

(IMC proposals 2007)

Solution. We begin with the following useful lemma.
Lemma. Let n be a nonnegative integer. Then

$$\sum_{i=0}^{n} \binom{n}{i} 5^i 2^{n-i} P_i = P_{3n}, \tag{4}$$

where P_n is defined as above.
Proof. Since the characteristic equation of $P_n = 2P_{n-1} + P_{n-2}, P_0 = 0, P_1 = 1$ is

$$x^2 - 2x - 1 = 0, \tag{5}$$

it follows that

$$P_n = \frac{1}{2\sqrt{2}} \left(a^n - b^n\right),$$

where $a = 1+\sqrt{2}$ and $b = 1-\sqrt{2}$ are the roots of (6). Therefore, taking into account that $a^3 = 5a+2$ and $b^3 = 5b+2$, we have

$$\begin{aligned} P_{3n} &= \frac{1}{2\sqrt{2}}(a^{3n} - b^{3n}) = \frac{1}{2\sqrt{2}}\left((5a+2)^n - (5b+2)^n\right) = \\ &= \frac{1}{2\sqrt{2}}\left(\sum_{i=0}^{n}\binom{n}{i}5^i 2^{n-i}a^i - \sum_{i=0}^{n}\binom{n}{i}5^i 2^{n-i}b^i\right) = \\ &= \sum_{i=0}^{n}\binom{n}{i}5^i 2^{n-i}\left(\frac{a^i - b^i}{2\sqrt{2}}\right) = \sum_{i=0}^{n}\binom{n}{i}5^i 2^{n-i}P_i \end{aligned}$$

and (5) follows. If we assume that $|z| < r_1$, we have

$$|P(z)| = \left|\sum_{i=0}^{n} a_i z^i\right| \geq |a_0| - \sum_{i=1}^{n}|a_i||z|^i > |a_0| - \sum_{i=1}^{n}|a_i|r_1^i = |a_0|\left(1 - \sum_{i=1}^{n}\left|\frac{a_i}{a_0}\right| r_1^i\right).$$

From (3), we obtain

$$\left|\frac{a_i}{a_0}\right| r_1^i \leq \left(\frac{5}{2}\right)^i \frac{2^i\binom{n}{i}P_i}{P_{3n}}, \tag{6}$$

for all $1 \leq i \leq n$. Substituting (8) into (7) and taking into account the previous lemma, we arrive at

$$|P(z)| > |a_0|\left(1 - \sum_{i=1}^{n}\left|\frac{a_i}{a_0}\right| r_1^i\right) \geq |a_0|\left(1 - \sum_{i=1}^{n}\left(\frac{5}{2}\right)^i \frac{2^i\binom{n}{i}P_i}{P_{3n}}\right) = 0.$$

Consequently, $P(z)$ does not have zeros in the region $\{z \in \mathbb{C} : |z| < r_1\}$. We now again appeal to the result mentioned before: modulus of all zeros of $P(z)$ is less or equal to the unique positive root of the equation

$$G(z) = |a_n|z^n - |a_{n-1}|z^{n-1} - \cdots - |a_1|z - |a_0| = 0.$$

Therefore, the second part of our statement will be proved if we show that $G(r_2) \geq 0$. Note that (5) implies that

$$\sum_{i=1}^{n}\left(\frac{5}{2}\right)^i \frac{2^i P_i\binom{n}{i}}{P_{3n}} = 1.$$

Combining (4) with the lemma stated at the beginning, we have

$$\left|\frac{a_{n-i}}{a_n}\right| \leq \left(\frac{5}{2}\right)^i \frac{2^i\binom{n}{i}P_i}{P_{3n}} r_2^i,$$

for all $1 \leq i \leq n$, and

$$G(r_2) = |a_n|\left(r_2^n - \sum_{i=1}^{n}\left|\frac{a_{n-i}}{a_n}\right| r_2^{n-i}\right) \geq |a_n|\left(r_2^n - \sum_{i=1}^{n}\left(\left(\frac{5}{2}\right)^i \frac{2^i\binom{n}{i}P_i}{P_{3n}} r_2^i\right) r_2^{n-i}\right) = |a_n|r_2^n\left(1 - \sum_{i=1}^{n}\left(\frac{5}{2}\right)^i \frac{2^i\binom{n}{i}P_i}{P_{3n}}\right) = 0,$$

and the conclusion follows.

Problems.

1. Let $P(x)$ be a polynomial of degree n with leading coefficient a_n. Suppose that $P(x)$ has n non-zero distinct roots $x_1, x_2, \ldots, x_n$. Prove that

$$\sum_{i=1}^{n} \frac{1}{x_i^2 P'(x_i)} = \frac{(-1)^{n-1}}{a_n \cdot x_1 x_2 \cdots x_n} \sum_{i=1}^{n} \frac{1}{x_i}.$$

2. Suppose that $P(x)$ and $Q(x)$ are given polynomials with $\deg Q < \deg P$ and $x_1, x_2, \ldots, x_n$ are distinct real roots of polynomial $P(x)$. Prove that

$$\frac{Q(x)}{P(x)} = \sum_{i=1}^{n} \frac{Q(x_i)}{P'(x_i)} \cdot \frac{1}{x - x_i}.$$

References

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