Junior Problems

J367. Let a and b be positive real numbers. Prove that

$$\frac{1}{4a}+\frac{3}{a+b}+\frac{1}{4b}\geq\frac{4}{3a+b}+\frac{4}{a+3b}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J368. Find the best constants α and β such that $\alpha<\dfrac{x}{2x+y}+\dfrac{y}{x+2y}\leq\beta$ for all $x,\, y\in(0,\,\infty)$.

Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

J369. Solve the equation

$$\sqrt{1+\frac{1}{x+1}}+\frac{1}{\sqrt{x+1}}=\sqrt{x}+\frac{1}{\sqrt{x}}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J370. Triangle ABC has side lengths $BC=a$, $CA=b$, $AB=c$. If

$$\left(a^2+b^2+c^2\right)^2=4a^2b^2+b^2c^2+4c^2a^2,$$

find all possible values of $\angle A$.

Proposed by Adrian Andreescu, Dallas, TX, USA

J371. Prove that for all positive integers n,

$$\binom{n+3}{2}+6\binom{n+4}{4}+90\binom{n+5}{6}$$

is the sum of two perfect cubes.

Proposed by Alessandro Ventullo, Milan, Italy

J372. In triangle ABC, $\frac{\pi}{7}<A\leq B\leq C<\frac{5\pi}{7}$. Prove that

$$\sin\frac{7A}{4}-\sin\frac{7B}{4}+\sin\frac{7C}{4}>\cos\frac{7A}{4}-\cos\frac{7B}{4}+\cos\frac{7C}{4}.$$

Proposed by Adrian Andreescu, Dallas, TX, USA

Senior Problems

S367. Solve in positive real numbers the system of equations

$$\begin{cases} \left(x^3+y^3\right)\left(y^3+z^3\right)\left(z^3+x^3\right)=8, \\ \dfrac{x^2}{x+y}+\dfrac{y^2}{y+z}+\dfrac{z^2}{z+x}=\dfrac{3}{2}. \end{cases}$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S368. Determine all positive integers n such that $\sigma(n) = n + 55$, where $\sigma(n)$ denotes the sum of the divisors of n.

Proposed by Alessandro Ventullo, Milan, Italy

S369. Given the polynomial $P(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ having n real roots (not necessarily distinct) in the interval $[0, 1]$, prove that $3a_1^2 + 2a_1 - 8a_2^2 \le 1$.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S370. Prove that in any triangle,

$$\left|3a^2-2b^2\right|m_a+\left|3b^2-2c^2\right|m_b+\left|3c^2-2a^2\right|m_c \ge \frac{8K^2}{R}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S371. Let ABC be a triangle and let M be the midpoint of the arc $\widehat{BAC}$. Let AL be the bisector of angle A, and let I be the incenter of triangle ABC. Line MI intersects the circumcircle of triangle ABC at K and line BC intersects the circumcircle of triangle AKL at P. If $PI \cap AK = \{X\}$, and $KI \cap BC = \{Y\}$, prove that $XY \parallel AI$.

Proposed by Bobojonova Latofat, Tashkent, Uzbekistan

S372. Prove that in any triangle,

$$\frac{2}{3}\left(m_am_b+m_bm_c+m_cm_a\right) \ge \frac{1}{4}\left(a^2+b^2+c^2\right)+\sqrt{3}K.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Undergraduate Problems

U367. Let $\{a_n\}_{n\geq 1}$ be the sequence of real numbers given by $a_1 = 4$ and $3a_{n+1} = (a_n+1)^3 - 5$, $n \geq 1$. Prove that a_n is a positive integer for all n, and evaluate $\sum_{n=1}^{\infty} \frac{a_n - 1}{a_n^2 + a_n + 1}$

Proposed by Albert Stadler, Herrliberg, Switzerland

U368. Let

$$x_n = \sqrt{2} + \sqrt[3]{\frac{3}{2}} + \cdots + \sqrt[n+1]{\frac{n+1}{n}}, \quad n = 1, 2, 3, \ldots.$$

Evaluate $\lim_{n\to\infty} \frac{x_n}{n}$.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U369. Prove that

$$\frac{648}{35}\sum_{k=1}^{\infty} \frac{1}{k^3(k+1)^3(k+2)^3(k+3^3)} = \pi^2 - \frac{6217}{630}.$$

Proposed by Albert Stadler, Herrliberg, Switzerland

U370. Evaluate

$$\lim_{x\to 0} \frac{\sqrt{1+2x}\cdot\sqrt[3]{1+3x}\cdots\sqrt[n]{1+nx} - 1}{x}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U371. Let n be a positive integer, and let $A_n = (a_{ij})$ be the $n \times n$ matrix where $a_{ij} = x^{(i+j-2)^2}$, x being a variable. Evaluate the determinant of A_n.

Proposed by Mehtaab Sawhney, Commack High School, New York, USA

U372. Let α, $\beta > 0$ be real numbers, and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x) \neq 0$, for all x in a neighborhood U of 0. Evaluate

$$\lim_{x\to +0} \frac{\int_0^{\alpha x} t^{\alpha} f(t)dt}{\int_0^{\beta x} t^{\beta} f(t)dt}$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Olympiad Problems

O367. Prove that for any positive integer $a > 81$ there are positive integers x, y, z such that

$$a = \frac{x^3 + y^3}{z^3 + a^3}$$

Proposed by Nairi Sedrakyan, Yerevan, Armenia

O368. Let a, b, c, d, e, f be real numbers such that $a+b+c+d+e+f = 15$ and $a^2+b^2+c^2+d^2+e^2+f^2 = 45$. Prove that $abcdef \leq 160$.

Proposed by Marius Stănean, Zalău, România

O369. Let a, b, $c > 0$. Prove that

$$\frac{a^2 + bc}{b + c} + \frac{b^2 + ca}{c + a} + \frac{c^2 + ab}{a + b} \geq \sqrt{3(a^2 + b^2 + c^2)}.$$

Proposed by An Zhen-Ping, Xianyang Normal University, China

O370. For any positive integer n we denote by $S(n)$ the sum of digits of n. Prove that for any integer n such that $\gcd(3, n) = 1$ and any k, $k > S(n)^2 + 7S(n) - 9$, there exists an integer m such that $n|m$ and $S(m) = k$.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

O371. Let $ABC(AB \neq BC)$ be a triangle, and let D and E be the feet of the altitudes from B and C, respectively. Denote by M, N, P the midpoints of BC, MD, ME, respectively. If $\{S\} = NP \cap BC$ and T is the point of intersection of DE with the line through A which is parallel to BC, prove that ST is tangent to the circumcircle of triangle ADE.

Proposed by Marius Stănean, Zalău, România

O372. A regular $n-$gon Γ_b of side b is drawn inside a regular $n-$gon Γ_a of side a such that the center of the circumcircle of Γ_a is not inside Γ_b. Prove that

$$b < \frac{a}{2\cos^2 \frac{\pi}{2n}}.$$

Proposed by Nairi Sedrakyan, Yerevan, Armenia