A New Method About Using Polynomial Roots and Arithmetic-Geometric Mean Inequality to Solve Olympiad Problems

The purpose of this article is to present a new method (and some useful lemmas) to solve a comprehensive class of olympiad inequalities by using polynomial roots with an unknown theorem which is similar to Arithmetic-Geometric Mean Inequality.

1. Introduction

The article consists of three main parts:

- At first a proof will be given to a new theorem which is very important to use the method correctly.
- Then a well-known inequality problem which was asked in International Mathematical Olympiads will be solved to understand the pure method completely. Subsequently several applications of this method with nice ideas will be seen.
- Lastly some useful and new lemmas, whose proof come from a similar way to the method, will be introduced with further examples.

2. Main Theorem

Theorem. *Let a, b, c be non-negative real numbers. For all real numbers t such that $t \geq \max[a,b,c]$ or $t \geq \frac{4}{9}(a+b+c)$, the following inequality holds:*

$$(t-a)(t-b)(t-c) \leq \left(\frac{3t-(a+b+c)}{3}\right)^3.$$

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Proof. i) If $t \geq \max[a,b,c]$, then all the factors on the left hand side are positive and it is just an Arithmetic-Geometric Mean Inequality.
ii) If $t \geq \frac{4}{9}(a+b+c)$.
When we arrange the inequality in the following way, it is enough to prove that

$$(t-a)(t-b)(t-c) \leq \left(\frac{3t-(a+b+c)}{3}\right)^3 = \left(t - \frac{a+b+c}{3}\right)^3 \Leftrightarrow$$

$$t^3 - t^2(a+b+c) + t(ab+bc+ca) - abc$$

$$\leq t^3 - t^2(a+b+c) + t\frac{(a+b+c)^2}{3} - \frac{(a+b+c)^3}{27} \Leftrightarrow$$

$$t(ab+bc+ca) - abc \leq t\frac{(a+b+c)^2}{3} - \frac{(a+b+c)^3}{27} \Leftrightarrow$$

$$\frac{(a+b+c)^3}{27} - abc \leq t\frac{(a+b+c)^2}{3} - t(ab+bc+ca) \Leftrightarrow$$

$$(a+b+c)^3 - 27abc \leq 9t((a+b+c)^2 - 3(ab+bc+ca)).$$

Here it is easy to see that the both sides of the inequality are positive. Because

$$t \geq \frac{4}{9}(a+b+c),$$

it is enough to prove that

$$4(a+b+c)((a+b+c)^2 - 3(ab+bc+ca)) \geq (a+b+c)^3 - 27abc \Leftrightarrow$$

$$4(a+b+c)^3 - 12(a+b+c)(ab+bc+ca) \geq (a+b+c)^3 - 27abc \Leftrightarrow$$
$$3(a+b+c)^3 + 27abc \geq 12(a+b+c)(ab+bc+ca) \Leftrightarrow$$
$$(a+b+c)^3 + 9abc \geq 4(a+b+c)(ab+bc+ca) \Leftrightarrow$$
$$(a^3+b^3+c^3) + 3(a^2b + a^2c + b^2a + b^2c + c^2a + c^b) + 6abc + 9abc$$
$$\geq 4(a^2b + a^2c + b^2a + b^2c + c^2a + c^2b) + 12abc \Leftrightarrow$$
$$a^3 + b^3 + c^3 + 3abc \geq a^2b + a^2c + b^2a + b^2c + c^2a + c^2b.$$

This is obviously true, because it is immediately Schur's Inequality of third degree. Equality holds for $a = b = c$.
Although for different numbers of variables we don't have such a good result, the followings are useful:
Theorem (Different Variations). *For all real numbers a, b and t, the following inequality holds:*

$$(t-a)(t-b) \leq \left(\frac{2t-(a+b)}{2}\right)^2.$$

Proof. Actually this inequality is equivalent to $(a-b)^2 \geq 0$ which is clearly true.
Theorem (Different Variations). *Let $n \geq 4$ be an integer and let $a_1, a_2, \ldots, a_n$ be arbitrary non-negative real numbers. Then for all real numbers t such that $t \geq \dfrac{a_1 + a_2 + \ldots + a_n}{2}$, the following inequality holds:*

$$(t-a_1)(t-a_2)\ldots(t-a_n) \leq \left(\frac{nt - (a_1 + a_2 + \ldots + a_n)}{n}\right)^n.$$

Proof. It is easy to see that the right-hand side of the inequality is always positive, because $nt \geq 2(a_1 + a_2 + \ldots + a_n) > a_1 + a_2 + \ldots + a_n$. So it is enough to look for two cases which make the left-hand side of the inequality positive. Otherwise the inequality will be obviously true.
i) If all factors in the left-hand side of the inequality are positive, then it is just an Arithmetic-Geometric Mean Inequality.
ii) If there exists a negative factor in the left-hand side of the inequality, there must be one more negative factor for being the whole product positive.

Without loss of generality we can suppose that the factors $t - a_1$ and $t - a_2$ are negative. Because $t \geq \dfrac{a_1 + a_2 + \ldots + a_n}{2}$, we have

$$(t - a_1) + (t - a_2) < 0 \Rightarrow a_1 + a_2 > 2t \geq a_1 + a_2 + \ldots + a_n \Rightarrow 0 > a_3 + \ldots + a_n$$

which is a contradiction, because $a_3, a_4, \ldots, a_n$ an are non-negative real numbers.
Hence the proof finished.

3. Sample Problems

How to apply it?

1. First of all the inequality in the problem is arranged to get an appropriate expression to use the method. For that, the inequality is often made normalized. Namely the inequality is written with a form using the terms $a + b + c$, $ab + bc + ca$, abc for example. (This step is not necessary always.)
2. Then a polynomial is defined whose roots are the variables in the problem.
3. After that the "Main Theorem" (or one of the other variations of this theorem) is applied on this polynomial. At the same time different results can be gotten using several ideas on the expression.
4. Finally setting an appropriate value of real number t into the expression immediately gives us the inequality to prove.

Problem 1. Prove that $0 \leq xy + yz + zx - 2xyz \leq \dfrac{7}{27}$, where x, y and z are non-negative real numbers satisfying $x + y + z = 1$.

(International Mathematical Olympiads 1984)

Solution. We will prove only the right-hand side of the inequality.
Let f be a cubic polynomial with real roots x, y, z.

$$\begin{aligned} f(t) = (t - x)(t - y)(t - z) &= t^3 - t^2(x + y + z) + t(xy + yz + zx) - xyz \\ &= t^3 - t^2 + t(xy + yz + zx) - xyz. \end{aligned}$$

Due to the "Main Theorem", we have

$$\begin{aligned} t^3 - t^2 + t(xy + yz + zx) - xyz &= (t-x)(t-y)(t-z) \\ &\leq \left(\frac{3t - (x+y+z)}{3}\right)^3 \\ &= \left(\frac{3t-1}{3}\right)^3 \end{aligned}$$

for all real numbers t such that $t \geq \frac{4}{9}(x+y+z) = \frac{4}{9}$. Setting $t = \frac{1}{2}$ into this inequality gives us that

$$\frac{1}{8} - \frac{1}{4} + \frac{1}{2}(xy + yz + zx) - xyz \leq \frac{1}{216} \Rightarrow$$

$$\frac{1}{2}(xy + yz + zx) - xyz \leq \frac{1}{216} - \frac{1}{8} + \frac{1}{4} = \frac{28}{216} = \frac{7}{54} \Rightarrow$$

$$xy + yz + zx - 2xyz \leq \frac{7}{27}.$$

Hence the proof finished. Equality holds for $x = y = z = \frac{1}{3}$.

Problem 2. Prove that $7(ab + bc + ca) \leq 2 + 9abc$, where a, b, c are positive real numbers satisfying $a + b + c = 1$.

(United Kingdom Mathematical Olympiads 1999)

Solution. Let f be a cubic polynomial with real roots a, b, c.

$$\begin{aligned} f(t) &= (t-a)(t-b)(t-c) = t^3 - t^2(a+b+c) + t(ab+bc+ca) - abc \\ &= t^3 - t^2 + t(ab + bc + ca) - abc. \end{aligned}$$

Due to the "Main Theorem", we have

$$\begin{aligned} t^3 - t^2 + t(ab + bc + ca) - abc &= (t-a)(t-b)(t-c) \\ &\leq \left(\frac{3t - (a+b+c)}{3}\right)^3 \\ &= \left(\frac{3t-1}{3}\right)^3 \end{aligned}$$

for all real numbers t such that $t \geq \frac{4}{9}(a+b+c) = \frac{4}{9}$. Setting $t = \frac{7}{9}$ into this inequality gives us that

$$\frac{343}{729} - \frac{49}{81} + \frac{7}{9}(ab+bc+ca) - abc \leq \frac{64}{729} \Rightarrow$$
$$\frac{7}{9}(ab+bc+ca) - abc \leq \frac{64}{729} - \frac{343}{729} + \frac{441}{729} = \frac{162}{729} = \frac{2}{9} \Rightarrow$$
$$7(ab+bc+ca) - 9abc \leq 2.$$

Hence the proof finished. Equality holds for $a = b = c = \frac{1}{3}$.

Problem 3. Let a, b, c be positive real numbers such that $a+b+c = 1$. Prove the inequality

$$a^2 + b^2 + c^2 + 3abc \geq \frac{4}{9}.$$

(Serbia Mathematical Olympiads 2008)

Solution. Because

$$a^2 + b^2 + c^2 = (a+b+c)^2 - 2(ab+bc+ca) = 1 - 2(ab+bc+ca),$$

it is enough to prove that

$$\frac{5}{9} \geq 2(ab+bc+ca) - 3abc \Leftrightarrow \frac{5}{27} \geq \frac{2}{3}(ab+bc+ca) - abc.$$

Let f be a cubic polynomial with real roots a, b, c.
Due to the "Main Theorem", we have

$$\begin{aligned} t^3 - t^2 + t(ab+bc+ca) - abc &= (t-a)(t-b)(t-c) \\ &\leq \left(\frac{3t - (a+b+c)}{3}\right)^3 \\ &= \left(\frac{3t-1}{3}\right)^3 \end{aligned}$$

for all real numbers t such that $t \geq \frac{4}{9}(a+b+c) = \frac{4}{9}$. Setting $t = \frac{2}{3}$ into this inequality gives us that

$$\frac{8}{27} - \frac{4}{9} + \frac{2}{3}(ab+bc+ca) - abc \leq \frac{1}{27} \Rightarrow$$

$$\frac{2}{3}(ab+bc+ca)-abc \le \frac{1}{27}-\frac{8}{27}+\frac{4}{9}=\frac{5}{27}.$$

Hence the proof finished. Equality holds for $a=b=c=\dfrac{1}{3}$.

Problem 4. Let a, b, c be non-negative real numbers such that $a+b+c=2$. Prove that

$$a^3+b^3+c^3+\frac{15abc}{4} \ge 2.$$

Solution. Because

$$a^3+b^3+c^3=(a+b+c)^3-3(a+b+c)(ab+bc+ca)+3abc=8-6(ab+bc+ca)+3abc,$$

it is enough to prove that

$$6 \ge 6(ab+bc+ca)-\frac{27abc}{4} \Leftrightarrow \frac{8}{9} \ge \frac{8}{9}(ab+bc+ca)-abc.$$

Let f be a cubic polynomial with real roots a, b, c.

$$\begin{aligned} f(t) &= (t-a)(t-b)(t-c) \\ &= t^3-t^2(a+b+c)+t(ab+bc+ca)-abc \\ &= t^3-2t^2+t(ab+bc+ca)-abc. \end{aligned}$$

Due to the "Main Theorem", we have

$$\begin{aligned} t^3-2t^2+t(ab+bc+ca)-abc &= (t-a)(t-b)(t-c) \\ &\le \left(\frac{3t-(a+b+c)}{3}\right)^3 \\ &= \left(\frac{3t-2}{3}\right)^3 \end{aligned}$$

for all positive real numbers t such that $t \ge \dfrac{4}{9}(a+b+c)=\dfrac{8}{9}$. Setting $t=\dfrac{8}{9}$ into this inequality gives us

$$\frac{512}{729}-\frac{128}{81}+\frac{8}{9}(ab+bc+ca)-abc \le \frac{8}{729} \Rightarrow$$

$$\frac{8}{9}(ab+bc+ca)-abc \le \frac{8}{729}-\frac{512}{729}+\frac{128}{81}=\frac{8}{9}.$$

Hence the proof finished. Equality holds for $a=b=c=\frac{2}{3}$ and $a=0$, $b=c=1$ or permutations.

Problem 5. Let a,b,c be positive real numbers such that $ab+bc+ca=1$. Prove that

$$abc+a+b+c \ge \frac{10\sqrt{3}}{9}.$$

Solution. Note that at most one of a,b,c can be greater than or equal to 1. We look for two cases:

i) If exactly one of a,b,c is greater than or equal to 1, then

$$\begin{aligned} abc+a+b+c &= (a-1)(b-1)(c-1)+ab+bc+ca+1 \\ &= (a-1)(b-1)(c-1)+2 \ge 2 > \frac{10\sqrt{3}}{9}. \end{aligned}$$

ii) If a,b,c are smaller than 1.

Let f be a cubic polynomial with real roots a,b,c.

$$\begin{aligned} f(t) &= (t-a)(t-b)(t-c) \\ &= t^3-t^2(a+b+c)+t(ab+bc+ca)-abc \\ &= t^3-t^2+t(ab+bc+ca)-abc. \end{aligned}$$

Due the "Main Theorem", we have

$$t^3-t^2(a+b+c)+t-abc=(t-a)(t-b)(t-c) \le \left(\frac{3t-(a+b+c)}{3}\right)^3.$$

Setting $t=1$ into this inequality (because a,b,c are smaller than 1 and $1=t>\max[a,b,c]$ there is no problem with that) and combining $(a+b+c)^2 \ge 3(ab+bc+ac)=3$ with the result gives us that

$$(a+b+c)+abc \ge 2-\left(\frac{3-(a+b+c)}{3}\right)^3 \ge 2-\left(\frac{3-\sqrt{3}}{3}\right)^3=\frac{10\sqrt{3}}{9}.$$

Hence the proof finished. Equality holds for $a = b = c = \frac{\sqrt{3}}{3}$.

Problem 6. Prove that

$$\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x} - 3\right) \geq \frac{9}{4}\left(\frac{x}{z} + \frac{y}{x} + \frac{z}{y} - 3\right)$$

where x, y, z are arbitrary positive real numbers.

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Solution. Let us substitute $\frac{x}{y} = a$, $\frac{y}{z} = b$ and $\frac{z}{x} = c$, where a, b, c are positive real numbers satisfying $abc = 1$. It is enough to prove that

$$(a+b+c)(a+b+c-3) \geq \frac{9}{4}(ab+bc+ca-3) \Leftrightarrow$$

$$4(a+b+c)^2 + 27 \geq 12(a+b+c) + 9(ab+bc+ca).$$

Let f be a cubic polynomial with real roots a, b, c.

$$\begin{aligned} f(t) = (t-a)(t-b)(t-c) &= t^3 - t^2(a+b+c) + t(ab+bc+ca) - abc \\ &= t^3 - t^2(a+b+c) + t(ab+bc+ca) - 1 \end{aligned}$$

Combining the "Main Theorem" with $a + b + c \geq 3\sqrt[3]{abc} = 3$ gives us that

$$\begin{aligned} t^3 - t^2(a+b+c) + t(ab+bc+ca) - 1 &\leq \left(\frac{3t-(a+b+c)}{3}\right)^3 \leq \left(\frac{3t-3}{3}\right)^3 \\ &= (t-1)^3 = t^3 - 3t^2 + 3t - 1 \Rightarrow \\ 3t + (ab+bc+ca) &\leq t(a+b+c) + 3 \end{aligned}$$

for all positive real numbers t such that $t \geq \frac{4}{9}(a+b+c)$. Setting $t = \frac{4}{9}(a+b+c)$ into this inequality yields

$$\begin{aligned} \frac{4}{3}(a+b+c) + (ab+bc+ca) &\leq \frac{4}{9}(a+b+c)^2 + 3 \Rightarrow \\ 12(a+b+c) + 9(ab+bc+ca) &\leq 4(a+b+c)^2 + 27 \end{aligned}$$

Hence the proof finished. Equality holds for $x = y = z$.

Problem 7. Let x, y, z be positive real numbers such that

$$x^2 + y^2 + z^2 + 2xyz = 1.$$

Prove that

$$x + y + z \leq \frac{3}{2}.$$

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Solution. Let f be a cubic polynomial with real roots x, y, z.

$$f(t) = (t - x)(t - y)(t - z) = t^3 - t^2(x + y + z) + t(xy + yz + zx) - xyz$$

When we arrange this equation using $xyz = \dfrac{1 - (x^2 + y^2 + z^2)}{2}$, we get

$$2(t - x)(t - y)(t - z) = 2t^3 - 2t^2(x + y + z) + 2t(xy + yz + zx) - 1 + (x^2 + y^2 + z^2).$$

Due to the "Main Theorem", we have

$$2t^3 - 2t^2(x + y + z) + 2t(xy + yz + zx) - 1 + (x^2 + y^2 + z^2) = 2(t - x)(t - y)(t - z)$$

$$\leq \frac{2}{27}(3t - (x + y + z))^3.$$

Setting $t = 1$ into this inequality (because $x, y, z < 1$ from the given condition and so $1 = t > \max[x, y, z]$, there is no problem with that) gives us that

$$1 - 2(x + y + z) + (x + y + z)^2 \leq \frac{2}{27}(3 - (x + y + z))^3$$

$$27 - 54(x + y + z) + 27(x + y + z)^2 \leq 54 - 54(x + y + z) + 18(x + y + z)^2 - 2(x + y + z)^3 \Rightarrow$$

$$2(x + y + z)^3 + 9(x + y + z)^2 - 27 \leq 0 \Rightarrow$$

$$(2(x + y + z) - 3)((x + y + z) + 3)^2 \leq 0.$$

It follows $x + y + z \leq \dfrac{3}{2}$. Hence the proof finished.

Equality holds for $x = y = z = \dfrac{1}{2}$.

Problem 8. Let a, b, c, d be positive real numbers such that $a+b+c+d=1$. Prove that

$$(ab+ac+ad+bc+bd+cd)+4abcd \leq 2(abc+bcd+cda+dab)+\frac{17}{64}.$$

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Solution. Let f be a polynomial of fourth degree with real roots a, b, c, d.

$$f(t)=(t-a)(t-b)(t-c)(t-d)$$

$$=t^4-t^3(a+b+c+d)+t^2(ab+ac+ad+bc+bd+cd)-t(abc+bcd+cda+dab)+abcd$$

$$=t^4-t^3+t^2(ab+ac+ad+bc+bd+cd)-t(abc+bcd+cda+dab)+abcd.$$

Due to the "Main Theorem", we have

$$t^4-t^3+t^2(ab+ac+ad+bc+bd+cd)-t(abc+bcd+cda+dab)+abcd$$

$$=(t-a)(t-b)(t-c)(t-d) \leq \left(\frac{4t-(a+b+c+d)}{4}\right)^4=\left(\frac{4t-1}{4}\right)^4$$

where t is a real number satisfying $t \geq \frac{1}{2}(a+b+c+d)=\frac{1}{2}$. Setting $t=\frac{1}{2}$ into this inequality gives us that

$$\frac{1}{16}-\frac{1}{8}+\frac{1}{4}(ab+ac+ad+bc+bd+cd)-\frac{1}{2}(abc+bcd+cda+dab)+abcd \leq \frac{1}{256} \Rightarrow$$

$$(ab+ac+ad+bc+bd+cd)+4abcd \leq 2(abc+bcd+cda+dab)+\frac{17}{64}.$$

Hence the proof finished. Equality holds for $a=b=c=d=\frac{1}{4}$.

Comment. Because the method is very general in terms of t, every problem, that were solved by now, can be generalized as well. That means exactly, there are many problems, that were asked in international or national mathematical competitions, but very similar to each other.

4. Useful Lemmas

Lemma 1. *For all real numbers a, b, c and positive real number k the following inequality holds:*

$$(k(a+b+c)-abc)^2 \leq (a^2+k)(b^2+k)(c^2+k).$$

Proof. Let f be a cubic polynomial with real roots a, b, c.

$$f(t) = (t-a)(t-b)(t-c) = t^3 - t^2(a+b+c) + t(ab+bc+ca) - abc.$$

Because

$$|A+Bi|^2 = (A+Bi)(A-Bi) = A^2+B^2 \geq A^2,$$

we have

$$\begin{aligned} |f(it)|^2 &= |i^3t^3 - i^2t^2(a+b+c) + it(ab+bc+ca) - abc|^2 \\ &= |-it^3 + t^2(a+b+c) + it(ab+bc+ca) - abc|^2 \\ &\geq (t^2(a+b+c) - abc)^2, \end{aligned}$$

$$\begin{aligned} |f(it)|^2 &= |(it-a)(it-b)(it-c)|^2 \\ &= |it-a|^2|it-b|^2|it-c|^2 \\ &= (it-a)(-it-a)(it-b)(-it-b)(it-c)(-it-c) \\ &= (a^2-i^2t^2)(b^2-i^2t^2)(c^2-i^2t^2) \\ &= (a^2+t^2)(b^2+t^2+(c^2+t^2) \end{aligned}$$

where $i^2 = -1$. Combining these, we get

$$(t^2(a+b+c)-abc)^2 \leq (a^2+t^2)(b^2+t^2)(c^2+t^2)$$

where t is an arbitrary real number. When we substitute k in place of t^2, we get the desired result.

Problem 9. Let a, b, c be positive real numbers such that $a^2+b^2+c^2 \leq 3$. Prove that

$$abc + 8 \geq 3(a+b+c).$$

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Solution. Setting $k = 3$ into this lemma and then applying Arithmetic-Geometric Mean Inequality gives us that

$$(3(a+b=c)-abc)^2 \leq (a^2+3)(b^2+3)(c^2+3) \leq \left(\frac{(a^2+b^2+c^2)+9}{3}\right)^3 \leq 64 \Rightarrow$$

$$3(a+b+c) \leq abc+8.$$

Hence the proof finished. Equality holds for $a = b = c = 1$.
Lemma 2. *For all real numbers a, b, c, d and positive real number k the following inequality holds:*

$$(k^2 - k(ab+ac+ad+bc+bd+cd)+abcd)^2 \leq (a^2+k)(b^2+k)(c^2+k)(d^2+k).$$

Proof. Let f be a polynomial of fourth degree with real roots a, b, c, d.

$$f(t) = (t-a)(t-b)(t-c)(t-d)$$

$$= t^4 - t^3(a+b+c+d) + t^2(ab+ac+ad+bc+bd+cd) - t(abc+bcd+cda+dab) + abcd.$$

Because

$$|A+Bi|^2 = (A+Bi)(A-Bi) = A^2+B^2 \geq A^2,$$

we have

$$(t^4 - t^2(ab+ac+ad+bc+bd+cd)+abcd)^2$$

$$\leq |t^4 + it^3(a+b+c+d) - t^2(ab+ac+ad+bc+bd+cd)$$

$$-it(abc+bcd+cda+dab)+abcd|^2$$

$$= |i^4t^4 - i^3t^3(a+b+c+d) + i^2t^2(ab+ac+ad+bc+bd+cd)$$

$$-it(abc+bcd+cda+dab)+abcd|^2$$

$$= |f(it)|^2 = |(it-a)(it-b)(it-c)(it-d)|^2$$

$$= |it-a|^2|it-b|^2|it-c|^2|it-d|^2$$

$$= (-a+it)(-a-it)(-b+it)(-b-it)(-c+it)(-c-it)(-d+it)(-d-it)$$

$$= (a^2 - i^2t^2)(b^2 - i^2t^2)(c^2 - i^2t^2)(d^2 - i^2t^2)$$

$$= (a^2 + t^2)(b^2 + t^2)(c^2 + t^2)(d^2 + t^2)$$

where $i^2 = -1$ and t is an arbitrary real number. Substituting k into the place of t^2 gives us the desired result.

Problem 10. Let a, b, c, d be positive real numbers such that

$$a^2 + b^2 + c^2 + d^2 = 1.$$

Prove that

$$ab + ac + ad + bc + bd + cd \leq \frac{5}{4} + 4abcd.$$

Romanian Mathematical Olympiads 2003, Shortlist

Solution. Setting $k = \dfrac{1}{4}$ into the given lemma yields

$$\left(\frac{1}{16} - \frac{1}{4}(ab + ac + ad + bc + bd + cd) + abcd\right)^2$$

$$\leq \left(\frac{1}{4} + a^2\right)\left(\frac{1}{4} + b^2\right)\left(\frac{1}{4} + c^2\right)\left(\frac{1}{4} + d^2\right).$$

From Arithmetic-Geometric Mean Inequality, we also have

$$\left(\frac{1}{4} + a^2\right)\left(\frac{1}{4} + b^2\right)\left(\frac{1}{4} + c^2\right)\left(\frac{1}{4} + d^2\right) \leq \left(\frac{1 + (a^2 + b^2 + c^2 + d^2)}{4}\right)^4 = \frac{1}{16}.$$

Therefore we get

$$\left(\frac{1}{16} - \frac{1}{4}(ab + ac + ad + bc + bd + cd) + abcd\right)^2 \leq \frac{1}{16} \Rightarrow$$

$$\frac{1}{16} - \frac{1}{4}(ab + ac + ad + bc + bd + cd) + abcd \geq -\frac{1}{4} \Rightarrow$$

$$\frac{5}{4} + 4abcd \geq ab + ac + ad + bc + bd + cd.$$

Hence the proof finished. Equality holds for $a = b = c = d = \dfrac{1}{2}$.

Problem 11. Let a, b, c, d be real numbers such that

$$(a^2+1)(b^2+1)(c^2+1)(d^2+1) = 16.$$

Prove that

$$-3 \leq ab + ac + ad + bc + bd + cd - abcd \leq 5.$$

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Solution. Setting $k = 1$ into the given lemma yields

$$(1-(ab+ac+ad+bc+bd+cd)+abcd)^2 \leq (a^2+1)(b^2+1)(c^2+1)(d^2+1) = 16.$$

Therefore

$$-4 \leq (ab + ac + ad + bc + bd + cd) - abcd - 1 \leq 4 \Rightarrow$$

$$-3 \leq ab + ac + ad + bc + bd + cd - abcd \leq 5.$$

Hence the proof finished. Equality holds for $a = b = c = d = 1$ or $a = b = 1$, $c = d = -1$ and permutations.

Problem 12. Let a, b, c, d be real numbers such that $b - d \geq 5$ and all zeros x_1, x_2, x_3 and x_4 of the polynomial

$$P(x) = x^4 + ax^3 + bx^2 + cx + d$$

are real. Find the smallest value the product $(x_1^2+1)(x_2^2+1)(x_3^2+1)(x_4^2+1)$ can take.

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Solution. The answer is 16. For example, equality holds for $P(x) = (x-1)^4$. Due to the Vieta's Theorems, we have

$$b = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4$$

and

$$d = x_1x_2x_3x_4.$$

Therefore

$$b - d \geq 5 \Rightarrow (x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4) - x_1x_2x_3x_4 \geq 5.$$

Also setting $k = 1$ into the given lemma yields

$$(1 - (x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4) - x_1x_2x_3x_4)^2$$
$$\leq (1 + x_1^2)(1 + x_2^2)(1 + x_3^2)(1 + x_4^2).$$

It follows

$$16 \leq (1 - (x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4) - x_1x_2x_3x_4)^2$$
$$\leq (1 + x_1^2)(1 + x_2^2)(1 + x_3^2)(1 + x_4^2).$$

Hence the proof finished.

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