# A Characterization of the Parallelogram

Paris Pamfilos
University of Crete, Greece
pamfilos@uoc.gr

**Abstract**

In this article we discuss a simple characterization of the parallelogram by considering dissections by lines through a fixed point.

## 1 The characteristic property

A parallelogram $ABCD$ has a symmetry center coinciding with the intersection point of its diagonals $O$. Every line through the point $O$ dissects the parallelogram into two quadrilaterals or triangles, which lie symmetrically with respect to $O$, and, consequently, have the same area (see Figure 1).

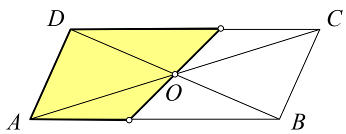


Figure 1: Symmetry of the parallelogram

One can, inversely, ask if there is another kind of convex quadrilateral with the same property. The answer is no, and this is the subject of the following theorem.

**Theorem 1.** *A convex quadrilateral $ABCD$ is a parallelogram, if and only if, there is a point $O$, such that every line through $O$ divides the quadrilateral into two equiareal polygons.*

Before to enter into the proof of the theorem, we discuss a simple lemma, which represents a key point in our arguments.

## 2 A property of the triangle

Consider a triangle $ABC$ and a point $D$ on base $BC$. A line through $D$ intersects the other sides $\{AB, AC\}$ at corresponding points $\{B', C'\}$. The triangles $\{BB'D, CC'D\}$ cannot have the same area for many directions of the line through $D$ (see Figure 2). This is made precise through the following lemma.

**Lemma 1.** *Under the conventions made above, there is no point $D$ on base $BC$ of the triangle, such that the triangles $\{DBB', DCC'\}$ have the same area for three or more lines through $D$.*

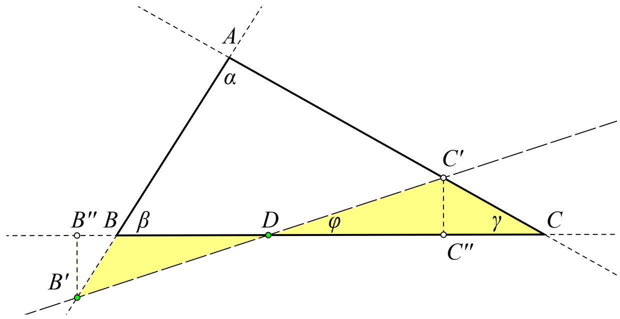


Figure 2: When the two triangles have the same area;

*Proof.* Denoting the projections of $\{B', C'\}$ on $BC$, respectively by $\{B'', C''\}$, the areas of the triangles are

$$\epsilon(DCC') = \frac{1}{2}|DC| \cdot |DC''| \tan(\phi) \quad \text{and} \quad \epsilon(DBB') = \frac{1}{2}|DB| \cdot |DB''| \tan(\phi).$$

Without loss of the generality, we assume here that the angle $\gamma$ is acute and the angle $\phi$ of the line intersecting the triangle is acute towards $C$. This, implying that $C''$ is inside $CD$ and $B$ is outside/inside $BD$, depending on whether the angle $\beta$ is acute/obtuse. With minor modifications in the formulas below the results are valid for all cases. We exclude also the case of a right-angled (at $B$ or $C$) triangle, for which the calculations below become even simpler.

In fact, calculating the lengths $\{|B'B''|, |C'C''|\}$ in two ways, we see that

$$|C'C''| = |DC''| \tan(\phi) = (|DC| - |DC''|) \tan(\gamma) \quad \Rightarrow \quad |DC''| = \frac{|DC| \tan(\gamma)}{\tan(\phi) + \tan(\gamma)},$$

$$|B'B''| = |DB''| \tan(\phi) = (|DB''| - |DB|) \tan(\beta) \quad \Rightarrow \quad |DB''| = \frac{|DB| \tan(\beta)}{\tan(\beta) - \tan(\phi)}.$$

Thus, if the areas of the two triangles are equal, the following equation must be valid:

$$\frac{|DC|^2 \tan(\gamma)}{\tan(\phi) + \tan(\gamma)} = \frac{|DB|^2 \tan(\beta)}{\tan(\beta) - \tan(\phi)} \quad \Leftrightarrow$$

$$\tan(\phi)(|DB|^2 \tan(\beta) + |DC|^2 \tan(\gamma)) = (|DC|^2 - |DB|^2) \tan(\beta) \tan(\gamma).$$

Consider the last equation as a linear equation in $x = \tan(\phi)$, for a fixed $D$. Then, if by assumption, the areas of the two triangles are equal for three different lines through $D$, we easily see that the above equation (or a similar one with modified signs) must be valid for two values of $\phi$. This implies that the corresponding coefficients of the linear equation must vanish. Investigating the different possibilities, we find that this is impossible.

For example, the vanishing of the right side of the last equation implies $|DB| = |DC|$, so point $D$ is the middle of $BC$, and this, looking at the left side, implies that $\tan(\beta) + \tan(\gamma) = 0$, which is impossible. A similar argument applies also to the other possible cases, regarding the locations of $\{B'', C''\}$ relative to the segments $\{BD, DC\}$. The details here are left as an exercise. □

## 3 Proof of the theorem

Assume now that $ABCD$ is a convex quadrilateral and $O$ is a point in its plane, such that every line through $O$ divides the quadrilateral in two equiareal polygons. This hypothesis trivially implies that point $O$ is inside the quadrilateral. But we can restrict its location further by thinking as follows. Assume that the intersection point $P$ of the diagonals of

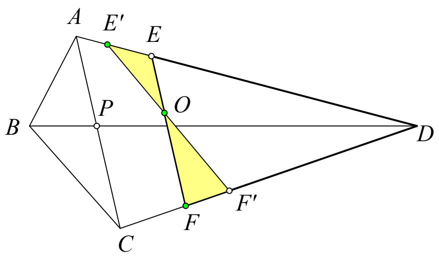


Figure 3: Location of $O$

the quadrilateral does not coincide with the middle of one diagonal, say the diagonal $BD$, and it is $|BP| < |PD|$ (see Figure 3). Then, we see easily that the ratio of the areas of the triangles

$$\frac{\epsilon(ABC)}{\epsilon(ACD)} = \frac{|BP|}{|PD|} \qquad \Rightarrow \qquad \epsilon(ABC) < \epsilon(ACD).$$

Consequently, we can find a parallel $EF$ to $AC$, lying between $AC$ and $D$, such that the area $EFD$ is half the area of the quadrilateral. It is then easily seen, that the point $O$ with the property, that every line through it divides the quadrilateral in two equiareal parts, if it exists, then it must be on $EF$. But then, drawing lines $E'F'$ through $O$, we can find infinitely many pairs of triangles $\{OEE', OFF'\}$ with equal areas. This contradicts the previous lemma and shows that $EF$ must coincide with $AC$, meaning that $O$ must be on $AC$. A similar argument shows that $O$ must be also on $BD$. Hence $O$ must coincide with the intersection point $P$ of the diagonals, implying the proof of the theorem.