Junior problems

J367. Let a and b be positive real numbers. Prove that

$$\frac{1}{4a}+\frac{3}{a+b}+\frac{1}{4b}\geq\frac{4}{3a+b}+\frac{4}{a+3b}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Nermin Hodzic, Dobosnica, Bosnia and Herzegovina

We can rewrite the inequality as

$$\frac{a+b}{4ab}+\frac{3}{a+b}\geq\frac{16(a+b)}{4ab+3(a+b)^2}\Leftrightarrow$$

$$\left(4ab+3(a+b)^2\right)\left(\frac{a+b}{4ab}+\frac{3}{a+b}\right)\geq 16(a+b).$$

Using the Cauchy-Schwarz inequality we have

$$\left(4ab+3(a+b)^2\right)\left(\frac{a+b}{4ab}+\frac{3}{a+b}\right)\geq\left(\sqrt{a+b}+3\sqrt{a+b}\right)^2=16(a+b).$$

Equality holds if and only if $a=b$.

Also solved by Henry Ricardo, New York Math Circle, Tappan, NY, USA; Bobojonova Latofat, Academic lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Ángel Plaza, Department of Mathematics, University of Las Palmas de Gran Canaria, Spain; Joel Schlosberg, Bayside, NY, USA; Albert Stadler, Herrliberg, Switzerland; Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; Yong Xi Wang, Affiliated High School of Shanxi University; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Ioan Viorel Codreanu, Satulung, Maramures, Romania; David E. Manes, Oneonta, NY, USA; Alan Yan, Princeton Junction, NJ, USA; David Stoner, Harvard University, Cambridge, MA, USA; Petros Panigyrakis, Evaggeliki Gymnasium, Athens, Greece; Nicuşor Zlota ,"Traian Vuia" Technical College, Focşani, Romania; Joachim Studnia, Lycée Condorcet, Paris, France; Dimitris Avramidis, Evaggeliki Gymnasium, Athens, Greece; Duy Quan Tran, University of Health Science, Ho Chi Minh City, Vietnam; Paul Revenant, Lycée Champollion, Grenoble, France; Polyahedra, Polk State College, FL, USA; Moubinool Omarjee Lycée Henri IV, Paris, France; Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA, USA; Daniel Lasaosa, Pamplona, Spain; Evgenidis Nikolaos, M.N. Raptou High School, Larissa, Greece; Arkady Alt, San Jose, CA, USA; WSA.

J368. Find the best constants α and β such that $\alpha < \dfrac{x}{2x+y} + \dfrac{y}{x+2y} \le \beta$ for all $x, y \in (0, \infty)$.

Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

Solution by Polyahedra, Polk State College, FL, USA

By the AM-HM inequality,

$$\frac{x}{2x+y} + \frac{y}{x+2y} = 2 - (x+y)\left(\frac{1}{2x+y} + \frac{1}{x+2y}\right) \le 2 - \frac{4(x+y)}{(2x+y)+(x+2y)} = \frac{2}{3},$$

with equality if and only if $x = y$. Hence $\beta = \frac{2}{3}$. On the other hand, assume that $x \ge y$. Then $2(2x+y) > 2x+y \ge x+2y$, thus

$$\frac{x}{2x+y} - \frac{1}{2} + \frac{y}{x+2y} = y\left(\frac{1}{x+2y} - \frac{1}{2(2x+y)}\right) > 0,$$

and this expression $\to 0$ as $y \to 0$ for fixed x. Hence $\alpha = \frac{1}{2}$.

Also solved by Paul Revenant, Lycée Champollion, Grenoble, France; WSA; Hyun Min Victoria Woo, Northfield Mount Hermon School, Mount Hermon, MA, USA; Ji Eun Kim, Tabor Academy, Marion, MA, USA; Yong Xi Wang, Affiliated High School of Shanxi University; Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; Albert Stadler, Herrliberg, Switzerland; David Stoner, Harvard University, Cambridge, MA, USA; Bobojonova Latofat, Academic lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Joel Schlosberg, Bayside, NY, USA; Jorge Ledesma, Faculty of Sciences UNAM, Mexico City, Mexico; Alan Yan, Princeton Junction, NJ, USA; Henry Ricardo, New York Math Circle, Tappan, NY, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Daniel Lasaosa, Pamplona, Spain; Nermin Hodzic, Dobosnica, Bosnia and Herzegovina; Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA, USA; Arkady Alt, San Jose, CA, USA.

J369. Solve the equation

$$\sqrt{1+\frac{1}{x+1}}+\frac{1}{\sqrt{x+1}}=\sqrt{x}+\frac{1}{\sqrt{x}}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Alessandro Ventullo, Milan, Italy

Clearly, $x>0$. The given equation can be written as

$$\frac{\sqrt{x+2}}{\sqrt{x+1}}+\frac{1}{\sqrt{x+1}}=\frac{x+1}{\sqrt{x}},$$

i.e.

$$\sqrt{x^2+2x}+\sqrt{x}=(x+1)\sqrt{x+1}.$$

The last equation can be written as

$$\frac{\sqrt{x}}{\sqrt{x+2}-1}=\sqrt{x+1},$$

which gives

$$\sqrt{x}+\sqrt{x+1}=\sqrt{(x+2)(x+1)}.$$

Squaring both sides and reordering, we get

$$x^2+x+1=2\sqrt{x^2+x},$$

i.e.

$$(\sqrt{x^2+x}-1)^2=0.$$

The last equation is equivalent to $x^2+x-1=0$, which solving for $x>0$ gives $x=\dfrac{-1+\sqrt{5}}{2}$.

Also solved by Moubinool Omarjee Lycée Henri IV , Paris France; Ángel Plaza, Department of Mathematics, University of Las Palmas de Gran Canaria, Spain; Bobojonova Latofat, Academic lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Yong Xi Wang, Affiliated High School of Shanxi University; Hyun Min Victoria Woo, Northfield Mount Hermon School, Mount Hermon, MA, USA; Joel Schlosberg, Bayside, NY, USA; Ji Eun Kim, Tabor Academy, Marion, MA, USA; Joe Currier, College at Brockport, SUNY, USA; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Vicente Vicario García, Sevilla, Spain; David E. Manes, Oneonta, NY, USA; Alan Yan, Princeton Junction, NJ, USA; David Stoner, Harvard University, Cambridge, MA, USA; Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; WSA; Nicuşor Zlota ,"Traian Vuia" Technical College, Focşani, Romania; Duy Quan Tran, University of Health Science, Ho Chi Minh City, Vietnam; Henry Ricardo, New York Math Circle, Tappan, NY, USA; Arpon Basu; Paul Revenant, Lycée Champollion, Grenoble, France; Polyahedra, Polk State College, FL, USA; Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA, USA; Albert Stadler, Herrliberg, Switzerland; Daniel Lasaosa, Pamplona, Spain; Evgenidis Nikolaos, M.N. Raptou High School, Larissa, Greece; Nermin Hodzic, Dobosnica, Bosnia and Herzegovina; Antoine Faisant, Lycée Pierre de Fermat, France; Arkady Alt, San Jose, CA, USA.

J370. Triangle ABC has sides lengths $BC = a, CA = b, AB = c$. If

$$(a^2 + b^2 + c^2)^2 = 4a^2b^2 + b^2c^2 + 4c^2a^2,$$

find all possible values of $\angle A$.

Proposed by Adrian Andreescu, Dallas, TX, USA

Solution by Nikolaos Evgenidis, M.N. Raptou High School, Larissa, Greece

Expanding the LHS and doing the maths we have

$$a^4 - 2a^2b^2 + b^4 + b^2c^2 + c^4 - 2c^2a^2 = 0.$$

By adding b^2c^2 to both sides, the above relation can be written as

$$(b^2 + c^2 - a^2)^2 = b^2c^2 \Leftrightarrow (b^2 + bc + c^2 - a^2)(b^2 - bc + c^2 - a^2) = 0.$$

Hence, either $b^2 + bc + c^2 = a^2$ or $b^2 - bc + c^2 = a^2$.
But from the Law of Cosines we know $a^2 = b^2 + c^2 - 2bc\cos\angle A$. This, combined with the above equations, implies either:

- $bc(1 + 2\cos\angle A) = 0$. Then $\cos\angle A = -\frac{1}{2} \Leftrightarrow \angle A = 120^o$ since $0 \leq A \leq 180$

or

- $bc(1 - 2\cos\angle A) = 0$. Then $\cos\angle A = \frac{1}{2} \Leftrightarrow \angle A = 60^o$ since $0 \leq A \leq 180$.

Therefore, the possible values of $\angle A$ are 60^o or 120^o.

Also solved by Moubinool Omarjee, Lycée Henri IV, Paris, France; Bobojonova Latofat, Academic lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; Yong Xi Wang, Affiliated High School of Shanxi University; Jorge Ledesma, Faculty of Sciences UNAM, Mexico City, Mexico; Hyun Min Victoria Woo, Northfield Mount Hermon School, Mount Hermon, MA, USA; Ji Eun Kim, Tabor Academy, Marion, MA, USA; Andreas Charalampopoulos, 4th Lyceum of Glyfada, Glyfada, Greece; Helen Kain, student, College at Brockport, SUNY; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Vicente Vicario García, Sevilla, Spain; Alan Yan, Princeton Junction, NJ, USA; David Stoner, Harvard University, Cambridge, MA, USA; WSA; Nicuşor Zlota ,"Traian Vuia" Technical College, Focşani, Romania; Duy Quan Tran, University of Health Science, Ho Chi Minh City, Vietnam; Panigyraki Chrysoula, Evaggeliki Gymnasium, Athens, Greece; Dimitris Avramidis, Evaggeliki Gymnasium, Athens, Greece; Paul Revenant, Lycée Champollion, Grenoble, France; Polyahedra, Polk State College, FL, USA; Adam Krause, student, College at Brockport, SUNY; Alok Kumar; Albert Stadler, Herrliberg, Switzerland; Daniel Lasaosa, Pamplona, Spain; Nermin Hodzic, Dobosnica, Bosnia and Herzegovina; Arkady Alt, San Jose, CA, USA.

J371. Prove that for all positive integers n,

$$\binom{n+3}{2}+6\binom{n+4}{4}+90\binom{n+5}{6}$$

is the sum of two perfect cubes.

Proposed by Alessandro Ventullo, Milan, Italy

Solution by Adnan Ali, Student in A.E.C.S-4, Mumbai, India

It is easily verified that

$$\binom{n+3}{2}+6\binom{n+4}{4}+90\binom{n+5}{6}=1^3+a_n^3, \tag{1}$$

where $a_n=\dfrac{(n+4)(n+1)}{2}$. Moreover a_n is an integer as $n+4$ and $n+1$ are always of different parities. (1) may be verified as follows

$$\binom{n+3}{2}+6\binom{n+4}{4}+90\binom{n+5}{6}$$

$$=\frac{4(n+3)(n+2)}{8}+\frac{2(n+4)(n+3)(n+2)(n+1)}{8}+\frac{(n+5)(n+4)(n+3)(n+2)(n+1)n}{8}$$

$$=\frac{(n+3)(n+2)}{2}\left(1+\frac{1}{2}(n+4)(n+1)+\frac{1}{4}(n+5)(n+4)(n+1)n\right)=(a_n+1)(1+a_n+a_n(a_n-2)),$$

and the proof is complete.

Also solved by Moubinool Omarjee Lycée Henri IV, Paris, France; Ángel Plaza, Department of Mathematics, University of Las Palmas de Gran Canaria, Spain; Bobojonova Latofat, Academic lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; Yong Xi Wang, Affiliated High School of Shanxi University; Jorge Ledesma, Faculty of Sciences UNAM, Mexico City, Mexico; Hyun Min Victoria Woo, Northfield Mount Hermon School, Mount Hermon, MA, USA; Ji Eun Kim, Tabor Academy, Marion, MA, USA; Adam Krause, student, College at Brockport, SUNY; Joel Schlosberg, Bayside, NY, USA; Joe Currier, student, College at Brockport, SUNY; Vicente Vicario García, Sevilla, Spain; David E. Manes, Oneonta, NY, USA; Alan Yan, Princeton Junction, NJ, USA; David Stoner, Harvard University, Cambridge, MA, USA; Arpon Basu, AECS-4, Mumbai, India; Paul Revenant, Lycée Champollion, Grenoble, France; Polyahedra, Polk State College, FL, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Albert Stadler, Herrliberg, Switzerland; Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA, USA; Daniel Lasaosa, Pamplona, Spain; Nermin Hodzic, Dobosnica, Bosnia and Herzegovina.

J372. In triangle ABC, $\frac{\pi}{7} < A \le B \le C < \frac{5\pi}{7}$. Prove that

$$\sin\frac{7A}{4} - \sin\frac{7B}{4} + \sin\frac{7C}{4} > \cos\frac{7A}{4} - \cos\frac{7B}{4} + \cos\frac{7C}{4}.$$

Proposed by Adrian Andreescu, Dallas, TX, USA

Solution by Alan Yan, Princeton Junction, NJ, USA

Let $X = \frac{7A}{4}, Y = \frac{7B}{4}, Z = \frac{7C}{4}$. We have that $\frac{\pi}{4} < X \le Y \le Z < \frac{5\pi}{4}$. Rearrange the original equality as

$$(\sin X - \cos X) - (\sin Y - \cos Y) + (\sin Z - \cos Z) > 0$$
$$\frac{\sqrt{2}}{2}(\sin X - \cos X) - \frac{\sqrt{2}}{2}(\sin Y - \cos Y) + \frac{\sqrt{2}}{2}(\sin Z - \cos Z) > 0$$
$$\sin\left(X - \frac{\pi}{4}\right) - \sin\left(Y - \frac{\pi}{4}\right) + \sin\left(Z - \frac{\pi}{4}\right) > 0$$

Let $U = X - \frac{\pi}{4}, V = Y - \frac{\pi}{4}, W = Z - \frac{\pi}{4}$. We have that $U + V + W = \pi$ and $0 < U \le V \le W \le \pi$. So, there exists a triangle with angles U, V, W. Let the circumradius of this triangle be R and let the sides corresponding to U, V, W be u, v, w, respectively. It suffices to prove that $\sin U - \sin V + \sin W > 0$.

Indeed,

$$\sin U - \sin V + \sin W > 0 \iff 2R(\sin U - \sin V + \sin W) > 0 \iff u - v + w > 0,$$

which is just the triangle inequality. ■

Also solved by Hyun Min Victoria Woo, Northfield Mount Hermon School, Mount Hermon, MA, USA; Ji Eun Kim, Tabor Academy, Marion, MA, USA; Yong Xi Wang, Affiliated High School of Shanxi University; Daniel Lasaosa, Pamplona, Spain; Nermin Hodzic, Dobosnica, Bosnia and Herzegovina; Polyahedra, Polk State College, FL, USA; David Stoner, Harvard University, Cambridge, MA, USA.

S367. Solve in positive real numbers the system of equations

$$\begin{cases} (x^3+y^3)(y^3+z^3)(z^3+x^3)=8, \\ \dfrac{x^2}{x+y}+\dfrac{y^2}{y+z}+\dfrac{z^2}{z+x}=\dfrac{3}{2}. \end{cases}$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by David Stoner, Harvard University, Cambridge, MA, USA

First, note that $\sum_{\text{cyc}} \frac{x^2}{x+y} = \sum_{\text{cyc}} \frac{y^2}{x+y}$, as their difference is $\sum_{\text{cyc}}(x-y)=0$. It follows that the second condition is equivalent to $\sum_{\text{cyc}} \frac{x^2+y^2}{x+y}=3$.

Now we claim that for positive reals a, b, the inequality $\frac{a^3+b^3}{2} \le (\frac{a^2+b^2}{a+b})^3$ holds, with equality occurring if and only if $a=b$. Indeed, this follows from the identity $2(a^2+b^2)^3-(a+b)^3(a^3+b^3)=(a-b)^4(a^2+b^2+ab)$.

Using this, we see that, for any x, y, z satisfying the system, we have by this claim and AM-GM:

$$\begin{aligned} 3 &= \sum \frac{x^2+y^2}{x+y} \\ &\ge 3\sqrt[3]{\left(\frac{x^2+y^2}{x+y}\right)\left(\frac{y^2+z^2}{y+z}\right)\left(\frac{z^2+x^2}{z+x}\right)} \\ &\ge 3\sqrt[9]{\left(\frac{x^3+y^3}{2}\right)\left(\frac{y^3+z^3}{2}\right)\left(\frac{z^3+x^3}{2}\right)} \\ &= 3. \end{aligned}$$

Therefore, equality must hold in the second inequality, which by our claim implies $x=y=z$. Plugging this into $\sum_{\text{cyc}} \frac{x^2+y^2}{x+y}=3$ gives $3x=3$, so $x=y=z=1$ is the sole solution to this system; it works, so we are done.

Also solved by Li Zhou, Polk State College, Winter Haven, FL, USA; Sutanay Bhattacharya, Bishnupur High School, West Bengal, India; Nermin Hodzic, Dobosnica, Bosnia and Herzegovina.

S368. Determine all positive integers n such that $\sigma(n) = n + 55$, where $\sigma(n)$ denotes the sum of the divisors of n.

Proposed by Alessandro Ventullo, Milan, Italy

Solution by Li Zhou, Polk State College, Winter Haven, FL, USA

Write the prime factorization of n as $= p_1^{e_1} \cdots p_k^{e_k}$.
If $k = 1$, then $2 \cdot 3^3 = 54 = \sigma(n) - n - 1 = p_1 \left(1 + \cdots + p_1^{e_1 - 2}\right)$, which means $p_1 = 2$, an impossibility.
If $k \geq 4$, then

$$\begin{aligned} \sigma(n) &= (1 + p_1 + \cdots + p_1^{e_1}) \cdots (1 + p_k + \cdots + p_k^{e_k}) > n + p_1p_2p_3 + p_2p_3p_4 + p_3p_4p_1 + p_4p_1p_2 \\ &\geq n + 2 \cdot 3 \cdot 5 + 3 \cdot 5 \cdot 7 + 5 \cdot 7 \cdot 2 + 7 \cdot 2 \cdot 3 > n + 55. \end{aligned}$$

Next, consider $k = 3$. If n is odd, then

$$\sigma(n) > n + p_1p_2 + p_2p_3 + p_3p_1 \geq n + 3 \cdot 5 + 5 \cdot 7 + 7 \cdot 3 > n + 55.$$

If n is even, then $p_1 = 2$. Since $55 = \sigma(n) - n$, $\sigma(n)$ must be odd. Hence e_2, e_3 must be even, thus

$$\sigma(n) > n + \left(1 + 3 + 3^2\right)\left(1 + 5 + 5^2\right) > n + 55.$$

Finally, consider $k = 2$. If $e_1 = e_2 = 1$, then $54 = \sigma(n) - n - 1 = p_1 + p_2$ and we get $\{p_1, p_2\} = \{7, 47\}$, $\{11, 43\}$, $\{13, 41\}$, $\{17, 37\}$, $\{23, 31\}$. If $(e_1, e_2) = (2, 1)$, then

$$\sigma(n) - n = \left(1 + p_1 + p_1^2\right)(1 + p_2) - p_1^2 p_2 = (1 + p_1)(1 + p_2) + p_1^2,$$

which is greater than 55 for $p_1 > 5$ and cannot equal 55 for $p_1 = 2, 3, 5$. If $e_1 = e_2 = 2$, then

$$\sigma(n) - n = \left(1 + p_2 + p_2^2\right)(1 + p_1) + p_1^2(1 + p_2) \geq 55,$$

with equality if and only if $\{p_1, p_2\} = \{2, 3\}$. If $e_1 \geq 3$, then

$$\sigma(n) - n \geq \left(1 + p_1 + p_1^2\right)(1 + p_2) + p_1^3,$$

which is greater than 55 for $p_1 \geq 3$. But if $p_1 = 2$, then e_2 must be even for $\sigma(n)$ to be odd. Thus

$$\sigma(n) - n \geq \left(1 + p_1 + p_1^2\right)\left(1 + p_2 + p_2^2\right) \geq 7 \cdot 13 > 55.$$

In conclusion, the solutions are $n = 7 \cdot 47 = 329$, $11 \cdot 43 = 473$, $13 \cdot 41 = 533$, $17 \cdot 37 = 629$, $23 \cdot 31 = 713$, and $2^2 \cdot 3^2 = 36$.

Also solved by Khurshid Juraev, Academic lyceum named after S.H.Sirojiddinov, Tashkent, Uzbekistan; David Stoner, Harvard University, Cambridge, MA, USA; Latofat Bobojonova, Academic lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; David E. Manes, Oneonta, NY, USA; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Nermin Hodzic, Dobosnica, Bosnia and Herzegovina; Albert Stadler, Herrliberg, Switzerland; Moubinool Omarjee, Lycée Henri IV, Paris, France.

S369. Given the polynomial $P(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ having n real roots (not necessarily distinct) in the interval $[0, 1]$, prove that $3a_1^2 + 2a_1 - 8a_2^2 \le 1$.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Latofat Bobojonova, Academic lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan

Comment. There must be a typo. Counter example: $P(x) = x^2 - 1.5x + 0.5$.

The original version should be :

Given the polynomial $P(x) = x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$ having n real roots (not necessarily distinct) in interval $[0, 1]$, prove that $3a_1^2 + 2a_1 - 8a_2 \le 1$.

Solution. Notice that $a_1 = -(x_1 + x_2 + ... + x_n)$ and $a_2 = \dfrac{(x_1 + x_2 + ... + x_n)^2 - (x_1^2 + x_2^2 + ... + x_n^2)}{2}$,

so we can rewrite our inequality as

$$3(x_1 + x_2 + ... + x_n)^2 - 2(x_1 + x_2 + ... + x_n) - 4\left((x_1 + x_2 + ... + x_n)^2 - (x_1^2 + x_2^2 + ... + x_n^2)\right) \le 1,$$

which is equivalent to

$$4(x_1^2 + x_2^2 + ... + x_n^2) - 2(x_1 + x_2 + ... + x_n) \le 1 + (x_1 + x_2 + ... + x_n)^2.$$

Since $x_1, x_2, ..., x_n \in [0, 1]$ we have that $x_1^2 + x_2^2 + ... + x_n^2 \le x_1 + x_2 + ... + x_n$, so

$$4(x_1^2 + x_2^2 + ... + x_n^2) - 2(x_1 + x_2 + ... + x_n) \le 4(x_1 + x_2 + ... + x_n) - 2(x_1 + x_2 + ... + x_n) =$$

$$= 2(x_1 + x_2 + ... + x_n) \le (x_1 + x_2 + ... + x_n)^2 + 1,$$

where the last inequality follows from $(x_1 + x_2 + ... + x_n - 1)^2 \ge 0$. Equality holds when $x_1 = 1$ and $x_2 = x_3 = ... = x_n = 0$.

Also solved by Khurshid Juraev, Academic lyceum named after S.H.Sirojiddinov, Tashkent, Uzbekistan; Nermin Hodzic, Dobosnica, Bosnia and Herzegovina; Yong Xi Wang, Affiliated High School of Shanxi University.

S370. Prove that in any triangle,

$$\left|3a^2 - 2b^2\right| m_a + \left|3b^2 - 2c^2\right| m_b + \left|3c^2 - 2a^2\right| m_c \geq \frac{8K^2}{R}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Latofat Bobojonova, academic lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan

Firstly, we observe a useful lemma.

Lemma. In any triangle it is true that $m_a \geq \dfrac{b^2+c^2}{4R}$.

Proof. Let ABC be triangle inscribed to the circle w. Let m_a intersect w at X $(X \neq A)$. By power of a point we have

$$AX = m_a + \frac{a^2}{4m_a} = \frac{4m_a^2 + a^2}{4m_a} = \frac{2b^2 + 2c^2}{4m_a} \leq 2R \Rightarrow m_a \geq \frac{b^2+c^2}{4R}.$$

So, the lemma is proved. Back to the main problem.

Using the lemma we get

$$|3a^2 - 2b^2|m_a + |3b^2 - 2c^2|m_b + |3c^2 - 2a^2|m_c \geq$$
$$\geq |3a^2 - 2b^2| \cdot \frac{b^2+c^2}{4R} + |3b^2 - 2c^2| \cdot \frac{c^2+a^2}{4R} + |3c^2 - 2a^2| \cdot \frac{a^2+b^2}{4R}.$$

And finally, using the Triangle Inequality and Heron's formula gives us the desired result:

$$|3a^2 - 2b^2|(b^2 + c^2) + |3b^2 - 2c^2|(c^2 + a^2) + |3c^2 - 2a^2|(a^2 + b^2) \geq$$

$$|(3a^2 - 2b^2)(b^2 + c^2) + (3b^2 - 2c^2)(c^2 + a^2) + (3c^2 - 2a^2)(a^2 + b^2)| =$$

$$= |2(2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4)| = 2|16K^2| = 32K^2.$$

Also solved by Arkady Alt, San Jose, CA, USA; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania.

S371. Let ABC be a triangle and let M be the midpoint of the arc BAC. Let AL be the bisector of angle A, and let I be the incenter of triangle ABC. Line MI intersects the circumcircle of triangle ABC at K and line BC intersects the circumcircle of triangle AKL at P. If $PI \cap AK = \{X\}$, and $KI \cap BC = \{Y\}$, prove that $XY \parallel AL$.

Proposed by Latofat Bobojonova, Tashkent, Uzbekistan

Solution by Khurshid Juraev, Academic lyceum named after S.H.Sirojiddinov, Tashkent, Uzbekistan

Firstly, we prove the well known lemma. After that we will show $\angle KXY = \angle KAI$.

Lemma. Let S be the midpoint of minor arc BC. Then KS, BC, and the line through I perpendicular to AS (let it be l) are concurrent.

Proof. $\angle IKS = \frac{\pi}{2}$ implies that l is tangent to (IKS), but we know this line is tangent to (BIC), so l is the radical axis of (IKS) and (BIC). BC is the radical axis of (ABC) and (BIC), and KS is the radical axis of (ABC) and (IKS), so these lines are concurrent at the radical center of (ABC), (IKS), and (BIC). □

Let the point of concurrency be P'. Then

$$\angle LP'K = \angle CBK - \angle BKP = \angle CAK - \angle BAS = \angle CAK - \angle SAC = \angle KAS.$$

This shows us that BC intersects (AKL) at P', so $P = P'$ and the Lemma implies that $PI \perp AI$. Thus

$$\angle PXK = \frac{\pi}{2} - \angle KAL = \frac{\pi}{2} - \angle KPL = \angle PYK$$

So $PXYK$ is cyclc quadrilateral. Finally, $\angle KXY = \angle KPY = \angle KAI$. We are done!

Also solved by Nermin Hodzic, Dobosnica, Bosnia and Herzegovina; Nikolaos Evgenidis, M.N. Raptou High School, Larissa, Greece; Prithwijit De, HBCSE, Mumbai, India; Li Zhou, Polk State College, Winter Haven, FL, USA.

S372. Prove that in any triangle,

$$\frac{2}{3}(m_am_b+m_bm_c+m_cm_a)\geq\frac{1}{4}(a^2+b^2+c^2)+\sqrt{3}K.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA;

Solution by Nguyen Viet Hung, HSGS, Hanoi University of Science, Vietnam

We have known that there exists a triangle whose side-lengths are m_a, m_b, m_c and its area is $K' = \frac{3}{4}K_{ABC}$. Applying the Hadwiger–Finsler inequality for this triangle we obtain

$$m_a^2+m_b^2+m_c^2\geq 4\sqrt{3}K'+(m_a-m_b)^2+(m_b-m_c)^2+(m_c-m_a)^2$$

or

$$2(m_am_b+m_bm_c+m_cm_a)\geq 4\sqrt{3}\cdot\frac{3}{4}K_{ABC}+m_a^2+m_b^2+m_c^2$$

On the other hand, easy to see that

$$m_a^2+m_b^2+m_c^2=\frac{3}{4}(a^2+b^2+c^2)$$

Thus we get

$$2(m_am_b+m_bm_c+m_cm_a)\geq 3\sqrt{3}K_{ABC}+\frac{3}{4}(a^2+b^2+c^2),$$

which is equivalent to the desired result, and we are done!

Also solved by David Stoner, Harvard University, Cambridge, MA, USA; Bobojonova Latofat, Academic lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Yong Xi Wang, Affiliated High School of Shanxi University; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Vicente Vicario García, Sevilla, Spain; Nicuşor Zlota ,"Traian Vuia" Technical College, Focşani, Romania; Khurshid Juraev, Academic lyceum named after S.H.Sirojiddinov, Tashkent, Uzbekistan; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Evgenidis Nikolaos, M.N. Raptou High School, Larissa, Greece; Nermin Hodzic, Dobosnica, Bosnia and Herzegovina; Arkady Alt, San Jose, CA, USA.

U367. Let $\{a_n\}_{n\geq 1}$ be the sequence of real numbers given by $a_1 = 4$ and $3a_{n+1} = (a_n+1)^3 - 5$, $n \geq 1$. Prove that a_n is a positive integer for all n, and evaluate

$$\sum_{n=1}^{\infty} \frac{a_n - 1}{{a_n}^2 + a_n + 1}.$$

Proposed by Albert Stadler, Herrliberg, Switzerland

Solution by Daniel Lasaosa, Pamplona, Spain

Define $a_n = 3b_n + 1$, or $b_1 = 1$, and for all $n \geq 1$,

$$b_{n+1} = 3b_n\left(b_n+1\right)^2 + b_n$$

therefore since b_1 is a positive integer, by trivial induction b_n is an integer for all n, hence so is a_n. Moreover,

$$\frac{a_n-1}{{a_n}^2+a_n+1} = \frac{b_n}{3{b_n}^2+3b_n+1} = \frac{b_n(b_n+1)}{b_{n+1}+1} = \frac{b_{n+1}-b_n}{3\left(b_n+1\right)\left(b_{n+1}+1\right)} =$$

$$= \frac{1}{3\left(b_n+1\right)} - \frac{1}{3\left(b_{n+1}+1\right)},$$

or

$$\sum_{n=1}^{N} \frac{a_n-1}{{a_n}^2+a_n+1} = \frac{1}{3\left(b_1+1\right)} - \frac{1}{3\left(b_{N+1}+1\right)} = \frac{1}{6} - \frac{1}{3\left(b_{N+1}+1\right)},$$

and since b_N clearly diverges to $+\infty$ because for any positive b_n we have $b_{n+1} > 3{b_n}^3$ and $b_1 = 1$, we finally conclude that

$$\sum_{n=1}^{\infty} \frac{a_n-1}{{a_n}^2+a_n+1} = \frac{1}{6} - \lim_{N\to\infty} \frac{1}{3\left(b_{N+1}+1\right)} = \frac{1}{6}.$$

Also solved by Yong Xi Wang, Affiliated High School of Shanxi University, Shanxi, China; Brian Bradie, Christopher Newport University, Newport News, VA, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Li Zhou, Polk State College, Winter Haven, FL, USA; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Arkady Alt, San Jose, CA, USA; Jorge Ledesma, UNAM, Mexico City, Mexico; Miguel Cidr/'as Senra, Pontevedra, Spain; Moubinool Omarjee , Lyc/'ee Henri IV , Paris, France; Nermin Hodzic, Dobosnica, Bosnia and Herzegovina; Kosmidis Dimitrios, National and Kapodistrian University of Athens, Greece.

U368. Let

$$x_n = \sqrt{2} + \sqrt[3]{\frac{3}{2}} + \cdots + \sqrt[n+1]{\frac{n+1}{n}}, \qquad n = 1, 2, 3, \ldots.$$

Evaluate $\lim_{n\to\infty} \frac{x_n}{n}$.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Li Zhou, Polk State College, USA

For $1 \leq k \leq n$, by the binomial theorem,

$$\left(1 + \frac{1}{k(k+1)}\right)^{k+1} = 1 + \frac{1}{k} + \cdots > \frac{k+1}{k}.$$

Hence,

$$1 < \sqrt[k+1]{\frac{k+1}{k}} < 1 + \frac{1}{k(k+1)} = 1 + \frac{1}{k} - \frac{1}{k+1},$$

and thus $n < x_n < n + 1 - \frac{1}{n+1}$. Therefore, by the squeeze theorem, $\lim_{n\to\infty} \frac{x_n}{n} = 1$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Yong Xi Wang, Affiliated High School of Shanxi University, Shanxi, China; Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA, USA; Hyun Min Victoria Woo, Northfield Mount Hermon School, Mount Hermon, MA, USA; Ji Eun Kim, Tabor Academy, Marion, MA, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Vicente Vicario García, Sevilla, Spain; Kosmidis Dimitrios, National and Kapodistrian University of Athens, Greece; Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Alessandro Ventullo, Milan, Italy; Arkady Alt, San Jose, CA, USA; Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; David Stoner, Harvard University, Cambridge, MA, USA; Henry Ricardo, New York Math Circle, Tappan, NY, USA; Joel Schlosberg, Bayside, NY, USA; Miguel Cidr/'as Senra, Pontevedra, Spain; Moubinool Omarjee Lycée Henri IV, Paris, France; Nermin Hodzic, Dobosnica, Bosnia and Herzegovina; Albert Stadler, Herrliberg, Switzerland.

U369. Prove that

$$\frac{648}{35}\sum_{k=1}^{\infty}\frac{1}{k^3(k+1)^3(k+2)^3(k+3)^3}=\pi^2-\frac{6217}{630}.$$

Proposed by Albert Stadler, Herrliberg, Switzerland

Solution by Li Zhou, Polk State College, USA

By partial fraction decomposition,

$$\begin{aligned}\frac{1}{[k(k+1)(k+2)(k+3)]^3} &= \frac{103}{1296}\left[\frac{1}{k}-\frac{1}{k+3}\right]-\frac{11}{432}\left[\frac{1}{k^2}+\frac{1}{(k+3)^2}\right]\\ &+\frac{1}{216}\left[\frac{1}{k^3}-\frac{1}{(k+3)^3}\right]-\frac{9}{16}\left[\frac{1}{k+1}-\frac{1}{k+2}\right]\\ &+\frac{3}{16}\left[\frac{1}{(k+1)^2}+\frac{1}{(k+2)^2}\right]-\frac{1}{8}\left[\frac{1}{(k+1)^3}-\frac{1}{(k+2)^3}\right].\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{k^3(k+1)^3(k+2)^3(k+3)^3} &= \frac{103}{1296}\left[1+\frac{1}{2}+\frac{1}{3}\right]-\frac{11}{432}\left[\frac{\pi^2}{3}-1-\frac{1}{2^2}-\frac{1}{3^2}\right]\\ &+\frac{1}{216}\left[1+\frac{1}{2^3}+\frac{1}{3^3}\right]-\frac{9}{16}\left[\frac{1}{2}\right]\\ &+\frac{3}{16}\left[\frac{\pi^2}{3}-1-1-\frac{1}{2^2}\right]-\frac{1}{8}\left[\frac{1}{2^3}\right]=\frac{35}{648}\left[\pi^2-\frac{6217}{630}\right].\end{aligned}$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Yong Xi Wang, Affiliated High School of Shanxi University; Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA, USA; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; Joel Schlosberg, Bayside, NY, USA; Jorge Ledesma, Faculty of Sciences UNAM, Mexico City, Mexico; Moubinool Omarjee Lycée Henri IV, Paris, France; Nermin Hodzic, Dobosnica, Bosnia and Herzegovina.

U370. Evaluate

$$\lim_{x\to 0}\frac{\sqrt{1+2x}\cdot\sqrt[3]{1+3x}\cdots\sqrt[n]{1+nx}-1}{x}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Alessandro Ventullo, Milan, Italy

Observe that for any $k=2,\ldots,n$ and for any x such that $|x|<1$, we have

$$\sqrt[k]{1+kx}=1+\frac{1}{k}kx+o(x^2)=1+x+o(x^2).$$

Hence,

$$\begin{aligned}\sqrt{1+2x}\cdot\sqrt[3]{1+3x}\cdot\ldots\cdot\sqrt[n]{1+nx}-1 &= (1+x+o(x^2))^{n-1}-1\\ &= (1+(n-1)x+o(x^2))-1\\ &= (n-1)x+o(x^2).\end{aligned}$$

Therefore,

$$\lim_{x\to 0}\frac{\sqrt{1+2x}\cdot\sqrt[3]{1+3x}\cdot\ldots\cdot\sqrt[n]{1+nx}-1}{x}=n-1.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Yong Xi Wang, Affiliated High School of Shanxi University; Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA, USA; Hyun Min Victoria Woo, Northfield Mount Hermon School, Mount Hermon, MA, USA; Ji Eun Kim, Tabor Academy, Marion, MA, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Vicente Vicario García, Sevilla, Spain; Li Zhou, Polk State College, Winter Haven, FL, USA; Kosmidis Dimitrios, National and Kapodistrian University of Athens, Greece; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Ángel Plaza, Department of Mathematics, University of Las Palmas de Gran Canaria, Spain; Arkady Alt, San Jose, CA, USA; Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; David Stoner, Harvard University, Cambridge, MA, USA; Henry Ricardo, New York Math Circle, Tappan, NY, USA; Joel Schlosberg, Bayside, NY, USA; Moubinool Omarjee Lycée Henri IV, Paris, France; Nermin Hodzic, Dobosnica, Bosnia and Herzegovina; Albert Stadler, Herrliberg, Switzerland.

U371. Let n be a positive integer, and let $A_n = (a_{ij})$ be the $n \times n$ matrix where $a_{ij} = x^{(i+j-2)^2}$, x being a variable. Evaluate the determinant of A_n.

Proposed by Mehtaab Sawhney, Commack High School, New York, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Denote by $D(n)$ the $n \times n$ determinant, and simplify its calculation through the following transformation (which clearly leaves the value of the determinant unchanged):

Step 1 For $i = n, n-1, \ldots, 2$ and in this order, subtract $x^{2i-3} = x^{(i-1)^2-(i-2)^2}$ times row $i-1$ from row i. Therefore, in position (i,j) for $i \geq 2$, we have $x^{(i+j-2)^2} - x^{(i+j-3)^2+2i-3} = x^{(i+j-3)^2+2i-3}\left(x^{2(j-1)}-1\right)$. For $j = 1$, this means that all elements of the first column are zero except for $(1,1)$, which is 1.

Step 2 Subtract $x^{(j-1)^2}$ times the resulting first column from column j for $j = 2, 3, \ldots, n$, or all elements in the first row and the first column are zero except for $(1,1)$, which is 1. Moreover, all elements such that $i \geq 2$ remain unchanged in this step, or their values are as after Step 1.

Step 3 Extract from column j a factor $\left(x^{2(j-1)}-1\right)$, or a prefactor of $\left(x^2-1\right)\left(x^4-1\right)\cdots\left(x^{2(n-1)}-1\right)$ is added in front of the determinant, while position (i,j) for $i,j \geq 2$ is now occupied by $x^{(i+j-3)^2+2i-3} = x^{(i+j-4)^2+2(2i+j-5)}$.

Step 4 Extract from column j a factor $x^{2(j-1)}$ and from row i a factor $x^{4(i-2)}$ for $i,j \geq 2$, or position (i,j) for $i,j \geq 2$ is now occupied by the same term as position $(i-1,j-1)$ in the $(n-1)\times(n-1)$ determinant, and a prefactor of $x^{4(0+1+\cdots+n-2)+2(1+2+\cdots+n-1)} = x^{(n-1)(3n-4)}$ is added in front of the determinant. All the first row and all the first column are still occupied by zeros, except for position $(1,1)$ which is occupied by 1.

From the previous transformation, it is clear that

$$D(n) = x^{(n-1)(3n-4)} \prod_{i=1}^{n-1}\left(x^{2i}-1\right) D(n-1)$$

Trivially, $D(1) = x^{(1+1-2)^2} = 1$, or after a simple inductive process, we conclude that

$$D(n) = x^{n(n-1)^2} \prod_{i=1}^{n-1}\left(x^{2i}-1\right)^{n-i}$$

Also solved by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Li Zhou, Polk State College, Winter Haven, FL, USA; Kosmidis Dimitrios, National and Kapodistrian University of Athens, Greece; Problem Solving Group, Department of Financial Engineering, University of the Aegean, Mitilini, Greece; Nermin Hodzic, Dobosnica, Bosnia and Herzegovina.

U372. Let $\alpha, \beta > 0$ be real numbers, and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x) \neq 0$, for all x in a neighborhood U of 0. Evaluate

$$\lim_{x \to 0^+} \frac{\int_0^{\alpha x} t^\alpha f(t) dt}{\int_0^{\beta x} t^\beta f(t) dt}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Alessandro Ventullo, Milan, Italy

Observe that $\lim_{x\to 0+} \int_0^{\alpha x} t^\alpha f(t)\ dt = \lim_{x\to 0^+} \int_0^{\beta x} t^\beta f(t)\ dt = 0$. Let $g(t) = t^\alpha f(t)$ and $h(t) = t^\beta f(t)$. We have

$$\frac{d}{dx} \int_0^{\alpha x} g(t)\ dt = g(\alpha x)\alpha = (\alpha x)^\alpha f(\alpha x)\alpha$$

and

$$\frac{d}{dx} \int_0^{\beta x} h(t)\ dt = h(\beta x)\beta = (\beta x)^\beta f(\beta x)\beta.$$

Clearly, $\dfrac{d}{dx} \int_0^{\beta x} h(t)\ dt \neq 0$ for all x in a neighborhood U of 0 $(x \neq 0)$. We have

$$\lim_{x\to 0^+} \frac{\frac{d}{dx} \int_0^{\alpha x} t^\alpha f(t)\ dt}{\frac{d}{dx} \int_0^{\beta x} t^\beta f(t)\ dt} = \begin{cases} 0 & \text{if } \alpha > \beta \\ 1 & \text{if } \alpha = \beta \\ +\infty & \text{if } \alpha < \beta. \end{cases}$$

By L'Hôpital's Rule we conclude that

$$\lim_{x\to 0^+} \frac{\int_0^{\alpha x} t^\alpha f(t)\ dt}{\int_0^{\beta x} t^\beta f(t)\ dt} = \begin{cases} 0 & \text{if } \alpha > \beta \\ 1 & \text{if } \alpha = \beta \\ +\infty & \text{if } \alpha < \beta. \end{cases}$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Vicente Vicario García, Sevilla, Spain; Li Zhou, Polk State College, Winter Haven, FL, USA; Kosmidis Dimitrios, National and Kapodistrian University of Athens, Greece; Ángel Plaza, Department of Mathematics, University of Las Palmas de Gran Canaria, Spain; Henry Ricardo, New York Math Circle, Tappan, NY, USA; Moubinool Omarjee Lycée Henri IV, Paris, France; Nermin Hodzic, Dobosnica, Bosnia and Herzegovina.

O367. Prove that for any positive integer $a > 81$ there are positive integers x, y, z such that

$$a = \frac{x^3 + y^3}{z^3 + a^3}.$$

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Solution by Li Zhou, Polk State College, USA

We show that the claim is true for all integers $a \geq 82$, $4 \leq a \leq 8$, and $25 \leq a \leq 71$. First, if $x = 3n(a - 3n^3)$, $y = 9n^4$, and $z = 9n^3 - a$, then

$$\frac{x^3 + y^3}{z^3 + a^3} = \frac{(3n)^3 \left[(a - 3n^3)^3 + (3n^3)^3\right]}{(9n^3 - a)^3 + a^3} = a.$$

Thus, it remains only to show that for each a above, there exists positive integer n such that $a - 3n^3 > 0$ and $9n^3 - a > 0$, that is, $\sqrt[3]{\frac{a}{9}} < n < \sqrt[3]{\frac{a}{3}}$. Indeed, for $4 \leq a \leq 8$, $n = 1$ works; and for $25 \leq a \leq 71$, $n = 2$ works. Finally, if $a \geq 82$, then there exists $k \geq 3$ such that $3k^3 < a \leq 3(k+1)^3$. Take $n = k$. Clearly, $n < \sqrt[3]{\frac{a}{3}}$. Moreover, $3k^3 - (k+1)^3 = 2k^2(k-3) + 3k(k-1) - 1 > 0$, that is, $\sqrt[3]{\frac{a}{9}} \leq \sqrt[3]{\frac{(k+1)^3}{3}} < k = n$, completing the proof.

Also solved by Moubinool Omarjee Lycée Henri IV, Paris, France; Albert Stadler, Herrliberg, Switzerland.

O368. Let a, b, c, d, e, f be real numbers such that $a+b+c+d+e+f = 15$ and $a^2+b^2+c^2+d^2+e^2+f^2 = 45$. Prove that $abcdef \leq 160$.

Proposed by Marius Stanean, Zalau, Romania

First solution by Evgenidis Nikolaos, M.N. Raptou High School, Larissa, Greece

We are going to use the Lagrange Multipliers method to solve the problem. Let $g_1 = a+b+c+d+e+f-15$ and $g_2 = a^2+b^2+c^2+d^2+e^2+f^2-45$. Then Lagrange's function is

$$F = abcdef - \lambda(a+b+c+d+e+f-15) - \mu(a^2+b^2+c^2+d^2+e^2+f^2-45).$$

For the first partial derivatives of the function we have

$$F'_a = bcdef - \lambda - 2\mu a,$$

$$F'_b = acdef - \lambda - 2\mu b,$$

$$F'_c = abdef - \lambda - 2\mu c,$$

$$F'_d = abcef - \lambda - 2\mu d,$$

$$F'_e = abcdf - \lambda - 2\mu e,$$

$$F'_f = abcde - \lambda - 2\mu f.$$

Since we want every partial derivative to be equal to zero, the above relations give

$$(b-a)(cdef + 2\mu) = 0$$

$$(c-a)(bdef + 2\mu) = 0$$

$$(d-a)(bcef + 2\mu) = 0$$

$$(e-a)(bcdf + 2\mu) = 0$$

$$(f-a)(bcde + 2\mu) = 0.$$

Now we have to examine separately some cases.

- *Case 1: all variables are equal ($a = b = c = d = e = f$).*
  This case is easily rejected since the given hypothesis cannot be satisfied simultaneously.

- *Case 2: 5 variables are equal, say $a = b = c = d = e$. Then $2\mu = -bcde$.*
  The given hypothesis give $\begin{cases} 5a + f = 15 \\ 5a^2 + f^2 = 45 \end{cases}$. This system of equations has two solution $a = 3, f = 0$ and $a = 2, f = 5$ for which values we find $F = 0$ and $F = 160$, respectively.

- *Case 3: 4 variables are equal, say $a = b = c = d$. Then $bcdf = bcde \Leftrightarrow e = f$.*
  Now we have to solve the following system of equations $\begin{cases} 4a + 2f = 15 \\ 4a^2 + 2f^2 = 45 \end{cases}$. The system has the following pair of solutions $a = \frac{15+2\sqrt{5}}{6}, b = \frac{15-4\sqrt{5}}{6}$ and $a = \frac{15-2\sqrt{5}}{6}, b = \frac{15+4\sqrt{5}}{6}$. Plugging in these values, we can find that $F < 160$ in both cases.

- *Case 4: 3 variables are equal, say $a = b = c$. Then $bcef = bcdf = bcde \Leftrightarrow d = e = f$.*
  As before, we have to solve the following system of equations that are deduced from the initial conditions of the problem $\begin{cases} 3a + 3f = 15 \\ 3a^2 + 3f^2 = 45 \end{cases}$. The solutions are $a = \frac{5+\sqrt{5}}{2}, b = \frac{5-\sqrt{5}}{2}$ and $a = \frac{5-\sqrt{5}}{2}, b = \frac{5+\sqrt{5}}{2}$.
  For these values it is $F = 125$.

Hence, we have proved that the maximum value of F is $F = 160$ or equivalently $abcdef \leq 160$, as we wished to prove.

Second solution by the author
From Cauchy-Schwarz's inequality, we have

$$5(45-f^2)=5(a^2+b^2+c^2+d^2+e^2)\geq(a+b+c+d+e)^2=(15-f)^2\iff$$

$$225-5f^2\geq 225-30f+f^2\iff f(f-5)\leq 0\iff 0\leq f\leq 5,$$

and similarly we obtain that $a,b,c,d,e\in[0,5]$.

The inequality is symmetric so we can assume, without losing generality of problem, that $0\leq a\leq b\leq c\leq d\leq e\leq f\leq 5$ and let $m=a+b$, $n=c+d$, $p=e+f\implies 0\leq m\leq n\leq p$ and $m+n+p=15$.

We denote $x=ab$, $y=cd$ and $z=ef\implies 0\leq x\leq y\leq z$.

Now we fix m, n, p (as constants) and let x,y,z vary (as variables) such that $x\leq\frac{m^2}{4}$, $y\leq\frac{n^2}{4}$, $z\leq\frac{p^2}{4}$ and $x+y+z=\frac{m^2+n^2+p^2-45}{2}$. In this case we must find maximum of xyz. The problem can be divided into four cases:

1. $\frac{m^2}{4}\leq\frac{n^2}{4}\leq\frac{p^2}{4}\leq\frac{m^2+n^2+p^2-45}{6}=\frac{x+y+z}{3}$ which means $x=y=z\implies a=b=c=d=e=f$ what can't happen.

2. $\frac{m^2}{4}\leq\frac{m^2+n^2+p^2-45}{6}\leq\frac{p^2}{4}$ such that $\frac{m^2}{4}+\frac{2n^2}{4}\leq\frac{m^2+n^2+p^2-45}{2}\iff m^2+2p^2\geq 90$.
I will prove the following

Lemma 1. If $0\leq x\leq y\leq z$ such that $x+y+z=s$ and $x\leq a\leq\frac{s}{3}$, $y\leq b$ and $a+2b\leq s$ then the maximum of xyz is obtained for $x=a$, $y=b$ and $z=s-a-b$.

Proof. Indeed we have from AM-GM inequality

$$a^2b^2(s-a-b)^2xyz=b(s-a-b)xa(s-a-b)yabz\leq\left(\frac{b(s-a-b)x+a(s-a-b)y+abz}{3}\right)^3=$$

$$\frac{[b(s-2a-b)x+a(s-a-2b)y+ab(x+y+z)]^3}{27}\leq\frac{[b(s-2a+b)a+a(s-a-2b)b+abs]^3}{27}=$$

$$a^3b^3(s-a-b)^3\implies xyz\leq ab(s-a-b).$$

□

According to lemma we have $xyz\leq\frac{m^2n^2(m^2+n^2+2p^2-90)}{64}$.
Denote $f(m,n,p)=m^2n^2(m^2+n^2+2p^2-90)$ and I will prove that

$$f(m,n,p)\leq f\left(\frac{m+n}{2},\frac{m+n}{2},p\right)\iff$$

$$\frac{(m+n)^6}{32}-m^2n^2(m^2+n^2)+(2p^2-90)\left(\frac{(m+n)^4}{16}-m^2n^2\right)\geq 0\iff$$

$$\frac{(m-n)^2}{16}\left[\frac{m^4+8m^3n-2m^2n^2+8mn^3+n^4}{2}+(2p^2-90)(m^2+6mn+n^2)\right]\geq 0\iff$$

$$\frac{(m-n)^2}{16}\left[\frac{(m^2+n^2)(m^2+6mn+n^2)+2mn(m-n)^2}{2}+(2p^2-90)(m^2+6mn+n^2)\right]\geq 0$$

which is true because

$$LHS\geq\frac{(m-n)^2}{16}\left[\frac{(m^2+n^2)(m^2+6mn+n^2)}{2}+(2p^2-90)(m^2+6mn+n^2)\right]\geq$$

$$\frac{(m-n)^2(m^2+6mn+n^2)}{16}\left[\frac{m^2+n^2}{2}+2p^2-90\right]\geq$$

$$\frac{(m-n)^2(m^2+6mn+n^2)(m^2+2p^2-90)}{16}\geq 0.$$

Further we must show that $f(m,m,p) \leq 4^5 \cdot 10$ where $2m + p = 15$ and $m^2 + 2p^2 - 90 \geq 0$. So $3m \leq 2m + p = 15 \Longrightarrow m \leq 5$ and $m^2 + 2(15-2m)^2 - 90 \geq 0 \Longleftrightarrow 3m^2 - 40m + 120 \geq 0 \Longleftrightarrow m \in \left[0, \frac{20-2\sqrt{10}}{3}\right] \cup \left[\frac{20+2\sqrt{10}}{3}, \infty\right]$. Therefore $m \in \left[0, \frac{20-2\sqrt{10}}{3}\right]$ and then we have to show that

$$m^4(2m^2 + 2(15-2m)^2 - 90) \leq 4^5 \cdot 10 \Longleftrightarrow$$

$$\left[m^2(m-6) - 32\right](m-4)^2(m+2) \leq 0$$

which is obviously true, with the equality when $m = n = 4 (x = y = 4) \Longrightarrow a = b = c = d = 2$ and $p = 7 \Longrightarrow e = 2, f = 5$.

3. $\frac{m^2}{4} \leq \frac{m^2+n^2+p^2-45}{6} \leq \frac{p^2}{4}$ such that $\frac{m^2}{4} + \frac{2n^2}{4} \geq \frac{m^2+n^2+p^2-45}{2} \Longleftrightarrow m^2 + 2p^2 \leq 90$.

I will prove the following

Lemma 2. If $0 \leq x \leq y \leq z$ such that $x + y + z = s$ and $x \leq a \leq \frac{s}{3}$, $y \leq b$ and $a + 2b \geq s$ then the maximum of xyz is obtained for $x = a$ and $y = z = \frac{s-a}{2}$.

Proof. In this case we have

$$xyz \leq x\frac{(y+z)^2}{4} = x\frac{(s-x)^2}{4}.$$

We define the function $f : [0, a] \mapsto \mathbb{R}$, $f(x) = x(s-x)^2$. We have $f'(x) = (s-x)(s-3x) \geq 0$ for $x \in [0, a]$. Therefore the maximum of f is obtained for $x = a$ and $y = z = \frac{s-a}{2} \leq b$. □

According to lemma we have that the maximum of xyz is obtained for $x = \frac{m^2}{4}$ and $y = z = \frac{m^2+2n^2+2p^2-90}{8}$. But from $y = z$ it follows that $c = d = e = f$ so $n = p$ and then remains to show that

$$m^2(m^2 + 4p^2 - 90)^2 \leq 4^6 \cdot 10$$

where $m + 2p = 15$ and $m^2 + 2p^2 \leq 90$. So $3m \leq 2m + p = 15 \Longrightarrow m \leq 5$ and $2m^2 + (15-m)^2 - 180 \leq 0 \Longleftrightarrow m^2 - 10m + 15 \geq 0 \Longleftrightarrow m \in \left[5 - \sqrt{10}, 5 + \sqrt{10}\right]$. Therefore $m \in \left[5 - \sqrt{10}, 5\right]$ and then we have to show that

$$m^2(m^2 + (15-m)^2 - 90)^2 \leq 4^6 \cdot 10 \Longleftrightarrow$$

$$m(2m^2 - 30m + 135) \leq 64\sqrt{10}.$$

Let the function $g : [5 - \sqrt{10}, 5] \mapsto \mathbb{R}$, $g(m) = m(2m^2 - 30m + 135)$. We have $g'(m) = 6m^2 - 60m + 135 = 6\left(m - 5 + \frac{\sqrt{10}}{2}\right)\left(m - 5 - \frac{\sqrt{10}}{2}\right)$, so the maximum value of g is for $m_0 = 5 - \frac{\sqrt{10}}{2}$, but $g(m_0) = 175 + 5\sqrt{10} < 64\sqrt{10} \Longleftrightarrow 175 < 59\sqrt{10}$ which is true.

4. $\frac{m^2+n^2+p^2-45}{6} \leq \frac{m^2}{4}$, but this case can not be held because this is equivalent with

$$m^2 + 90 \geq 2n^2 + 2p^2 \Longleftrightarrow$$

$$a^2 + 2ab + b^2 + 90 \geq 2c^2 + 4cd + 2d^2 + 2e^2 + 4ef + 2f^2 \Longleftrightarrow$$

$$a^2 + b^2 + 2ab + 90 \geq 2(45 - a^2 - b^2) + 4cd + 4ef \Longleftrightarrow$$

$$3(a^2 + b^2) + 2ab \geq 4cd + 4ef$$

and this is true only if $a = b = c = d = e = f$, what can't happen, because $3a^2 \leq 3cd$, $3b^2 \leq 3ef$ and $2ab \leq cd + ef$.

Also solved by Moubinool Omarjee Lycée Henri IV, Paris, France.

O369. Let $a, b, c > 0$. Prove that

$$\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b} \geq \sqrt{3(a^2+b^2+c^2)}.$$

Proposed by An Zhen-Ping, Xianyang Normal University, China

Solution by Arkady Alt, San Jose, CA, USA

Let $p := ab+bc+ca, q := abc$ and let $a+b+c=1$(due to homogeneity).

Then

$$\sum_{cyc}\frac{a^2+bc}{b+c}=\frac{1}{p-q}\sum_{cyc}\left(a^2+bc\right)(1-b)(1-c)=\frac{1}{p-q}\sum_{cyc}\left(a^2+bc\right)(a+bc)=$$

$$\frac{1}{p-q}\left(a^3+b^3+c^3+3abc+abc(a+b+c)+a^2b^2+b^2c^2+c^2a^2\right)=$$

$$\frac{1+3q-3p+3q+q+p^2-2q}{p-q}=\frac{1+5q-3p+p^2}{p-q}$$

and inequality becomes

$$\frac{1-3p+p^2+5q}{p-q} \geq \sqrt{3(1-2p)}.$$

Also note that $0 < p \leq \dfrac{1}{3}$ (because $ab+bc+ca \leq \dfrac{(a+b+c)^2}{3}$) and $q \geq \dfrac{(1-p)(4p-1)}{6}$

(Schure Inequality $\sum_{cyc} a^2(a-b)(a-c) \geq 0$ in 1-p-q notation).

Since $\dfrac{1-3p+p^2+5q}{p-q}$ increasing in $q>0$ and $q \geq \max\left\{0, \dfrac{(1-p)(4p-1)}{6}\right\}$ then

$$\frac{1-3p+p^2+5q}{p-q} \geq \frac{1-3p+p^2+5\cdot\dfrac{(1-p)(4p-1)}{6}}{p-\dfrac{(1-p)(4p-1)}{6}}=\frac{1+7p-14p^2}{4p^2+p+1}$$

and suffices to prove inequalities:

$$\frac{1+7p-14p^2}{4p^2+p+1} \geq \sqrt{3(1-2p)} \iff \left(1+7p-14p^2\right)^2 \geq 3(1-2p)\left(4p^2+p+1\right)^2$$

for $p \in [1/4, 1/3]$

and

$$\frac{1-3p+p^2}{p} \geq \sqrt{3(1-2p)} \text{ for } p \in (0, 1/4].$$

For $p \in [1/4, 1/3]$ we have $\left(1+7p-14p^2\right)^2 - 3(1-2p)\left(4p^2+p+1\right)^2 =$

$$2\left(3p-1\right)\left(2p^2+5p-1\right)\left(8p^2-p-1\right) = 2\left(1-3p\right)\left(2p^2+5p-1\right)\left(1+p-8p^2\right) \geq 0$$

because

$$2p^2+5p-1 \geq 2\cdot\left(\frac{1}{4}\right)^2 + 5\cdot\frac{1}{4} - 1 = \frac{3}{8}$$

and for $p \in (0, 1/4]$ we have $\left(1+7p-14p^2\right)^2 - 3p^2\left(1-2p\right) = 196p^4 - 190p^3 + 18p^2 + 14p + 1 =$
$196p^4 - 190p^3 + 18p^2 + 14p + 1 = 1 - 4p + 18p\left(1-4p\right) + 90p^2\left(1-4p\right) + 170p^3 + 196p^4 > 0.$

Also solved by Yong Xi Wang, Affiliated High School of Shanxi University, Shanxi, China; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; WSA; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Ahn, Young Seob, Seoul, Korea; David Stoner, Harvard University, Cambridge, MA, USA; Duy Quan Tran, University of Health Science, Ho Chi Minh City, Vietnam; Khurshid Juraev, Academic lyceum named after S.H.Sirojiddinov, Tashkent, Uzbekistan; Bobojonova Latofat, Academic lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Moubinool Omarjee Lycée Henri IV, Paris, France; Nermin Hodzic, Dobosnica, Bosnia and Herzegovina; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Evgenidis Nikolaos, M.N. Raptou High School, Larissa, Greece; Paul Revenant, Lycée Champollion, Grenoble, France.

O370. For any positive integer n we denote by $S(n)$ the sum of digits of n. Prove that for any integer n such that $\gcd(3,n)=1$ and any integer k, $k > S(n)^2 + 7S(n) - 9$, there exists an integer m such that $n|m$ and $S(m)=k$.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Solution by the author

Without loss of generality we can suppose that $(n,10)=1$. Let $\mathcal{M} = \left\{S(m)|m \vdots n\right\}$, where $S(m)$denotes the sum of the m number's digits. Let's now prove the following properties.

Property 1. If $a;b \in \mathcal{M}$, then $a+b \in \mathcal{M}$. Indeed, let $a = S(A)$, $b = S(B)$ and A, $B \vdots n$, then $10^k A + B \vdots m$ and $S(10^k A + B) = S(A) + S(B) = a + b \in \mathcal{M}$, where k is the number of digits in B.

Corollary 1. If $a,b \in \mathcal{M}$ and x,y are non negatived integers, then $ax+by \in \mathcal{M}$. it is evident that $S(n) \in \mathcal{M}$.

Property 2. If $(n,10)=1$, then $S(n)+9 \in \mathcal{M}$. Indeed, for two numbers in 1, 10, 10^2, ..., 10^n with the same remainder after division by n, their difference $10^i - 10^j = 10^j(10^{i-j}-1)$ will contain n, from which we find that $10^{i-j} - 1 \vdots n (i>j)$, so there exists a natural number k such that $10^k - 1 \vdots n$.

Now let $n = \overline{a_l a_{l-1} ... a_1}$. Consider the following number

$$m = \sum_{a_i \in A} 10^{i-1} \cdot 10^{a_1 + ... + ai-1+1)k} + ... + 10^{a_1+a_2+...+ai-1+a_i)k})$$

$$+10^{l-1} \cdot (10^{a_1+...+ai-1+1)k} + ... + 10^{a_1+a_2+...+al-1+a_l-1)k})$$

$$+10^{l-2} \cdot (10^{a_1+a_2+...+al+1)k} + ... + 10^{a_1+a_2+...+al-1+a_l+10)k}),$$

where A are nonzero digits from $a_1 a_2 ... a a_{l-1}$. It is easy to see that $S(m) = a_1 + a_2 + ... + a_{l-1} + a_l - 1 + 10 = S(n) + 9$ and

$$m \equiv a_1 + 10a_2 + ... + 10^{l-2} a_{l-1} + 10^{l-1}(a_l - 1) + 10^{l-2} \cdot 10 \pmod{n}$$

hence $m \vdots n$. So $S(n)+9 \in \mathcal{M}$.

Lemma. If $a,b \in \mathcal{N}$ and $(a,b)=1$, then any number $c > ab - a - b$ can be represented as $ax+by$, where x and y are non negative integers.

Indeed, since $(a,b)=1$, then there exist integer numbers u and v such that $au+bv=1$. Without loss of generality we can suppose that $0 < u \le b$. Indeed, suppose that $pb < u \le (p+1)b$, then in that case $a(u-pb) + b(v+pa) = 1$.

Now, let $c = ab - a - b + d$, where $d > 0$, then from the above proven it follows that d may be represented in the form $ax_1 + by_1$, where $0 < x_1 \le b$, consequently $c = a(x_1 - 1) + b(a + y_1 - 1)$. If $a + y_1 - 1 < 0$, then $d = ax_1 + by_1 \le ax_1 + b(-a) \le ab - ab = 0$, which is impossible.

The lemma is proved.

Since $(n,3)=1$, then $(S(n),3)=1$ and $(S(n), S(n)+9)=1$, consequently by the corollary, properties and lemma for any

$$k > S(n)(S(n)+9 - S(n) - (S(n)+9) = S^2(n) + 7S(n) - 9.$$

O371. Let $ABC(AB \neq BC)$ be a triangle, and let D and E be the feet of altitudes from B and C, respectively. Denote by M, N, P, the midpoints of BC, MD, ME, respectively. If $\{S\} = NP \cap BC$ and T is the point of intersection of DE with the line through A, which is parallel to BC, prove that ST is tangent to the circumcircle of triangle ADE.

Proposed by Marius Stănean, Zalău, România

Solution by Adnan Ali, Student in A.E.C.S-4, Mumbai, India

It is well-known that MD and ME are tangents to $\odot(ADE)$. It is also clearly seen that the line through A, parallel to BC is also tangent to $\odot(ADE)$ at A. Let $\odot(ADE) \cap AM = K \neq A$. Then because T lies on the polar of M w.r.t $\odot(ADE)$ and TA is tangent to $\odot(ADE)$ at A, by La Hire's Theorem, we conclude that TK is tangent to $\odot(ADE)$ at K.

Thus, it is sufficient to prove that SK is tangent to $\odot(ADE)$ at K. Denote by H, the orthocenter of $\triangle ABC$ and let $DE \cap BC = F$. Then we claim that H is the orthocenter of $\triangle AMF$. The proof of our claim is as follows:

Proof: Consider the cyclic quadrilateral $BEDC$. Then from Brocard's Theorem, M, the center of $\odot(BEDC)$ is the orthocenter of $\triangle HAF$. So, $MH \perp AF$ and we also know by definition that $AH \perp MF$. Thus, H is the orthocenter of $\triangle AMF$, completing the proof of our claim.

Since AH is the diameter of $\odot(AEHD)$, $\angle HKA = 90^\circ$ and so F, H, K are collinear. Also from the converse of midpoint theorem, S is the midpoint of MF. Moreover $\angle FKA = 90^\circ \Rightarrow \triangle FKM$ is right-angled at K. So S is the circumcenter of $\triangle FKM$ and we have $\angle SKF = \angle SFK = \angle MFH$. But we also have $\angle MAH = 90^\circ - \angle LMA = 90^\circ - \angle FMK = \angle MFK = \angle MFH$, where $L = AH \cap BC$. Thus, $\angle SKF = \angle SFK = \angle MFH = \angle MAH = \angle KAH$. Denote by G, the midpoint of AH. Then $\angle KAH = \angle GAK = \angle GKA \Rightarrow \angle GKA = \angle SKF$. This implies that $\angle GKS = \angle AKF$, and because $\angle AKF = 90^\circ$, we conclude that SK is tangent to $\odot(ADE)$ at K, completing our proof.

Also solved by David Stoner, Harvard University, Cambridge, MA, USA; Khurshid Juraev, Academic lyceum named after S.H.Sirojiddinov, Tashkent, Uzbekistan; Bobojonova Latofat, Academic lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Evgenidis Nikolaos, M.N. Raptou High School, Larissa, Greece.

O372. A regular n-gon Γ_b of side b is drawn inside a regular n-gon Γ_a of side a such that the center of the circumcircle of Γ_a is not inside Γ_b. Prove that

$$b < \frac{a}{2\cos^2\frac{\pi}{2n}}.$$

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Solution by Evgenidis Nikolaos, M.N. Raptou High School, Larissa, Greece

It is well known that $2\cos^2\frac{\pi}{2n} = \cos\frac{\pi}{n} + 1$.

Also by the Law of Sines in any triangle $A_iA_jA_k$ and $B_iB_jB_k$ it is obvious that $a = 2R_a\sin\frac{\pi}{n}$ and $b = 2R_b\sin\frac{\pi}{n}$, where R_a, R_b are the circumradius of Γ_a, Γ_b, respectively and $A_i, A_j, A_k, B_i, B_j, B_k$ are vertices of the polygons with $1 \le i \ne j \ne k \le n$.

Then, the given inequality can be written equivalently in the following way

$$b < \frac{a}{2\cos^2\frac{\pi}{2n}} \Leftrightarrow 2R_b\sin\frac{\pi}{n} < \frac{2R_a\sin\frac{\pi}{n}}{\cos\frac{\pi}{n}+1} \Leftrightarrow R_b(\cos\frac{\pi}{n}+1) < R_a.$$

However, it is obvious that $R_b\cos\frac{\pi}{n}$ is the apothem, h_b, of regular n-gon, Γ_b. Hence, it suffices to prove that $R_a > R_b + h_b$. Then, we will prove that the circumcircle of Γ_b is inside or tangent to the circumcircle of Γ_a, since that would be equivalent to

$$R_a \ge R_b + O_aO_b (1),$$

and it is easy to see that $O_aO_b > h_b$, as O_a does not lie inside Γ_b, where O_a, O_b are the circumcenters of Γ_a, Γ_b respectively..

Assume for the sake of contradiction, that the two circumcircles intersect. Let K be a point (not a vertex) lying on a minor arc A_mA_l of Γ_a and inside the circumcircle of Γ_b, where m, l are consecutive vertices of the regular n-gon.

Then, it is $\angle A_mKA_l = \frac{\pi(n-1)}{n}$. Furthermore, Γ_a is convex and Γ_b lies inside it, so $\angle A_mKA_l > \angle B_iKB_j$ for any vertices B_i, B_j of Γ_b. But $\angle B_iKB_j \le \frac{\pi(n-1)}{n}$, since K lies inside the circumcircle of Γ_b, which is a contradiction since we should have $\angle A_mKA_l > \frac{\pi(n-1)}{n}$, obviously false.

Therefore, the two circumcircles do not intersect and the result follows from (1).

Also solved by Bobojonova Latofat, Academic lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Khurshid Juraev, Academic lyceum named after S.H.Sirojiddinov, Tashkent, Uzbekistan.