Junior Problems

J373. Let a, b, c be real numbers greater than -1. Prove that

$$\left(a^2+b^2+2\right)\left(b^2+c^2+2\right)\left(c^2+a^2+2\right) \geq (a+1)^2(b+1)^2(c+1)^2.$$

Proposed by Adrian Andreescu, Dallas, TX, USA

J374. Let a, b, c be positive real numbers such that $a+b+c \geq 3$. Prove that

$$abc+2 \geq \frac{9}{a^3+b^3+c^3}.$$

Proposed by Mehmet Berke, İşler, Denizli, Turkey

J375. Solve in real numbers the equation

$$\sqrt[3]{x}+\sqrt[3]{y}=\frac{1}{2}+\sqrt{x+y+\frac{1}{4}}.$$

Proposed by Adrian Andreescu, Dallas, TX, USA

J376. Let α, β, γ be the angles of a triangle. Prove that

$$\frac{1}{5-4\cos\alpha}+\frac{1}{5-4\cos\beta}+\frac{1}{5-4\cos\gamma} \geq 1.$$

Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia

J377. Let ABC be a triangle with $\angle A \leq 90°$. Prove that

$$\sin^2\frac{A}{2} \leq \frac{m_a}{2R} \leq \cos^2\frac{A}{2}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J378. Let P be a point in the interior of the triangle ABC such that $\angle BAP = 105°$, and let D, E, F be the intersection of BP, CP, DE with the side AC, AB, BC, respectively. Assume that the point B lies between C and F and that $\angle BAF = \angle CAP$. Find $\angle BAC$.

Proposed by Marius Stănean, Zalău, România

Senior Problems

S373. Let x, y, z be positive real numbers. Prove that

$$\sum_{cyc} \frac{1}{xy+2z^2} \le \frac{xy+yz+zx}{xyz(x+y+z)}.$$

Proposed by Tolibjon Ismoilov, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan

S374. Let a, b, c be positive real numbers. Prove that at least one of the numbers

$$\frac{a+b}{a+b-c}, \quad \frac{b+c}{b+c-a}, \quad \frac{c+a}{c+a-b}$$

is not in the interval (1, 2).

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S375. Let a, b, c be nonnegative real numbers such that $ab+bc+ca=a+b+c>0$. Prove that

$$a^2+b^2+c^2+5abc \ge 8.$$

Proposed by An Zhen-Ping, Xianyang Normal University, China

S376. Solve in integers the equation $x^5-2xy+y^5=2016$.

Proposed by Adrian Andreescu, Dallas, TX, USA

S377. If z is a complex number with $|z| \ge 1$, prove that

$$\frac{|2z-1|^5}{25\sqrt{5}} \ge \frac{|z-1|^4}{4}.$$

Proposed by Florin Stănescu, Găești, România

S378. In a triangle, let m_a, m_b, m_c be the lengths of the medians, w_a, w_b, w_c be the lengths of the angle bisectors, and r and R be the inradius and circumradius, respectively. Prove that

$$\frac{m_a}{w_a}+\frac{m_b}{w_b}+\frac{m_c}{w_c} \le \left(\sqrt{\frac{R}{r}}+\sqrt{\frac{r}{R}}\right)^2.$$

Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia

Undergraduate Problems

U373. Prove the following inequality holds for all positive integers $n \geq 2$,

$$\left(1+\frac{1}{1+2}\right)\left(1+\frac{1}{1+2+3}\right)\cdots\left(1+\frac{1}{1+2+\cdots+n}\right) < 3.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U374. Let p and q be complex numbers such that two of the zeros a, b, c of the polynomial $x^3 + 3px^2 + 3qx + 3pq = 0$ are equal. Evaluate $a^2b + b^2c + c^2a$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U375. Let

$$a_n = \sum_{k=1}^{n} \sqrt[k]{\frac{\left(k^2+1\right)^2}{k^4+k^2+1}}, \quad n = 1,\ 2,\ 3,\ \ldots.$$

Determine $\lfloor a_n \rfloor$ and evaluate $\lim_{n\to\infty} \frac{a_n}{n}$.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U376. Evaluate

$$\lim_{n\to\infty}\left(1+\sin\frac{1}{n+1}\right)\left(1+\sin\frac{1}{n+2}\right)\cdots\left(1+\sin\frac{1}{n+n}\right).$$

Proposed by Marius Cavachi, Constanța, România

U377. Let m and n be positive integers and let

$$f_k(x) = \underbrace{\sin(\sin(\cdots(\sin x)\cdots))}_{k \text{ times}}.$$

Evaluate

$$\lim_{x\to 0}\frac{f_m(x)}{f_n(x)}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U378. Let $f : [0,\, 1] \to \mathbb{R}$ be a continuous function. Prove that

$$\frac{(-1)^{n-1}}{(n-1)!}\int_0^1 f(x)\ln^{n-1} x\, dx = \int_0^1\int_0^1\cdots\int_0^1 f(x_1x_2\cdots x_n)dx_1dx_2\cdots dx_n.$$

Proposed by Albert Stadler, Herrliberg, Switzerland

Olympiad Problems

O373. Let $n \geq 3$ be a natural number. On a $n \times n$ table we perform the following operation: choose a $(n-1) \times (n-1)$ square and add or subtract 1 to all its entries. At the beginning all the entries in the table are 0. Is it possible after a finite number of operations to obtain all the numbers from 1 to n^2 in the table?

Proposed by Alessandro Ventullo, Milan, Italy

O374. Prove that in any triangle,

$$\max\left(|A-B|,\ |B-C|,\ |C-A|\right) \leq \arccos\left(\frac{4r}{R}-1\right).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O375. Let a, b, c, d, e, f be real numbers such that $ad-bc=1$ and $e,\ f \geq \frac{1}{2}$. Prove that

$$\sqrt{e^2\left(a^2+b^2+c^2+d^2\right)+e\left(ac+bd\right)}+\sqrt{f^2\left(a^2+b^2+c^2+d^2\right)-f\left(ac+bd\right)} \geq (e+f)\sqrt{2}.$$

Proposed by Marius Stănean, Zalău, România

O376. Let $a_1,\ a_2,\ \ldots,\ a_{100}$ be a permutation of the numbers $1,\ 2,\ \ldots,\ 100$. Let $S_1=a_1$, $S_2=a_1+a_2$, $\ldots, S_{100}=a_1+a_2+\cdots+a_{100}$. Find the maximum possible number of perfect squares among the numbers $S_1,\ S_2,\ \ldots,\ S_{100}$.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

O377. Let $a_1,\ a_2,\ \ldots,\ a_n,\ b_1,\ b_2,\ \ldots,\ b_n$ be positive real numbers such that $a_ib_i>1$ for all $i \in \{1,\ 2,\ \ldots,\ n\}$. Denote

$$a=\frac{a_1+a_2+\ldots+a_n}{n} \qquad \text{and} \qquad b=\frac{b_1+b_2+\ldots+b_n}{n}.$$

Prove that

$$\frac{1}{\sqrt{a_1b_1-1}}+\frac{1}{\sqrt{a_2b_2-1}}+\cdots+\frac{1}{\sqrt{a_nb_n-1}} \geq \frac{n}{\sqrt{ab-1}}$$

Proposed by Marius Stănean, Zalău, România

O378. Consider a convex hexagon $ABCDEF$ such that $AB \parallel DE$, $BC \parallel EF$, and $CD \parallel FA$. Let M, N, K be the intersections of lines BD and AE, AC and DF, CE and BF, respectively. Prove that the perpendiculars from M, N, K to the lines AB, CD, EF respectively, are concurrent.

Proposed by Nairi Sedrakyan, Yerevan, Armenia