A Nice Theorem on Mixtilinear Incircles

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Abstract

There are three mixtilinear incircles and three mixtilinear excircles in an arbitrary triangle. In this paper, we will present many properties of mixtilinear incircles along with a famous theorem involving concyclic points and its proof.

A mixtilinear circle of a triangle is a circle tangent to two sides and the circumcircle of the triangle. If this circle is tangent to the circumcircle internally, it is called a mixtilinear incircle of the triangle. Otherwise, it is called a mixtilinear excircle of the triangle.

THEOREM. Given a triangle ABC, let ω, Ω, Ω_A, be the incircle, circumcircle, and mixtilinear incircle opposite point A, respectively. Let Ω_A be tangent to Ω at T_A. Suppose that P is a point on Ω such that the tangents from P to ω intersect BC at points X, Y. Then P, T_A, X, Y are concyclic.

Before the proof of the theorem, we need some theorems and properties about mixtilinear incircles from [1], [2], [3]. We will present them with proofs. Suppose that ω, Ω, Ω_A, are the incircle, circumcircle, and mixtilinear incircle opposite A of a triangle ABC and T_A is the mixtilinear point opposite A. Let I, O be the incenter and the circumcenter of the triangle ABC respectively, and let AI intersect Ω at points A and A_1. Let A_2 be the diametrically opposite point to A_1 on Ω. Analogously, define T_B, T_C, Ω_B, Ω_C and B_1, B_2, C_1, C_2.

Theorem 1. Let Ω_A be tangent to the sides AB, AC at points M, N, respectively. Then

(i) M, I, N are collinear;

(ii) A_2, I, T_A are collinear;

Proof. There is a homothety centered at T_A that sends Ω to Ω_A. (See *Picture 1*). So B_1, N, T_A and C_1, M, T_A are collinear and $MN \parallel B_1C_1$. Since AM, AN are tangent lines to Ω_A from A, we get that T_AA is the symmedian of triangle MT_AN. We have

$$\angle C_1T_AA = \frac{\angle ACB}{2} = \angle B_1T_AA_2$$

and

$$\angle B_1T_AA = \frac{\angle ABC}{2} = \angle C_1T_AA_2.$$

This shows that the line T_AA_2 is the isogonal conjugate of the line T_AA with respect to $\angle MT_AN$. Hence T_AA_2 is the median line of triangle MT_AN.

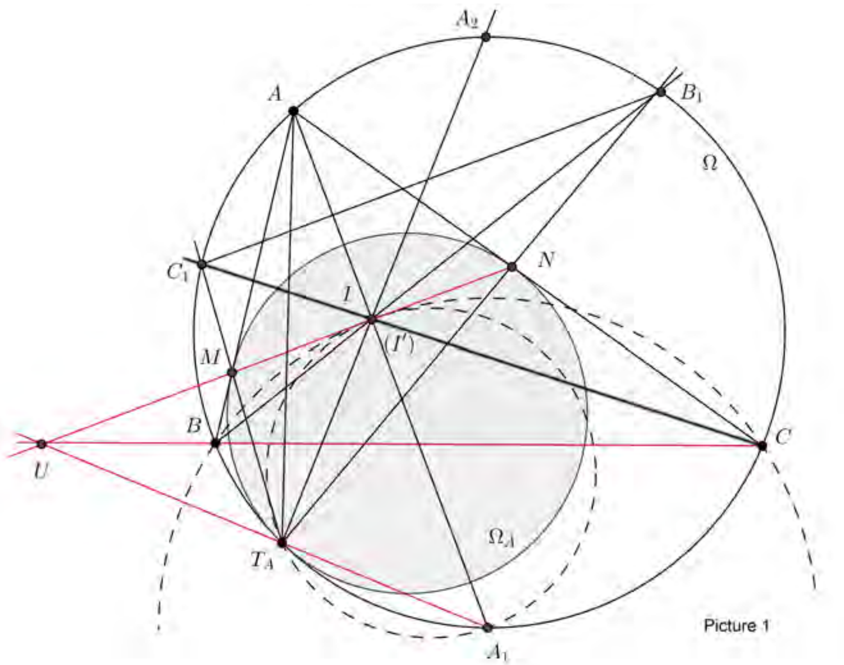
Let I' be the midpoint of MN. Then

$$\angle BT_AI' = \angle CT_AI' = 90^\circ - \frac{\angle BAC}{2} = \angle AMN = \angle ANM$$

and the quadrilaterals $BT_AI'M$ and $CT_AI'N$ are cyclic. So

$$\angle MBI' = \angle MT_AI' = \angle C_1T_AA_2 = \frac{\angle ABC}{2}, \qquad \angle NCI' = \angle NT_AI' = \angle B_1T_AA_2 = \frac{\angle ACB}{2}$$

and $I \equiv I'$. Hence M, I, N are collinear. Since T_A, I', A_2 are collinear, we get that the points T_A, I, A_2 are also collinear. This proves *Theorem 1.*



Picture 1

Theorem 2. Suppose that M, N are defined in the same way as the previous problem (tangency points of Ω_A to AB, AC). Then the lines MN, BC and T_AA_1 are concurrent.

Proof. We have three circles (BIC), (T_AIA_1) and Ω. From *Theorem 1* we get I is midpoint of MN, $IA_1 \perp MN$ and $IT_A \perp T_AA_1$. We can see that the circumcenter of (BIC) is the point A_1 and the circumcenter of (T_AIA_1) id the midpoint of IA_1. So, the circles (BIC) and (T_AIA_1) are tangents each to other at point I. Thus, MN is the radical axis of the circles (BIC) and (T_AIA_1). Moreover, BC is the radical axis of the circles (BIC), Ω and T_AA_1 is the radical axis of the circles Ω, (T_AIA_1). Hence the lines MN, BC and T_AA_1 are concurrent at point U (See *Picture 1*). This proves *Theorem 2.*

Proof of THEOREM. Consider an inversion with center I and radius r (the radius of the incircle of triangle ABC). Suppose that T_A inverts to the point K. Also, let D, E, F be the tangency points of the incircle with sides BC, CA, AB respectively. We easily get that $B_1C_1 \parallel EF$ and $AI \perp EF$, so $EF \parallel MN$. Moreover from *Theorem 1* and *Theorem 2*, we get that the lines MN, BC, T_AA_1 are concurrent at a point U, that

$$\angle IT_AU = \angle IDU = 90^\circ$$

and that U, I, D, T_A are concyclic (See *Picture 2*).

Now let H' be the orthocenter of triangle DEF. Then $DH' \perp MN$ and since $UIDT_A$ is cyclic we get

$$\angle IDK = \angle IDH' = 90^\circ - \angle DIU = \angle DUI = \angle IT_AD.$$

And since $IK \cdot IT_A = ID^2 = r^2$ this implies that lines DH' and IT_A intersect at K.

Now, since $\triangle IKD \sim \triangle IDT_A$ we have that

$$KD = \frac{ID}{IT_A} \cdot DT_A = r \cdot \frac{DT_A}{IT_A}. \qquad (*)$$

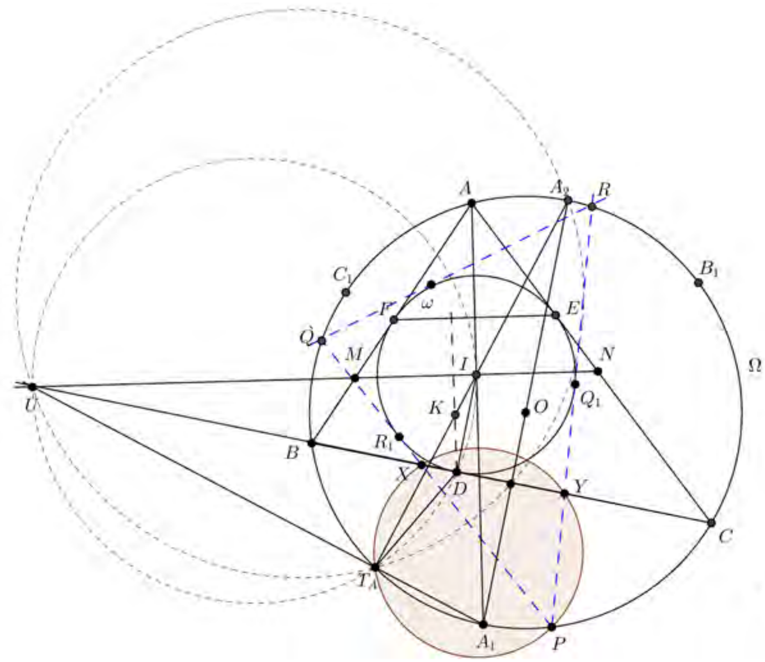
The line A_1A_2 passes through O and is perpendicular to BC. So, since $UIDT_A$ is cyclic

$$\angle IA_2A_1 = \angle T_AA_2A_1 = \angle T_AUD = \angle T_AID$$

and since $\angle A_1IU = \angle IT_AU = 90^\circ$ we get that

$$\angle A_2IA_1 = 180^\circ - \angle T_AIA_1 = 180^\circ - \angle T_AUI = \angle T_ADI.$$

Hence $\triangle IDT_A \sim \triangle A_1IA_2$.



Picture 2

From $(*)$ we have

$$KD = r \cdot \frac{DT_A}{IT_A} = r \cdot \frac{IA_1}{A_1A_2} = r \cdot \sin\frac{\angle BAC}{2} = r \cdot \cos\left(90^\circ - \frac{\angle BAC}{2}\right) = \frac{DH'}{2}$$

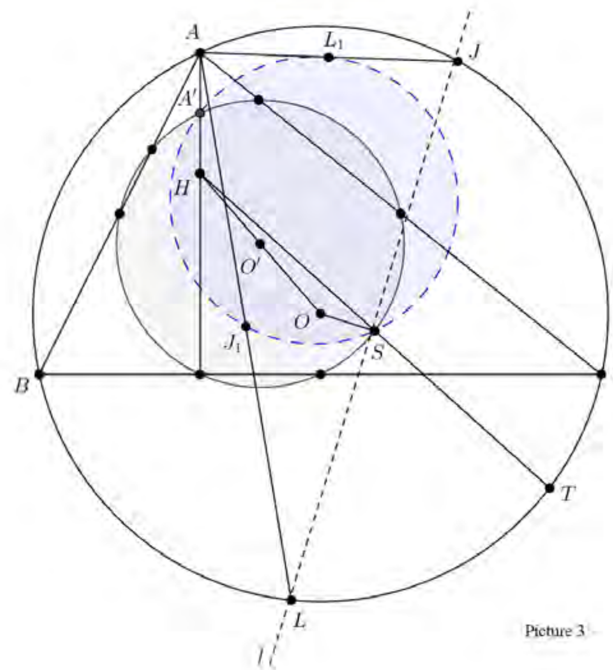
and thus K is the midpoint of DH'.

Lemma 1. Let ABC be an arbitrary triangle and let H be its orthocenter. Let O be the circumcenter of triangle ABC and let O' be the center of the nine-point circle of triangle ABC (the midpoint of segment OH). Suppose that a point S lies on the nine-point circle of triangle ABC and let l be the line that passes through S such that $OS \perp l$. If l cuts the circumcircle of triangle ABC at points L, J then the nine-point circle of triangle ALJ passes through the midpoint of AH.

Proof of Lemma 1. Let ray HS intersect the circumcircle of triangle ABC at point T. Obviously, $HS = ST$ and $LHJT$ is parallelogram. So

$$\angle LHJ = \angle LTJ = 180^\circ - \angle BAC.$$

Let A', J_1 and L_1 be the midpoints of segments AH, AL and AJ, respectively. Then $J_1A' \parallel LH$, $L_1A' \parallel JH$ and $\angle J_1A'L_1 = \angle LHJ = 180^\circ - \angle BAC = 180^\circ - \angle J_1PL_1$. Hence the points J_1, A', L_1 and P are concyclic. Therefore, the midpoint of AH lies on the ninepoint circle of triangle ALJ, as desired. (See *Picture 3*). This proves *Lemma 1.*



Picture 3

Now, we will use the *Lemma 1* and prove the *THEOREM.* Let $PQ \cap (I) = R_1$ and $PR \cap (I) = Q_1$ (See the *Picture 2*). Suppose that after inversion about (I)

$$P \to P', \qquad X \to X', \qquad Y \to Y', \qquad A \to A'.$$

Then P', X', Y', A' are the midpoints of segments Q_1R_1, DR_1, DQ_1 and EF, respectively. We have $T_A \to K$, $(O) \to (DEF)^{\frac{1}{2}}$ where, K be the midpoint of DH' and $(DEF)^{\frac{1}{2}}$ is nine-point circle of the triangle DEF. Obviously, $(DEF)^{\frac{1}{2}}$ passes through points K and P'. And, line Q_1R_1 passes through P' and satisfies $IP' \perp Q_1R_1$. Therefore from *Lemma 1*, the point K lies on ninepoint circle of the triangle DQ_1R_1. So the points K, X', Y' and P' are concyclic.

Hence the points P, T_A, X and Y are also concyclic. This completes the proof of *THEOREM.*

References

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[2] K.L.Nguyen, J.C.Salazar, On Mixtilinear Incircles and excircles, *Forum Geometricorum*, Vol. 6, 2006.

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