A Fine Use of Transformations

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1 Introduction

In this paper, we discuss a very fundamental, yet underexposed, idea to geometry. An *affine transformation* between two affine spaces U, V is a transformation $\varphi : U \to V$ such that for any three collinear points A, B, and C, the points $A' = \varphi(A)$, $B' = \varphi(B)$, and $C' = \varphi(C)$ are collinear, and

$$\frac{CA}{BA} = \frac{C'A'}{B'A'}$$

In the scope of this paper, we can think of an affine transformation of the plane – we will work only with affine transformations between $\mathbb{R}^2$ and itself – to be any transformation that preserves collinearity and ratios of segments.

Theorem 1.1. *All affine transformations can be written as* $\varphi(x, y) = (a_1x + b_1y + c_1, a_2x + b_2y + c_2)$.

Proof. Suppose φ is an affine transformation, and let $\psi(P) = \varphi(P) - \varphi(0)$. Then 0, A, and αA are mapped to $\varphi(0)$, $\varphi(A)$, and $\varphi(\alpha A) = \varphi(0) + \alpha(\phi(A) - \phi(0))$, so $\psi(\alpha A) = \alpha\psi(A)$. Also, $\psi(A + B) = 2\psi\left(\frac{A+B}{2}\right) = 2\left(\psi\left(\frac{A}{2}\right) + \psi\left(\frac{B}{2}\right)\right) = \psi(A) + \psi(B)$, so ψ is a linear map, as desired. □

Corollary. *We can classify all affine transformations as a combination rotations, translations, dilations, and shears. A shear is a transformation of the form* $(x, y) \mapsto (x + ky, y)$.

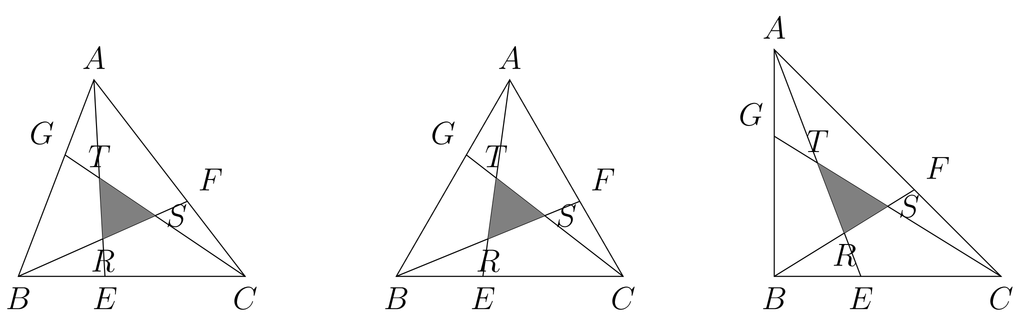
The most important property of affine transformations is that they preserve ratios of lengths and areas. This often allows us to make very magical assumptions about a configuration, like that a triangle is equilateral, a parallelogram is a square, or an ellipse is a circle.

2 Some Applications

We begin with a problem of lines.

Example 2.1 (Putnam 2001)**.** $\triangle ABC$ has an area 1. Points E, F, G lie, respectively, on sides BC, CA, AB such that AE bisects BF at point R, BF bisects CG at point S, and CG bisects AE at point T. Find the area of $\triangle RST$.

Solution. When Samuel Zbarsky presented his solution to this problem at the Carnegie Mellon Putnam Seminar, he began it with, "We assume that $\triangle ABC$ is both equilateral and isosceles-right." That is, a shear along BC will preserve all of the ratios in the problem, so we swap back and forth between whichever one makes calculations easiest. Consider the following three diagrams:



By symmetry in the equilateral case, we can see that $\frac{AG}{AB} = \frac{BE}{BC} = \frac{CF}{CA} = r$. In the isosceles-right case, we can use the fact that CG bisects AE to obtain the identity $(1-r)\left(1-\frac{1}{2}r\right) = \frac{1}{2} \implies r = \frac{3-\sqrt{5}}{2}$. Now we know that $\frac{CT}{CG} = \frac{1}{2(1-r)}$, so we can compute $\frac{ST}{SG} = \frac{r}{1-r}$. Also, since $BS = SG$, we have

$$\frac{SR}{SB} = \frac{BS - BR}{BS} = 1 - \frac{BR}{GS} = 1 - \frac{BF}{CG} = 1 - \frac{\sqrt{r^2 + (1-r)^2}}{\sqrt{1^2 + (1-r)^2}} = r$$

Finally, we have

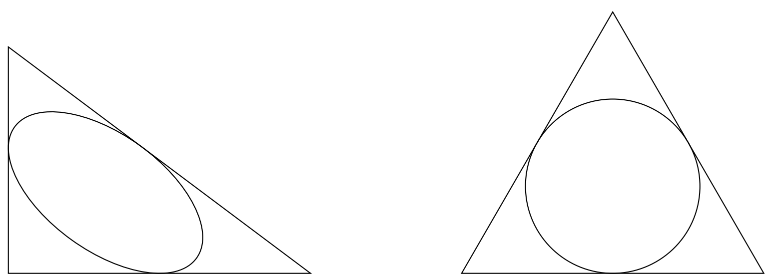
$$[RST] = \frac{ST}{SG} \cdot \frac{SR}{SB} \cdot [BSG] = \frac{1}{2} \cdot \frac{r^2}{1-r} \cdot [BCG] = \frac{r^2}{2} \cdot [ABC] = \frac{7 - 3\sqrt{5}}{4} \cdot [ABC]$$

□

Affine transformations are excellent for problems involving ellipses, since an ellipse is the image of a circle under an affine transformation.

Example 2.2 (Lehigh 2015)**.** What is the area of the largest ellipse that can be inscribed in a triangle with sides 3, 4, and 5?

Solution. We can perform an affine transformation mapping the $3-4-5$ triangle to an equilateral triangle:



The ellipse of maximal area in the original triangle will map to the ellipse of maximal area in the new triangle, which will be its incircle by symmetry. In the equilateral triangle, we compute the ratio of the incircle to total area:

$$\frac{\pi r^2}{K} = \frac{\pi (K/s)^2}{K} = \frac{\pi K}{s^2} = \frac{\pi(\ell^2\sqrt{3}/4)}{(3\ell/2)^2} = \frac{\pi}{3\sqrt{3}}$$

Since the transformation preserves ratios of areas, we just need to scale back to get the desired ellipse's area: $\frac{\pi}{3\sqrt{3}} \cdot \frac{1}{2}(3)(4) = \frac{2\pi}{\sqrt{3}}$. This ellipse of maximal area is called the *Steiner inellipse*, and its center is the centroid of the triangle. □

3 Some More Gems

Try the following problems using the right affine transformations.

Exercise 3.1 (Putnam 1994)**.** Let A be the area of the region in the first quadrant bounded by the line $y = \frac{1}{2}x$, the x-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$. Find the positive number m such that A is equal to the area in the first quadrant bounded by the line $y = mx$, the y-axis, and the ellipse $\frac{1}{9}x^2 + y^2$.

Exercise 3.2 (Morocco 2015)**.** A line passes through parallelogram $ABCD$ and intersects the segments AB, AC, and AD at E, F, and G, respectively. Prove that $\frac{AB}{AE} + \frac{AD}{AG} = \frac{AC}{AF}$.

Exercise 3.3. In $\triangle ABC$ let D be the foot of the perpendicular from A to BC, and let X be any point on AD. BX and CX intersect AC and AB at E and F, respectively. Show that $\angle EDX = \angle XDF$.

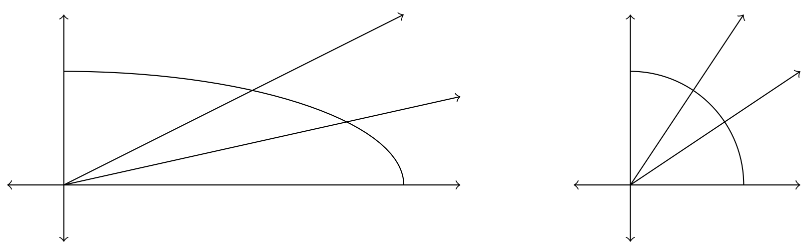
Exercise 3.4 (AIME 2015)**.** A block of wood has the shape of a right circular cylinder with radius 6 and height 8, and its entire surface has been painted blue. Points A and B are chosen on the edge of one of the circular faces of the cylinder so that $\widehat{AB}$ on that face measures $120°$. The block is then sliced in half along the plane that passes through point A, point B, and the center of the cylinder, revealing a flat, unpainted face on each half. The area of one of these unpainted faces is $a \cdot \pi + b\sqrt{c}$, where a, b, and c are integers and c is not divisible by the square of any prime. Find $a + b + c$.

Exercise 3.5 (CMIMC 2016)**.** Let $\mathcal{P}$ be the unique parabola in the xy-plane which is tangent to the x-axis at $(5, 0)$ and to the y-axis at $(0, 12)$. We say a line ℓ is "$\mathcal{P}$-friendly" if the x-axis, y-axis, and $\mathcal{P}$ divide ℓ into three segments, each of which has equal length. If the sum of the slopes of all $\mathcal{P}$-friendly lines can be written in the form $-\frac{m}{n}$ for m and n positive relatively prime integers, find $m + n$.

4 Solutions

Exercise 3.1 (Putnam 1994)**.** Let A be the area of the region in the first quadrant bounded by the line $y = \frac{1}{2}x$, the x-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$. Find the positive number m such that A is equal to the area in the first quadrant bounded by the line $y = mx$, the y-axis, and the ellipse $\frac{1}{9}x^2 + y^2$.

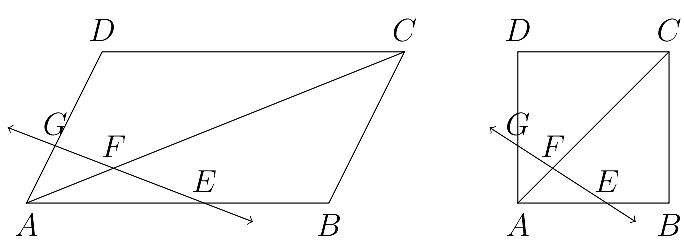
Solution. We could solve this problem using a nasty integral. Instead, we scale along the x-axis, transforming the ellipse into a circle:



The $y = \frac{1}{2}x$ line is transformed into the line $y = \frac{3}{2}x$. By symmetry, the $y = mx$ line must be transformed into the line $y = \frac{2}{3}x$. Transforming back, we get $m = \frac{2}{9}$. □

Exercise 3.2 (Morocco 2015)**.** A line passes through parallelogram $ABCD$ and intersects the segments AB, AC, and AD at E, F, and G, respectively. Prove that $\frac{AB}{AE} + \frac{AD}{AG} = \frac{AC}{AF}$.

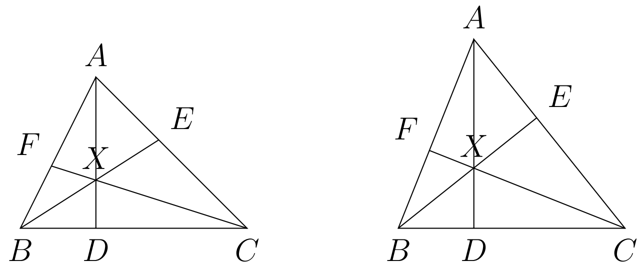
Solution. Perform an affine transformation mapping the parallelogram to the square $[0, 1]^2$.



Now we can let $E = (u, 0)$ and $G = (0, v)$ to get $F = \left(\frac{uv}{u+v}, \frac{uv}{u+v}\right)$. We conclude with the relation $\frac{1}{u} + \frac{1}{v} = \sqrt{2} \cdot \frac{u+v}{uv\sqrt{2}}$. □

Exercise 3.3. In $\triangle ABC$ let D be the foot of the perpendicular from A to BC, and let X be any point on AD. BX and CX intersect AC and AB at E and F, respectively. Show that $\angle EDX = \angle XDF$.

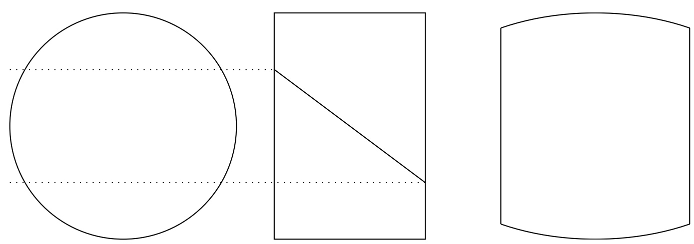
Solution. Perform a dilation centered at D along the AD axis so that X is the orthocenter of ABC. Note that this would preserve the fact that DE and DF are reflections over AD.



Since $BDXF$ and $CDXE$ are cyclic, $\angle EDX = \angle ECX = 90° - \angle A = \angle XBF = \angle XDF$. $\square$

Exercise 3.4 (AIME 2015)**.** A block of wood has the shape of a right circular cylinder with radius 6 and height 8, and its entire surface has been painted blue. Points A and B are chosen on the edge of one of the circular faces of the cylinder so that $\widehat{AB}$ on that face measures $120°$. The block is then sliced in half along the plane that passes through point A, point B, and the center of the cylinder, revealing a flat, unpainted face on each half. The area of one of these unpainted faces is $a \cdot \pi + b\sqrt{c}$, where a, b, and c are integers and c is not divisible by the square of any prime. Find $a + b + c$.

Solution. The trace of the cut is a truncated ellipse.



The length of one axis is the diameter of the circle, 12. We can calculate the length of the other axis from similar triangles to be 20. Furthermore, it is truncated at a distance 5 from the center. It remains to use this information to calculate the area of this truncated ellipse.

We perform an affine transformation mapping the ellipse to a truncated circle of radius 6. Then this circle is truncated at distance $\frac{6}{10} \cdot 5 = 3$ from the center, so the two arcs of the truncated circle each have measure $60°$. This means the area of the truncated circle is

$$2 \cdot 6^2 \cdot \pi \cdot \frac{60°}{360°} + 2 \cdot \frac{1}{2} \cdot 6^2 \cdot \frac{\sqrt{3}}{2} = 12\pi + 18\sqrt{3}$$

Scaling back, that is $\frac{10}{6}(12\pi + 18\sqrt{3}) = 20\pi + 30\sqrt{3}$. $\square$

Exercise 3.5 (CMIMC 2016)**.** Let $\mathcal{P}$ be the unique parabola in the xy-plane which is tangent to the x-axis at $(5, 0)$ and to the y-axis at $(0, 12)$. We say a line ℓ is "$\mathcal{P}$-friendly" if the x-axis, y-axis, and $\mathcal{P}$ divide ℓ into three segments, each of which has equal length. If the sum of the slopes of all $\mathcal{P}$-friendly lines can be written in the form $-\frac{m}{n}$ for m and n positive relatively prime integers, find $m + n$.

Solution. Scale along the y-axis so that $(0, 12)$ is replaced with $(0, 5)$. The answer will be $\frac{12}{5}$ times as big as this new problem's answer. Rotate the parabola $45°$, and using the angle addition formulas, we find that for each slope m in this new frame, the slope in the original frame is $\frac{m-1}{m+1}$.

We are now looking at the parabola that is tangent to the lines $y = \pm x$ at $\left(\pm\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}}\right)$. Simple calculations reveal this to be the parabola $\mathcal{P}' : y = \frac{\sqrt{2}}{10}x^2 + \frac{5\sqrt{2}}{4}$. Let $A = (a, -a)$ and $B = (b, b)$, and we want $\left(\frac{1}{3}a + \frac{2}{3}b, -\frac{1}{3}a + \frac{2}{3}b\right)$ and $\left(\frac{2}{3}a + \frac{1}{3}b, -\frac{2}{3}a + \frac{1}{3}b\right)$ to be on $\mathcal{P}'$:

$$\begin{cases} -\frac{1}{3}a + \frac{2}{3}b = \frac{\sqrt{2}}{10}\left(\frac{1}{3}a + \frac{2}{3}b\right)^2 + \frac{5\sqrt{2}}{4} \\ -\frac{2}{3}a + \frac{1}{3}b = \frac{\sqrt{2}}{10}\left(\frac{2}{3}a + \frac{1}{3}b\right)^2 + \frac{5\sqrt{2}}{4} \end{cases}$$

Subtracting, we get $a + b = \frac{\sqrt{2}}{10}(a+b)(b-a)$. $a + b = 0$ means that the line is horizontal, which gives way to two solutions – easily verifiable from the intermediate value theorem. Otherwise, $b - a = 5\sqrt{2} \implies m = \frac{b+a}{b-a} = \frac{\sqrt{2}}{5}b - 1$. Thus, in the original frame the slope is $1 - \frac{5\sqrt{2}}{b}$. We can use Vieta's formulas in the original equation,

$$-\frac{1}{3}(b - 5\sqrt{2}) + \frac{2}{3}b = \frac{\sqrt{2}}{10}\left(\frac{1}{3}(b - 5\sqrt{2}) + \frac{2}{3}b\right)^2 + \frac{5\sqrt{2}}{4}$$

to evaluate the sum of these two slopes to be $2 - 36$. Adding the two horizontal lines that each contribute -1 to the sum, we get that the sum of all of the slopes is -36, which scales back to $-36 \cdot \frac{12}{5} = -\frac{432}{5}$. $\square$

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