Junior Problems

J379. Prove that for any nonnegative real numbers a, b, c the following inequality holds:

$$(a-2b+4c)(-2a+4b+c)(4a+b-2c) \leq 27abc.$$

Proposed by Adrian Andreescu, Dallas, Texas

J380. Let $x_1, x_2, \ldots, x_n$ be nonnegative real numbers such that

$$x_1 + x_2 + \cdots + x_n = 1.$$

(a) Find the minimum value of

$$x_1\sqrt{1+x_1} + x_2\sqrt{1+x_2} + \cdots + x_n\sqrt{1+x_n}.$$

(b) Find the maximum value of

$$\frac{x_1}{1+x_2} + \frac{x_2}{1+x_3} + \cdots + \frac{x_n}{1+x_1}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J381. Let x, y, z be positive real numbers such that $x+y+z=3$. Prove that

$$\frac{xy}{4-y} + \frac{yz}{4-z} + \frac{zx}{4-x} \leq 1.$$

Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia

J382. Find all triples (x, y, z) of real numbers with $x, y, z > 1$ satisfying

$$\left(\frac{x}{2} + \frac{1}{x} - 1\right)\left(\frac{y}{2} + \frac{1}{y} - 1\right)\left(\frac{z}{2} + \frac{1}{z} - 1\right) = \left(1 - \frac{x}{yz}\right)\left(1 - \frac{y}{zx}\right)\left(1 - \frac{z}{xy}\right)$$

Proposed by Alessandro Ventullo, Milan, Italy

J383. Let ABC be a triangle with $AB = AC$ and $\angle BAC = 72^\circ$. Let D and E be the points on sides AB and AC, respectively, such that $\angle ACD = 12^\circ$ and $\angle ABE = 30^\circ$. Prove that $DE = CE$.

Proposed by Marius Stănean, Zalău, România

J384. In triangle ABC, $A < B < C$. Prove that

$$\cos\frac{A}{2}\csc\frac{B-C}{2} + \cos\frac{B}{2}\csc\frac{C-A}{2} + \cos\frac{C}{2}\csc\frac{A-B}{2} < 0.$$

Proposed by Titu Andreescu, University of Texas at Dallas

Senior Problems

S379. Prove that in any triangle ABC

$$\cos 3A + \cos 3B + \cos 3C + \cos(A - B) + \cos(B - C) + \cos(C - A) \geq 0.$$

Proposed by Titu Andreescu, University of Texas at Dallas

S380. Let a, b, c be real numbers such that $abc = 1$. Prove that

$$\frac{a + ab + 1}{(a + ab + 1)^2 + 1} + \frac{b + bc + 1}{(b + bc + 1)^2 + 1} + \frac{c + ca + 1}{(c + ca + 1)^2 + 1} \leq \frac{9}{10}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S381. Let $ABCD$ be a cyclic quadrilateral and M and N be the midpoints of the diagonals AC and BD. Prove that

$$MN \geq \frac{1}{2}|AC - BD|.$$

Proposed by Titu Andreescu, University of Texas at Dallas

S382. Prove that in any triangle ABC the following inequality holds:

$$\frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} + \frac{r}{R} \leq 2.$$

Proposed by Florin Stănescu, Găeşti, România

S383. Solve in positive integers the equation

$$x^6 - y^6 = 2016xy^2.$$

Proposed by Adrian Andreescu, Dallas, Texas

S384. Let ABC be a triangle with circumcenter O and orthocenter H. Let D, E, F be the feet of the altitudes from A, B, C, respectively. Let K be the intersection of AO with BC and L be the intersection of AO with EF. Furthermore, let T be the intersection of AH and EF, and let S be the intersection of KT and DL. Prove that BC, EF, SH are concurrent.

Proposed by Bobojonova Latofat and Khurshid Juraev, Tashkent, Uzbekistan

Undergraduate Problems

U379. Let a, b, c be nonnegative real numbers. Prove that

$$a^3+b^3+c^3-3abc \geq k|(a-b)(b-c)(c-a)|,$$

where $k=\left(\frac{27}{4}\right)^{1/4}(1+\sqrt{3})$ and that k is the best possible constant.

Proposed by Albert Stadler, Herrliberg, Switzerland

U380. Prove that for all positive real numbers a, b, c the following inequality holds:

$$\frac{1}{4a}+\frac{1}{4b}+\frac{1}{4c}+\frac{1}{2a+b+c}+\frac{1}{2b+c+a}+\frac{1}{2c+a+b} \geq \frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U381. Find all positive integers n such that

$$\sigma(n)+d(n)=n+100.$$

(We denote by $\sigma(n)$ the sum of the divisors of n and by $d(n)$ the number of divisors of n.)

Proposed by Alessandro Ventullo, Milan, Italy

U382. Prove that

$$\int_0^1 \prod_{k=1}^{\infty}\left(1-x^k\right)\,dx=\frac{4\pi\sqrt{3}}{\sqrt{23}}\frac{\sinh\frac{\pi\sqrt{23}}{3}}{\cosh\frac{\pi\sqrt{23}}{2}}$$

Proposed by Albert Stadler, Herrliberg, Switzerland

U383. Let $n \geq 2$ be an integer and A and B be two $n \times n$ matrices with complex entries such that $A^2=B^2=O$ with $A+B$ being invertible. Prove that n is even and $\operatorname{rank}(AB)^k=n/2$, for all $k \geq 1$.

Proposed by Florin Stănescu, Găeşti, România

U384. Let m and n be positive integers. Evaluate

$$\lim_{x\to 0}\frac{(1+x)\left(1+\frac{x}{2}\right)^2\cdots\left(1+\frac{x}{m}\right)^m-1}{(1+x)\sqrt{1+2x}\cdots\sqrt[n]{1+nx}-1}$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Olympiad Problems

O379. Let a, b, c, d be real numbers such that $a^2+b^2+c^2+d^2=4$. Prove that

$$\frac{2}{3}(ab+bc+cd+da+ac+bd) \le (3-\sqrt{3})abcd + 1 + \sqrt{3}$$

Proposed by Marius Stănean, Zalău, România

O380. Let ABC be a triangle with orthocenter H. Let X and Y be points on side BC such that $\angle BAX = \angle CAY$. Let E and F be the feet of the altitudes from B and C, respectively. Let T and S be the intersections of EF with AX and AY, respectively. Prove that X, Y, S, T are concyclic. Furthermore, prove that H lies on the polar of A with respect to this circle.

Proposed by Bobojonova Latofat and Khurshid Juraev, Tashkent, Uzbekistan.

O381. Let a, b, c be positive real numbers. Prove that

$$\frac{a^3+b^3+c^3}{3} \ge \frac{a^2+bc}{b+c} \cdot \frac{b^2+ca}{c+a} \cdot \frac{c^2+ab}{a+b} \ge abc.$$

Proposed by An Zhen-ping, Xianyang Normal University, China

O382. Prove that in any triangle ABC

$$\left(\frac{m_a+m_b+m_c}{3}\right)^2 - \frac{m_a m_b m_c}{m_a+m_b+m_c} \le \frac{a^2+b^2+c^2}{6}$$

Proposed by Titu Andreescu, University of Texas at Dallas

O383. Let a, b, c be positive real numbers. Prove that

$$\frac{a+b}{6c} + \frac{b+c}{6a} + \frac{c+a}{6b} + 2 \ge \sqrt{\frac{a+b}{2c}} + \sqrt{\frac{b+c}{2a}} + \sqrt{\frac{c+a}{2b}}$$

Proposed by Marius Stănean, Zalău, România

O384. Let ω_1 and ω_2 be circles intersecting at points A and B. Let CD be their common tangent such that C, D lie on ω_1, ω_2, respectively; and A is closer to CD than B. Let CA and CB intersects ω_2 at A, E and B, F, respectively. Lines DA and DB intersects ω_1 at A, G and B, H, respectively. Let P be the intersection of CG and DE and Q be the intersection of EG and FH. Prove that A, P, Q lie on the same line.

Proposed by Anton Vassilyev, Astana, Kazakhstan