About a Fibonacci Problem

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In the history of mathematics it is known that one of the first arithmetic problem which made Fibonacci famous was the following: find a (rational) number x such that $x^2 + x$ and $x^2 - x$ are simultaneously perfect squares (in $\mathbb{Q}$). He presented the solution $x = \frac{25}{24}$, for which

$$x^2 - x = \frac{5^2}{24^2} \text{ and } \mathrm{x}^2 + \mathrm{x} = \frac{35^2}{24^2}.$$

We do not know how he proceeded to find this solution. Also, there is no information if Fibonacci knew other solutions. We present a general method to find all numbers which are solutions of this problem.

Expressed algebraically, this Fibonacci problem asks to find all rational solutions of the system:

$$\begin{aligned} x^2 - x &= y^2 \\ x^2 + x &= z^2 \end{aligned} \tag{1}$$

The system (1) is clearly equivalent to the following:

$$\begin{aligned} 2x^2 &= y^2 + z^2 \\ x^2 - x &= y^2 \end{aligned} \tag{2}$$

Since (1) and (2) are equivalent, we may work with them simultaneously. It is clear that we can restrict the problem to find only positive solutions $x, y, z > 0$. Assume that

$$x = \frac{a}{d}; y = \frac{b}{d}; z = \frac{c}{d},$$

where a, b, c, d are positive integers. From (2), we obtain the system

$$\begin{aligned} 2a^2 &= b^2 + c^2 \\ a^2 - ad &= b^2, \end{aligned} \tag{3}$$

and also we may consider the additional equation

$$a^2 + ad = c^2 \tag{4}$$

First, we prove that we may assume that $\gcd(a, d) = 1$. Indeed, if $\gcd(a, d) = \delta$, $\delta > 1$, we arrive at $\delta^2 | b^2$ and $\delta^2 | c^2$. Then, $\delta | b$ and $\delta | c$, and the fractions $y = \frac{b}{d}; \frac{c}{d}$ can be simplified. Now, from $\gcd(a, d) = 1$, we have $\gcd(a, a - d) = 1$. From the equation $a^2 - ad = b^2$, we obtain that a and $a - d$ are perfect squares. Denote

$$\begin{aligned} a &= A^2 \\ a - d &= B^2, \end{aligned} \tag{5}$$

where $\gcd(A, B) = 1$ and $b = AB$. Again, from $\gcd(a, d) = 1$ we have that $\gcd(a, a + d) = 1$ and using the same argument, but for the equation (4), one obtains that

$$\begin{aligned} a + d &= C^2 \\ c &= AC, \end{aligned} \tag{6}$$

where $\gcd(A, C) = 1$. From these equations, we obtain that A, B, C satisfy the Diophantine equation

$$B^2 + C^2 = 2A^2 \tag{7}$$

Lemma. The general integer solution of the equation

$$u^2 + v^2 = 2t^2,$$

with gcd(u,t)=gcd$(v,t) = 1$, is given by relations

$$u = -m^2 + 2mn + n^2;\ v = m^2 + 2mn - n^2;\ t = m^2 + n^2,$$

where m, n are integers of distinct parities and pairwise prime.

Proof. Divide the equation by t^2 and denote $\frac{u}{t} = x$, $\frac{v}{t} = y$. Then, the problem reduces to finding the rational points (x, y) on the circle satisfying

$$x^2 + y^2 = 2.$$

We proceed by a standard method. The point $A(-1,-1)$ lies on the circle. Taking the pencil of lines

$$L_\lambda : y + 1 = \lambda(x + 1)$$

through the point A, we observe that every line L_λ intersects the circle again at a rational point if and only if λ is rational. Reciprocally, if a line of the pencil intersects the circle at a rational point, then λ is rational. Then, by intersecting the circle with L_λ, we get a second point of coordinates

$$x = \frac{-\lambda^2 + 2\lambda + 1}{\lambda^2 + 1},\ \ y = \frac{\lambda^2 + 2\lambda - 1}{\lambda^2 + 1}.$$

If we denote $\lambda = \frac{m}{n}$ where m, n are relatively prime integers, we obtain

$$x = \frac{-m^2 + 2mn + n^2}{m^2 + n^2},\ \ y = \frac{m^2 + 2mn - n^2}{m^2 + n^2}.$$

This proves the lemma.

Going back to our problem, the lemma implies that the equation (7) has the general solution

$$\begin{aligned} A &= m^2 + n^2 \\ B &= -m^2 + 2mn + n^2 \\ C &= m^2 + 2mn - n^2, \end{aligned} \tag{8}$$

where m, n are relatively prime integers of distinct parities. Moreover, since $C > B$, we have $m > n$. Returning to the original problem we have

$$\begin{aligned} a &= A^2 = (m^2 + n^2)^2 \\ d &= A^2 - B^2 = 4mn(m^2 - n^2) \\ b &= AB = (m^2 + n^2)(-m^2 + 2mn + n^2) \\ c &= AC = (m^2 + n^2)(m^2 + 2mn - n^2). \end{aligned} \tag{9}$$

In conclusion, the general solution to Fibonacci problem is

$$x = \frac{(m^2 + n^2)^2}{4mn(m^2 - n^2)},$$

$$y = \frac{(m^2 + n^2)(-m^2 + 2mn + n^2)}{4mn(m^2 - n^2)},$$

$$z = \frac{(m^2 + n^2)(m^2 + 2mn - n^2)}{4mn(m^2 - n^2)}.$$

For $m = 2$ and $n = 1$, we obtain

$$x = \frac{25}{24};\ y = \frac{5}{24};\ z = \frac{35}{24},$$

which is a Fibonacci solution. For example, for $m = 3$ and $n = 2$, we have

$$x = \frac{169}{120};\ y = \frac{91}{120};\ z = \frac{221}{120}.$$